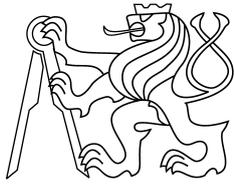




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PROGRAMME

Thursday, December 10, 2009

13:00-13:30 **Petr Hájek**: An MV-algebra for vagueness

13:30-14:00 **Denisa Diaconescu**: On the forcing semantics for MTL logic

14:00-14:30 **Claudia Mureşan**: Co-Stone Residuated Lattices

15:00-15:30 **Kateřina Helisová**: MCMC simulation of a random set

15:30-16:00 **Roman Frič**: Notes on fuzzy random variables

16:30-17:00 **Milan Petřík** and Peter Sarkoci: Web-geometric approach to models of fuzzy logic

17:00-17:30 **Mirko Navara**: Several approaches to conditional probability

Friday, December 11, 2009

13:00-13:30 **Jan Paseka**: Non-orthomodular lattice effect algebras possessing state

13:30-14:00 **Eva Drobná**, Ferdinand Chovanec, and Ol'ga Nánásiová: Independence in D-posets

14:00-14:30 **Ferdinand Chovanec**: Graphical Representation of MV-algebra pastings

15:00-15:30 **Tomáš Kroupa**: Games, states, and ordered algebraic structures

15:30-16:00 Milan Matoušek and **Pavel Pták**: Abstract symmetric difference

16:00-16:30 **Milan Matoušek**: Orthocomplemented lattices with a symmetric difference

Graphical representation of MV-algebra pastings

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We deal with a construction of some difference posets [1] via a method of a pasting of MV-algebras. We give some re-formulations of the basic notions of an MV-algebra pasting introduced in [2] and we present a considerable generalization of the Greechie's Loop Lemma [3] that gives the necessary and sufficient conditions under which a pasting of Boolean algebras (so-called Greechie logic) is an orthomodular lattice.

We suggest a generalization of Greechie diagrams used in MV-algebra pastings.

Acknowledgments. This work has been supported by the Slovak Research and Development Agency under the contract No. APVV-0071-06.

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On the semantics for MTL logic

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Monoidal t-norm based logic (MTL logic) is the logic of left-continuous t-norms and was introduced by Esteva and Godo [1]. We define the weak forcing value $|\varphi|_{\mathcal{X}}$ of an MTL formula φ in a complete MTL-algebra \mathcal{X} . Besides the truth value $\|\varphi\|_{\mathcal{X}}$, $|\varphi|_{\mathcal{X}}$ constitutes an alternative way to evaluate the formula φ in \mathcal{X} . The weak forcing value is a refinement of the notion of validity in a Kripke model (in the sense of [2] and [3]).

It is of great interest to compare the two kinds of semantics: the truth value and the weak forcing. We conclude that the axioms (A2), (A6), (A7), (A9) and (A10) are not valid in the new semantics.

In [2] and [3], it was proved that the r-forcing is a more adequate notion for reflecting the logical structure of MTL. Arising from r-forcing, we shall define the \mathcal{X} -valued forcing property and the forcing value $[\varphi]_{\mathcal{X}}$ of a formula of MTL in a complete MTL-algebra \mathcal{X} . The first notion is obtained from an \mathcal{X} -valued weak forcing property f by adding a condition that homogenizes the action of f w.r.t. the elements of X . Then one can define in a natural way the forcing value $[\varphi]_{\mathcal{X}}$, resulting a semantics $[\cdot]_{\mathcal{X}}$ distinct from $|\cdot|_{\mathcal{X}}$.

One of the main results of the above quoted papers asserts that the Kripke completeness (defined by means of r-forcing) coincides with the usual algebraic completeness of MTL. We shall extend this result by proving that $[\cdot]_{\mathcal{X}}$ and $\|\cdot\|_{\mathcal{X}}$ coincide.

We also give an algebraic proof for the fact that the forcing value coincides with the truth value by using the notion of forcing operator.

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Independence in D-posets

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Problems and results concerning the definition of the conditional probability, conditional system and independence of events on orthomodular lattices are extended to a more general structural basis, on a D-poset. Substantial differences in a comprehensive characterization of a conditional system are given.

Conditional probabilities on a classical measurable space are studied in several different ways, but result in equivalent theories. The classical probability theory does not describe the causality model.

The situation changes when non-standard spaces are considered. For example, it is well known that the set of random events in quantum mechanical experiments is a more general structure than Boolean algebra. In the quantum logic approach the set of random events is assumed to be a quantum logic L . Such a model can be found not only in the quantum theory, but also in economics, biology, etc.

A conditional state on a D-poset [1] (as it is the most general algebraic structure describing random events) is defined using Renyi's approach (or Bayesian principle). This approach helps us to define such independence of events that admits the situation completely different from that known in the classical probability theory. Namely, if an event A is independent of an event B , then the event B can be dependent on the event A [2], [3] (problem of causality).

Comparison of conditional probability on regular and non-regular quantum structures is also specified. As it will be shown itself, the question of a conditional set, which is essential for the definition of a conditional state as a function of two variables, is much more complicated than in orthomodular lattices where the maximal conditional set contains all non-zero elements. The characterization of a maximal conditional set in a D-poset cannot be such plain.

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Notes on fuzzy random variables

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Let $(\Omega_i, \mathbf{A}_i, p_i)$, $i = 1, 2$, be probability spaces. We study measure preserving measurable maps $f : \Omega_1 \rightarrow \Omega_2$ and their fuzzy generalizations (measure preserving fuzzy random variables in the sense of Bugajski and Gudder, cf. [2]) called extended random maps. We study simple categorical situations and discuss relationships between such maps and conditional probabilities.

Background information can be found in [4], [3], [1].

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An MV-algebra for vagueness

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The philosophers criticize sharp break between true cases (having truth value 1) and borderline cases (< 1). Here we offer a fuzzy semantics (a particular non-standard MV-algebra) where this is (somehow) handled. Łukasiewicz propositional and predicate logic with its standard, general and witnessed semantics is assumed to be known (and will be briefly recalled.)

Take the lexicographic product of two copies of the ordered additive group of reals (operations coordinatewise, ordering lexicographic); this is a linearly ordered Abelian group. Now our MV-chain K_{Re} is the interval $[(0, 0), (1, 0)]$ with the usual operations; thus a pair $(x, y) \in Re^2$ is in K_{Re} iff $(x = 0 \text{ and } y \geq 0)$ or $(0 < x < 1 \text{ and } y \text{ arbitrary})$ or $(x = 1 \text{ and } y \leq 0)$. For each $(x, y) \in K_{Re}$ let $\beta(x, y)$ be the set of all elements of K_{Re} with the first coordinate x (the neighborhood of (x, y) or the set of elements infinitely near to (x, y)). The set $\beta(1, 0)$ is the set of degrees of almost full truth, say ft-degrees. Identify $(0, 0)$ with 0 and $(1, 0)$ with 1.

Recall that the operation $(a, b) * (c, d)$ is $\max[(0, 0), (a + c - 1, b + d)]$, in more details, $(a, b) * (c, d)$ is equal to

$$\begin{aligned} &(a + c - 1, b + d) \text{ if } a + c > 1, \\ &(0, \max(0, b + d)) \text{ if } a + c = 1, \\ &(0, 0) \text{ if } a + c < 1. \end{aligned}$$

The set of all $(x, 0) \in K_{Re}$ is (the domain of) a subalgebra of K_{Re} isomorphic with the standard MV-algebra $[0, 1]_L$. The mapping $f(x, y) = x$ (projection to the first coordinate) is a homomorphism of K_{Re} to (the copy of) $[0, 1]_L$. From this it easily follows (for propositional calculus):

THEOREM 1. *The following sets are equal: the set $1Taut([0, 1]_L)$ of standard L-1-tautologies, the set $1Taut(K_{Re})$ of K_{Re} -1-tautologies, and the set $\beta(1)Taut(K_{Re})$ of K_{Re} - $\beta(1)$ -tautologies. The same for satisfiability instead of tautologicity.*

Now turn to predicate calculus. The morphism f preserves all existing infinite sups and infs BUT be aware of what sups and infs in K_{Re} exist: a non-empty set $X \subseteq K_{Re}$ has infimum iff $f(X)$ has a minimum; similarly for sup and max (easy to see). Consequently,

LEMMA 2. *The homomorphic image $f(M)$ of a safe K_{Re} -structure is a witnessed $[0, 1]_L$ -structure. And each witnessed $[0, 1]_L$ -structure is a safe K_{Re} -structure modulo the embedding above.*

Now recall that the set of standard 1-tautologies of Łukasiewicz predicate calculus is equal to the set of standard witnessed 1-tautologies and similarly for satisfiables. (See my [2].) Our last lemma then gives:

THEOREM 3. (1) *A formula φ of predicate calculus is a $\beta(1)$ - K_{Re} -tautology (i.e. has a value in $\beta(1)$ for each model over K_{Re}) iff it is a (witnessed) standard 1-tautology of Łukasiewicz predicate calculus $L\forall$.*

(2) *φ is $\beta(1)$ - K_{Re} -satisfiable (i.e. has a value form $\beta(1)$ in some model over K_{Re}) iff it is (1)- K_{Re} -satisfiable iff it is satisfiable in a (witnessed) standard model over $L\forall$.*

COROLLARY 4. *The set $\beta(1)TAUT\forall(K_{Re})$ is Π_2 -complete and the set $\beta(1)SAT\forall(K_{Re}) = (1)SAT\forall(K_{Re})$ is Π_1 -complete.*

PROBLEM 5. What is the arithmetical complexity of the set $(1)TAUT\forall(K_{Re})$ of 1-tautologies over K_{Re} ? Is the set $(1)TAUT\forall(K_{Re})$ of 1-tautologies over K_{Re} equal to $\beta(1)TAUT\forall(K_{Re})$?

These results will be compared with the results on Chang's MV-algebra and Komori's MV-algebras as presented in the paper [3].

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MCMC simulation of a random set

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Monte Carlo Markov Chain (MCMC) simulations are very often used in stochastic geometry. This contribution concerns a method used for simulation of a random set given by a union of interacting discs where the interactions are described by a probability density of the modeled set with respect to a process of discs without interactions (so-called random-disc Boolean model). First, the simulation algorithm will be described, then some simplifying tools will be shown and finally, the usage of the simulations for deriving statistical inference will be briefly mentioned.

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Games, states, and ordered algebraic structures

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Theory of cooperative games was established by von Neumann and Morgenstern. Aumann and Shapley [1] generalized the original model by considering a possibly infinite set of players. A particular solution concept in cooperative game theory is the Shapley value, which is a positive and a symmetric linear operator from a space of (non-additive) functions to the space of measures (states). The existence and uniqueness of this operator is a key problem for various classes of games and coalition structures. This problem was solved in [3] for games on tribes and even in a more general setting in [2]. Ordered algebraic structures and associated games thus constitute a natural framework for studying coalition games, which is demonstrated in [4] on MV-algebras.

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Orthocomplemented lattices with a symmetric difference

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We investigate the relationship of ODLs with the class of orthomodular lattices (\mathcal{OML}). In particular, we study the variety \mathcal{OML}_Δ of the OMLs induced by ODLs. We see that $\mathcal{OML}_\Delta \neq \mathcal{OML}$ and we find out how this inequality is related to Z_2 -states on OMLs.

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Abstract symmetric difference

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In abstracting the set-theoretic symmetric difference, we introduce an algebra $(X, \wedge, \vee, \perp, 0, 1, \Delta)$, where $(X, \wedge, \vee, \perp, 0, 1)$ is an orthocomplemented lattice and Δ is a binary operation subject to the following axioms:

$$(D_1) \quad x\Delta(y\Delta z) = (x\Delta y)\Delta z,$$

$$(D_2) \quad x\Delta 1 = x^\perp, \quad 1\Delta x = x^\perp,$$

$$(D_3) \quad x\Delta y \leq x \vee y.$$

In the talk we survey some basic properties of these algebras and comment on the link with orthomodular lattices.

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Co-Stone residuated lattices

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Residuated lattices are the algebraic counterpart of monoidal logic; they include MTL-algebras, BL-algebras and MV-algebras. We recall that a *residuated lattice* is an algebraic structure $(A, \vee, \wedge, \odot, \rightarrow, 0, 1)$, with the first 4 operations binary and the last two constant, such that $(A, \vee, \wedge, 0, 1)$ is a bounded lattice, $(A, \odot, 1)$ is a commutative monoid and the following property, called *the law of residuation*, is satisfied: for all $a, b, c \in A$, $a \leq b \rightarrow c$ iff $a \odot b \leq c$, where \leq is the partial order of the lattice $(A, \vee, \wedge, 0, 1)$.

In [11] we gave an axiomatic purely algebraic definition of the reticulation of a residuated lattice, that we proved to be equivalent to the general notion of reticulation applied to residuated lattices, and which turned out to be very useful in practice. In the article that this abstract is based on, we present several applications for the reticulation, related to co-Stone algebras, applications in the form of transfers of properties between the category of bounded distributive lattices and the category of residuated lattices through the reticulation functor. This transfer of properties between different categories is the very purpose of the reticulation.

The co-Stone structures were introduced by us as being dual notions to Stone structures. Let A be a bounded distributive lattice or a residuated lattice; the definitions we are about to give are valid for both types of structures. For any non-empty subset X of A , the *co-annihilator* of X is the set $X^\top = \{a \in A \mid (\forall x \in X) a \vee x = 1\}$. In the case when X consists of a single element x , we denote the co-annihilator of X by x^\top and call it the *co-annihilator* of x . Also, we will denote $X^{\top\top} = (X^\top)^\top$ and $x^{\top\top} = (x^\top)^\top$. Let us remark that, for A a bounded distributive lattice or a residuated lattice and for any $X \subseteq A$, X^\top is a filter of A . We will denote the Boolean center of a bounded distributive lattice or a residuated lattice A by $B(A)$.

DEFINITION 1. Let A be a bounded distributive lattice or a residuated lattice. Then A is said to be *co-Stone* (respectively *strongly co-Stone*) iff, for all $x \in A$ (respectively all $X \subseteq A$), there exists an element $e \in B(A)$ such that $x^\top = \langle e \rangle$ (respectively $X^\top = \langle e \rangle$).

Concerning co-Stone and strongly co-Stone structures (by structure we mean here bounded distributive lattice or residuated lattice), the first question that arises

is whether they exist. Naturally, any strongly co-Stone structure is co-Stone and any complete co-Stone structure is strongly co-Stone. The answer to the question above is given by the fact that the trivial structure is strongly co-Stone and, moreover, any chain is strongly co-Stone, because a chain A clearly has all co-annihilators equal to $\{1\} = \langle 1 \rangle$, except for 1^\top , which is equal to $A = \langle 0 \rangle$.

We prove the fact that a residuated lattice is co-Stone iff its reticulation is co-Stone and the same is valid for strongly co-Stone structures, then we obtain a structure theorem for m -co-Stone residuated lattices, by transferring through the reticulation a known characterization of m -co-Stone bounded distributive lattices to residuated lattices. This is the first major example of a result that can be transferred through the reticulation functor from the category of bounded distributive lattices to the category of residuated lattices. It also permits us to state that a residuated lattice is m -co-Stone iff its reticulation is m -co-Stone. Here is the characterization of m -co-Stone residuated lattices that we are referring to:

THEOREM 2. *Let m be an infinite cardinal. Then the following are equivalent:*

- (I) *for each subset X of A with $|X| \leq m$, there exists an element $e \in B(A)$ such that $X^\top = \langle e \rangle$;*
- (II) *A is a co-Stone residuated lattice and $B(A)$ is an m -complete Boolean algebra;*
- (III) *$A_{\top\top} = \{a^{\top\top} | a \in A\}$ is an m -complete Boolean sublattice of the lattice of filters of A ;*
- (IV) *for all $a, b \in A$, $(a \vee b)^\top = a^\top \vee b^\top$ and, for each subset X of A with $|X| \leq m$, there exists an element $x \in A$ such that $X^{\top\top} = x^\top$;*
- (V) *for each subset X of A with $|X| \leq m$, $X^\top \vee X^{\top\top} = A$.*

We bring an argument for our choice of the definition of the co-Stone structures over another definition for them that can be found in mathematical literature, for instance in [5]: the fact that the notion with our definition is transferrable through the reticulation (while the alternate one is not and does not coincide with ours).

We then define the strongly co-Stone hull of a residuated lattice, in accordance with its definition for MV-algebras from [7] and for bounded distributive lattices from [6], show that it is preserved by the reticulation functor and exemplify its calculation for a finite residuated lattice. For proving that the reticulation functor for residuated lattices preserves the strongly co-Stone hull we are using the fact that it preserves inductive limits, that we proved in [13].

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Several approaches to conditional probability

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It is highly desirable to generalize the notion of conditional probability to events which are vague (fuzzy) or not simultaneously observable (quantum). The respective probability models have been introduced and studied for several decades. However, models of *conditional* probability cause problems. In quantum systems, one measurement disables the test of some events which were originally testable. In fuzzy systems, the property

$$P(A|A) = 1$$

is usually violated if A is not sharp (Boolean) [3, 4]. One possible way is to introduce the conditional probability as a system of probabilities satisfying some modification of the Rényi conditions [1, 6]. However, this does not allow to explain a given probability as a conditional one.

We summarize the recent approaches [2, 5], their advantages and drawbacks.

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Non-orthomodular lattice effect algebras possessing state

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Common generalizations of MV-algebras [2] and orthomodular lattices are lattice effect algebras [4]. An *effect algebra* $(E; \oplus, 0, 1)$ is a set E with two special elements $0, 1$ and a partial binary operation \oplus which is commutative and associative at which these equalities hold if one of their sides exists. Moreover, to every element $a \in E$ there exists a unique element $a' \in E$ with $a \oplus a' = 1$ and if $a \oplus 1$ exists then $a = 0$. In every effect algebra we can define a partial order by $a \leq b$ iff there exists $c \in E$ with $a \oplus c = b$ (we set $c = b \ominus a$). If $(E; \leq)$ is a lattice (a complete lattice) then $(E; \oplus, 0, 1)$ is called a *lattice effect algebra* (a *complete lattice effect algebra*).

Effect algebras are a generalization of many structures which arise in quantum physics (see [1]) and in mathematical economics (see [3]). In approach to the mathematical foundations of physics the fundamental notions are states, observables, and symmetries. D.J. Foulis in [5] showed how effect algebras arise in physics and how they can be used to tie together the observables, states, and symmetries employed in the study of physical systems.

In spite of the fact that effect algebras are very natural algebraic structures to be carriers of states and probability measures, in above mentioned non-classical cases of sets of events, there are even finite effect algebras admitting no states, hence no probabilities. The smallest of them has only nine elements (see [8]). Modular complete lattice effect algebras offer some possibility to eliminate such unfavourable situation (see [9]).

We can prove

THEOREM 1. [7] *Let E be an Archimedean atomic lattice effect algebra, $c \in C(E)$, c finite in E , $c \neq 0$, $[0, c]$ a modular lattice. Then there exists an (o) -continuous state ω on E which is subadditive.*

Moreover, for non-orthomodular (i.e., $S(E) \neq E$) lattice effect algebras E we obtain

THEOREM 2. [6] *Let E be a sharply dominating Archimedean atomic lattice effect algebra with $B(E) \neq C(E)$. Then there exists an extremal (o) -continuous state ω on E which is subadditive.*

THEOREM 3. [7] *Let E be a modular Archimedean atomic lattice effect algebra with $S(E) \neq E$. Then there exists an (o) -continuous state ω on E , which is subadditive.*

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Web-geometric approach to models of fuzzy logic

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Web geometry [1], which is usually understood as a branch of differential geometry, offers several concepts and tools which are known to characterize algebraic properties of *loops* in a surprisingly transparent geometric way. These are known as *closure conditions*; in particular, the associativity of loops is characterized by the *Reidemeister closure condition* [3]. In order to apply the web-geometric concepts in the framework of non-standard logics, we have to adapt the theory in a way which allows for dealing with mathematical structures more general than loops. Trying to keep the setup as general as possible, we mimic the standard machinery of web geometry in the case of totally ordered magmas.

Recall that a *totally ordered magma*, or a *togma* for short, is a structure (M, \odot, \leq) where (M, \leq) is a chain and (M, \odot) is a magma [2] with operation isotone with respect to \leq . We introduce the notion *local togma* at the point $(u, v) \in M \times M$ and we explain how its associativity is characterized by an adopted version of the Reidemeister closure condition. As a result, we give a visual characterization of associative togmas with a neutral element which are, actually, tomonoids. Using this tool we are able to discuss certain properties of MTL-chains geometrically.

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