Generic AVL Tree Implementation (GAVL)

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Implementation of AVL tree with some interesting features. All algorithms for insertion, deletion, traversal etc. are implemented without function call recursion or tree node stacks. The only exception is routine to check correctness of tree used for debugging purposes. The implementation has more advantages. It can work with partially unbalanced trees with balance factor (height difference) outside range <-1,1>. The balancing subroutine is common for insertion and deletion. Provided macros enables to build trees with strict item type checking which avoids necessity to use type-casting from (void *).

Table of Contents

1. Introduction **1**

2. Functions Description **2**

Generic AVL tree 2

struct gavl_node 2

struct gavl_root 3

gavl_node2item 3

gavl_node2item_safe 4

gavl_node2key 5

gavl_next_node 6

gavl_prev_node 6

gavl_balance_one 7

gavl_insert_primitive_at 7

gavl_delete_primitive 8

gavl_cut_first_primitive 9

gavl_first_node 10

gavl_last_node 11

gavl_is_empty 11

gavl_search_node 12

gavl_find 13

gavl_find_first 14

gavl_find_after 14

gavl_insert_node_at 15

gavl_insert_node 16

gavl_insert 17

gavl_delete_node 18

gavl_delete 18

gavl_delete_and_next_node 19

gavl_cut_first 20

Custom AVL Tree Instances 21

GAVL_CUST_NODE_INT_IMP 23

3. Examples of GAVL Usage **25**

Generic AVL Tree 25

Customized AVL Tree 27

4. Used Algorithms Background **29**

Tree Balancing Operations 29

Tree Balancing Case 1 29

Tree Balancing Case 2 30

Height of the AVL Tree 32

List of Figures

4-1. Tree balancing case 1 29
4-2. Tree balancing case 2 31

List of Equations

4-1. 29
4-2. 29
4-3. 30
4-4. 30
4-5. 30
4-6. 30
4-7. 31
4-8. 31
4-9. 31
4-10. 31
4-11. 32
4-12. 32
4-13. 32
4-14. 33
4-15. 33
4-16. 33

Chapter 1. Introduction

There are more concepts how to store data items sorted by some key values. The most commonly used ones are:

- sorted conntinuous arrays of items or pointers to items
- binary search trees (splash tree, AVL-tree, RB-tree)
- more childs per node search trees (B-tree)

The binary search trees are considered most efficient for in memory operations when frequent tree updates are present. If update frequency is very low, sorted arrays of pointers could be more efficient. If the node retrieval operations are time consuming the n-nary search trees could be used.

The provided GAVL library tries to offer fast and API friendly set of functions and macros usable for in-memory stored AVL trees of user specified items. The tree node chaining informations must be added to each stored item for holding of the tree node topology. The GAVL library can use user specified field of items or small malloced structure for holding of node chaining information.

The tree search, insert and delete functions can work with and enforce tree with unique items or can be used in mode when more items with same search key value are allowed. The balancing and other functions are written such way, that they tolerate even degraded tree which does not fulfill AVL tree definition. The operations do not result in further degradation or program exceptions. Even future tree operations tends to enhance balancing if the height differences (balance factors) of degraded tree has been correctly updated.

Chapter 2. Functions Description

Generic AVL tree

struct gavl_node

Name

`struct gavl_node` — Structure Representing Node of Generic AVL Tree

Synopsis

```
struct gavl_node {
  struct gavl_node * left;
  struct gavl_node * right;
  struct gavl_node * parent;
  int hdiff;
};
```

Members

left

pointer to left child or NULL

right

pointer to right child or NULL

parent

pointer to parent node, NULL for root

hdiff

difference of height between left and right child

Description

This structure represents one node in the tree and links *`left`* and *`right`* to nodes with lower and higher value of order criterion. Each tree is built from one type of items defined by user. User can decide to include node structure inside item representation or GAVL can malloc node structures for each inserted item. The GAVL allocates memory space with capacity

sizeof(gavl_node_t)+sizeof(void*) in the second case. The item pointer is stored following node structure (void**)(node+1);

struct gavl_root

Name

`struct gavl_root` — Structure Representing Root of Generic AVL Tree

Synopsis

```
struct gavl_root {
  gavl_node_t * root_node;
  int node_offs;
  int key_offs;
  gavl_cmp_fnc_t * cmp_fnc;
};
```

Members

root_node

pointer to root node of GAVL tree

node_offs

offset between start of user defined item representation and included GAVL node structure. If negative value is stored there, user item does not contain node structure and GAVL manages standalone ones with item pointers.

key_offs

offset to compared (ordered) fields in the item representation

cmp_fnc

function defining order of items by comparing fields at offset ***key_offs***.

gavl_node2item

Name

gavl_node2item — Conversion from GAVL Tree Node to User Defined Item

Synopsis

```
void * gavl_node2item (const gavl_root_t * root, const gavl_node_t * node);
```

Arguments

root

GAVL tree root

node

node belonging to *root* GAVL tree

Return Value

pointer to item corresponding to node

gavl_node2item_safe

Name

gavl_node2item_safe — Conversion from GAVL Tree Node to User Defined Item

Synopsis

```
void * gavl_node2item_safe (const gavl_root_t * root, const gavl_node_t * node);
```

Arguments

root

GAVL tree root

node

node belonging to *root* GAVL tree

Return Value

pointer to item corresponding to node

gavl_node2key

Name

`gavl_node2key` — Conversion from GAVL Tree Node to Ordering Key

Synopsis

```
void * gavl_node2key (const gavl_root_t * root, const gavl_node_t * node);
```

Arguments

root

GAVL tree root

node

node belonging to *root* GAVL tree

Return Value

pointer to key corresponding to node

gavl_next_node

Name

`gavl_next_node` — Returns Next Node of GAVL Tree

Synopsis

```
gavl_node_t * gavl_next_node (const gavl_node_t * node);
```

Arguments

node

node for which accessor is looked for

Return Value

pointer to next node of tree according to ordering

gavl_prev_node

Name

`gavl_prev_node` — Returns Previous Node of GAVL Tree

Synopsis

```
gavl_node_t * gavl_prev_node (const gavl_node_t * node);
```

Arguments

node

node for which predecessor is looked for

Return Value

pointer to previous node of tree according to ordering

gavl_balance_one

Name

`gavl_balance_one` — Balance One Node to Enhance Balance Factor

Synopsis

```
int gavl_balance_one (gavl_node_t ** subtree);
```

Arguments

subtree

pointer to pointer to node for which balance is enhanced

Return Value

returns nonzero value if height of subtree is lowered by one

gavl_insert_primitive_at

Name

gavl_insert_primitive_at — Low Lewel Routine to Insert Node into Tree

Synopsis

```
int gavl_insert_primitive_at (gavl_node_t ** root_nodep, gavl_node_t * node, gavl_node_t
* where, int leftright);
```

Arguments

root_nodep

pointer to pointer to GAVL tree root node

node

pointer to inserted node

where

pointer to found parent node

leftright

left (>=1) or right (<=0) branch

Description

This function can be used for implementing AVL trees with custom root definition. The value of the selected ***left*** or ***right*** pointer of provided ***node*** has to be NULL before insert operation, i.e. node has to be end node in the selected direction.

Return Value

positive value informs about success

gavl_delete_primitive

Name

gavl_delete_primitive — Low Lewel Deletes/Unlinks Node from GAVL Tree

Synopsis

```
int gavl_delete_primitive (gavl_node_t ** root_nodep , gavl_node_t * node );
```

Arguments

root_nodep

pointer to pointer to GAVL tree root node

node

pointer to deleted node

Return Value

positive value informs about success.

gavl_cut_first_primitive

Name

gavl_cut_first_primitive — Low Lewel Routine to Cut First Node from Tree

Synopsis

```
gavl_node_t * gavl_cut_first_primitive (gavl_node_t ** root_nodep );
```

Arguments

root_nodep

pointer to pointer to GAVL tree root node

Description

This enables fast delete of the first node without tree balancing. The resulting tree is degraded but height differences are kept consistent. Use of this function can result in height of tree maximally one greater the tree managed by optimal AVL functions.

Return Value

returns the first node or NULL if the tree is empty

gavl_first_node

Name

`gavl_first_node` — Returns First Node of GAVL Tree

Synopsis

```
gavl_node_t * gavl_first_node (const gavl_root_t * root);
```

Arguments

root

GAVL tree root

Return Value

pointer to the first node of tree according to ordering

gavl_last_node

Name

`gavl_last_node` — Returns Last Node of GAVL Tree

Synopsis

```
gavl_node_t * gavl_last_node (const gavl_root_t * root);
```

Arguments

root

GAVL tree root

Return Value

pointer to the last node of tree according to ordering

gavl_is_empty

Name

`gavl_is_empty` — Check for Empty GAVL Tree

Synopsis

```
int gavl_is_empty (const gavl_root_t * root);
```

Arguments

root

GAVL tree root

Return Value

returns non-zero value if there is no node in the tree

gavl_search_node

Name

`gavl_search_node` — Search for Node or Place for Node by Key

Synopsis

```
int gavl_search_node (const gavl_root_t * root, const void * key, int mode, gavl_node_t
** nodep);
```

Arguments

root

GAVL tree root

key

key value searched for

mode

mode of the search operation

nodep

pointer to place for storing of pointer to found node or pointer to node which should be parent of inserted node

Description

Core search routine for GAVL trees searches in tree starting at *root* for node of item with value of item field at offset *key_off* equal to provided **key* value. Values are compared by function pointed by **cmp_fnc* field in the tree *root*. Integer *mode* modifies search algorithm: GAVL_FANY .. finds node of any item with field value **key*, GAVL_FFIRST .. finds node of first item with **key*, GAVL_FAFTER .. node points after last item with **key* value, reworded - index points at first item with higher value of field or after last item

Return Value

Return of nonzero value indicates match found. If the *mode* is ored with GAVL_FCMP, result of last compare is returned.

gavl_find

Name

gavl_find — Find Item for Provided Key

Synopsis

```
void * gavl_find (const gavl_root_t * root, const void * key);
```

Arguments

root

GAVL tree root

key

key value searched for

Return Value

pointer to item associated to key value.

gavl_find_first

Name

`gavl_find_first` — Find the First Item with Provided Key Value

Synopsis

```
void * gavl_find_first (const gavl_root_t * root, const void * key);
```

Arguments

root

GAVL tree root

key

key value searched for

Description

same as above, but first matching item is found.

Return Value

pointer to the first item associated to key value.

gavl_find_after

Name

`gavl_find_after` — Find the First Item with Higher Key Value

Synopsis

```
void * gavl_find_after (const gavl_root_t * root, const void * key);
```

Arguments

root

GAVL tree root

key

key value searched for

Description

same as above, but points to item with first key value above searched ***key***.

Return Value

pointer to the first item associated to key value.

gavl_insert_node_at

Name

gavl_insert_node_at — Insert Existing Node to Already Computed Place into GAVL Tree

Synopsis

```
int gavl_insert_node_at (gavl_root_t * root, gavl_node_t * node, gavl_node_t * where,
int leftright);
```

Arguments

root

GAVL tree root

node

pointer to inserted node

where

pointer to found parent node

leftright

left (1) or right (0) branch

Return Value

positive value informs about success

gavl_insert_node

Name

`gavl_insert_node` — Insert Existing Node into GAVL Tree

Synopsis

```
int gavl_insert_node (gavl_root_t * root, gavl_node_t * node, int mode);
```

Arguments

root

GAVL tree root

node

pointer to inserted node

mode

if mode is `GAVL_FAFTER`, multiple items with same key can be used, else strict ordering is required

Return Value

positive value informs about success

gavl_insert

Name

`gavl_insert` — Insert New Item into GAVL Tree

Synopsis

```
int gavl_insert (gavl_root_t * root, void * item, int mode);
```

Arguments

root

GAVL tree root

item

pointer to inserted item

mode

if mode is GAVL_FAFTER, multiple items with same key can be used, else strict ordering is required

Return Value

positive value informs about success, negative value indicates malloc fail or attempt to insert item with already defined key.

gavl_delete_node

Name

`gavl_delete_node` — Deletes/Unlinks Node from GAVL Tree

Synopsis

```
int gavl_delete_node (gavl_root_t * root, gavl_node_t * node);
```

Arguments

root

GAVL tree root

node

pointer to deleted node

Return Value

positive value informs about success.

gavl_delete

Name

`gavl_delete` — Delete/Unlink Item from GAVL Tree

Synopsis

```
int gavl_delete (gavl_root_t * root, void * item);
```

Arguments

root

GAVL tree root

item

pointer to deleted item

Return Value

positive value informs about success, negative value indicates that item is not found in tree defined by root

gavl_delete_and_next_node

Name

`gavl_delete_and_next_node` — Delete/Unlink Item from GAVL Tree

Synopsis

```
gavl_node_t * gavl_delete_and_next_node (gavl_root_t * root, gavl_node_t * node);
```

Arguments

root

GAVL tree root

node

pointer to actual node which is unlinked from tree after function call, it can be unalocated or reused by application code after this call.

Description

This function can be used after call `gavl_first_node` for destructive traversal through the tree, it cannot be combined with `gavl_next_node` or `gavl_prev_node` and root is emptied after the end of traversal. If the tree is used after unsuccessful/unfinished traversal, it must be balanced again. The height differences are inconsistent in other case. If traversal could be interrupted, the function `gavl_cut_first` could be better choice.

Return Value

pointer to next node or NULL, when all nodes are deleted

gavl_cut_first

Name

`gavl_cut_first` — Cut First Item from Tree

Synopsis

```
void * gavl_cut_first (gavl_root_t * root);
```

Arguments

root

GAVL tree root

Description

This enables fast delete of the first item without tree balancing. The resulting tree is degraded but height differences are kept consistent. Use of this function can result in height of tree maximally one greater the tree managed by optimal AVL functions.

Return Value

returns the first item or NULL if the tree is empty

Custom AVL Tree Instances

The provided macros allows to define all AVL tree functions for tree with user defined root and item types. The next defined function names are prefixed by custom selected *cust_prefix*:

cust_item_t * cust_prefix_node2item(const *cust_root_t* * *root*, const gavl_node_t * *node*);

Conversion from AVL tree node to user custom defined item

cust_key_t * cust_prefix_node2key (const *cust_root_t* * *root*, const gavl_node_t * *node*);

Conversion from AVL tree node to pointer to defined ordering key

int cust_prefix_search_node (const *cust_root_t* * *root*, const *cust_item_t* * *key*, gavl_node_t ** *nodep*);

Search for node or place for node by key

cust_item_t * cust_prefix_find (const *cust_root_t* * *root*, const *cust_key_t* * *key*);

Pointer to item with key value or NULL

```
cust_item_t * cust_prefix_find_first (const cust_root_t * root, const cust_key_t *
key);
```

Same as above, but first matching item is found.

```
cust_item_t * cust_prefix_find_after (const cust_root_t * root, const cust_key_t *
key);
```

Finds the first item with key value higher than value pointed by ***key*** or returns `NULL`.

```
int cust_prefix_insert (cust_root_t * root, cust_item_t * item);
```

Insert new item into AVL tree. If the operation is successful positive value is returned. If key with same key value already exists negative value is returned

```
int cust_prefix_delete_node (cust_root_t * root, gavl_node_t * node);
```

Deletes/unlinks node from custom AVL tree

```
int cust_prefix_delete (cust_root_t * root, cust_item_t * item);
```

Deletes/unlinks item from custom AVL tree

```
gavl_node_t * cust_prefix_first_node (const cust_root_t * root);
```

Returns the first node or `NULL` if the tree is empty

```
gavl_node_t * cust_prefix_last_node (const cust_root_t * root);
```

Returns last node or `NULL` if the tree is empty

```
cust_item_t * cust_prefix_first(const cust_root_t * root);
```

Returns pointer to the first item of the tree or `NULL`

```
cust_item_t * cust_prefix_last(const cust_root_t * root);
```

Returns pointer to last item of the tree or NULL

```
cust_item_t * cust_prefix_next(const cust_root_t * root , const cust_item_t * item);
```

Returns pointer to next item of the tree or NULL if there is no more items

```
cust_item_t * cust_prefix_prev(const cust_root_t * root , const cust_item_t * item);
```

Returns pointer to previous item of the tree or NULL if there is no more items

```
int cust_prefix_is_empty (const cust_root_t * root);
```

Returns non-zero value if there is no node in the tree

```
cust_item_t * cust_prefix_cut_first (cust_root_t * root);
```

Returns the first item or NULL if the tree is empty. The returned item is unlinked from the tree without tree rebalance

GAVL_CUST_NODE_INT_IMP

Name

GAVL_CUST_NODE_INT_IMP — Implementation of new custom tree with internal node

Synopsis

```
GAVL_CUST_NODE_INT_IMP ( cust_prefix , cust_root_t , cust_item_t , cust_key_t ,
 cust_root_node , cust_item_node , cust_item_key , cust_cmp_fnc);
```

Arguments

cust_prefix

defines prefix for builded function names

cust_root_t

user defined structure type of root of the tree

cust_item_t

user defined structure type of items stored in the tree

cust_key_t

type of the key used for sorting of the items

cust_root_node

the field of the root structure pointing to the tree root node

cust_item_node

the field of item structure used for chaining of items

cust_item_key

the key field of item structure defining order of items

cust_cmp_fnc

the keys compare function

Description

There are two macros designed for building custom AVL trees. The macro `GAVL_CUST_NODE_INT_DEC` declares functions for custom tree manipulations and is intended for use in header files. The macro `GAVL_CUST_NODE_INT_IMP` builds implementations for non-inlined functions declared by `GAVL_CUST_NODE_INT_DEC`. The *cust_cmp_fnc* is used for comparison of item keys in the search and insert functions. The types of two input arguments of *cust_cmp_fnc* functions must correspond with *cust_key_t* type. Return value should be positive for case when the first pointed key value is greater then second, negative for reverse case and zero for equal pointed values.

Chapter 3. Examples of GAVL Usage

Generic AVL Tree

This chapter describes use of generic AVL tree. This tree has advantage, that same functions could operate with any user provided items, but operations return type is void* and needs type-casting to user item type.

```
#include <malloc.h>
#include "ul_gavl.h"
```

The above code fragment includes basic GAVL declarations.

```
typedef struct test1_item {
  int my_val;
  /* more user data ... */
} test1_item_t;
```

New item type is declared. Type have to contain or be connected to some key value. The integer number *my_val* will be used as key value in this example. The tree root instance definition follows.

```
gavl_root_t test1_tree={
  .root_node = NULL,
  .node_offs = -1,
  .key_offs = UL_OFFSETOF(test1_item_t, my_val),
  .cmp_fnc = gavl_cmp_int
};
```

The field *root_node* have to be initialized to `NULL`. The value -1 for the *node_offs* field directs GAVL functions to allocate external node structure for each inserted item. The macro `UL_OFFSETOF` computes offset of *my_val* field in test1_item_t item. The last assignment select one of predefined functions for key values comparison.

Almost same declarations and definitions could be used if the node chaining structure is contained inside item structure. That is shown in the next code fragments.

```
typedef struct test2_item {
  gavl_node_t my_node;
  int my_val;
  /* more user data ... */
} test2_item_t;
```

The item declaration contains field with gavl_node_t type in this case. This field is used for storage of tree topology information.

```
gavl_root_t test2_tree={
  .root_node = NULL,
  .node_offs = UL_OFFSETOF(test2_item_t, my_node),
  .key_offs = UL_OFFSETOF(test2_item_t, my_val),
  .cmp_fnc = gavl_cmp_int
```

```
};
```

The field ***node_offs*** contains offset of the node structure inside item now. Next fragments are part of the function working with defined tree. There would be no difference in the functions calls for items with external nodes.

```
cust2_item_t *item;
int i;
gavl_node_t *node;
```

Declare some local variables first. Insert 100 items into tree then.

```
for(i=0;i<items_cnt;i++){
  item=malloc(sizeof(test2_item_t));
  item->my_val=i;
  if(gavl_insert(&test2_tree,item,0)<0)
    printf("gavl_insert error\n");
}
```

The tree is expected to store items with unique key values. If more items with same key values should be allowed, then the value of ***mode*** parameter in `gavl_insert` function call have to be changed from 0 to `GAVL_FAFTER`.

```
for(i=0;i<102;i++){
  if(!(item=(cust2_item_t *)gavl_find(&test2_tree,&i)))
    printf("no item %d\n",i);
  else
    if(item->my_val!=i)
      printf("gavl_find mismatch\n");
}
```

Some test of retrieving item corresponding to specified key. The tree in order traversal is in the next code sample.

```
node=gavl_first_node(&test2_tree)
while(node){
  item=(cust2_item_t *)gavl_node2item(&test2_tree,node);
  /* do something with item
   * insert and delete allowed there except delete of the actual item
   */
  node=gavl_next_node(node);
  /* do something with item
   * insert and delete allowed there except delete of next item
   */
}
```

Example of the item deletion is in the next fragment. The function `gavl_delete` guarantees that trial of deleting item, which is not part of the tree, is detected and negative value is returned. There is one prerequisite for this behavior for items with internal nodes that have never been part of any tree. The field ***parent*** of node structure stored in the item have to be initialized to `NULL` value.

```
for(i=0;i<80;i++){
```

```
    if(!(item=(cust2_item_t *)gavl_find(&test2_tree,&i)))
      continue;
    if(gavl_delete(&test2_tree,item)<0)
      printf("gavl_delete error\n");
    free(item);
  }
```

The function `gavl_cut_first` can significantly simplify deletion of all nodes of the tree without need of traversal over the tree. This function does not rebalance the tree, but keeps tree consistent.

```
  while((item=(cust2_item_t *)gavl_cut_first(&test2_tree))!=NULL)
    free(item);
```

Customized AVL Tree

The function templates are provided for faster and type-safe custom tree definition and manipulation. These templates expect user defined items with internal node structures and key comparison function with key pointer input parameters. The type of these parameters is required to be pointer to the key type.

Next code fragment is intended to be part of the declarations of user defined custom tree and items for some data storage purpose. The header with these definitions should be included from all modules directly working with specified tree.

```
#include "ul_gavl.h"

typedef int cust_key_t;

typedef struct cust2_item {
  cust2_key_t my_val;
  gavl_node_t my_node;
  /* more user item data ... */
} cust2_item_t;

typedef struct cust2_root {
  gavl_node_t *my_root;
  /* more user root/tree data ... */
} cust2_root_t;

int cust_cmp_fnc(cust_key_t *a, cust_key_t *b);
```

As can be seen from above code fragment, the key field with user declared type (cust_key_t) and internal node structure are required for items. The only one field with type gavl_node_t* is required for the custom tree root structure.

The use of the macro `GAVL_CUST_NODE_INT_DEC` declares all tree manipulation functions for the custom tree. The function names are prefixed by prefix specified as the first parameter of macro invocation. This declaration can be included in user header files as well.

```
GAVL_CUST_NODE_INT_DEC(cust2, cust2_root_t, cust2_item_t, cust2_key_t,\
  my_root, my_node, my_val, cust_cmp_fnc)
```

The implementation of the custom tree functions is realized by use of macro `GAVL_CUST_NODE_INT_IMP` with same parameters as for `GAVL_CUST_NODE_INT_DEC` macro.

```
#include "ul_gavlcust.h"

GAVL_CUST_NODE_INT_IMP(cust2, cust2_root_t, cust2_item_t, cust2_key_t,\
  my_root, my_node, my_val, cust_cmp_fnc)

cust2_root_t cust2_tree;
```

Static tree root instance is included at last line of above code fragment as well.

The next sample function for the custom tree shows same operation as have been described for generic tree.

```
void test_cust_tree(void)
{
  int i;
  cust2_item_t *item;

  /* build tree */
  for(i=1;i<=100;i++){
    item=malloc(sizeof(cust2_item_t));
    item->my_val=i;
    if(cust2_insert(&cust2_tree,item)<0)
      printf("cust2_insert error\n");
  }

  /* traverse tree */
  printf("Custom tree cust2:\n");
  for(item=cust2_first(&cust2_tree);item;item=cust2_next(&cust2_tree,item))
    printf("%d ",item->my_val);
  printf("\n");

  /* delete part of the tree */
  for(i=1;i<=80;i++){
    item=cust2_find(&cust2_tree,&i);
    if(cust2_delete(&cust2_tree,item)<0)
      printf("cust2_delete error\n");
    free(item);
  }

  /* traverse in reverse order */
  printf("Custom tree cust2:\n");
  for(item=cust2_last(&cust2_tree);item;item=cust2_prev(&cust2_tree,item))
    printf("%d ",item->my_val);
  printf("\n");

  /* delete all remaining items */
  while((item=cust2_cut_first(&cust2_tree))!=NULL)
    free(item);
}
```

Chapter 4. Used Algorithms Background

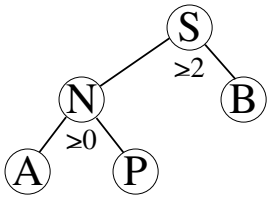
Tree Balancing Operations

This section describes used algorithms for keeping AVL tree balanced.

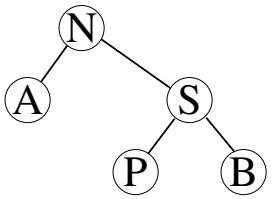
Next convention is used for height and balance computations for node X . The height of subtree starting at X is labeled as $x = height(\mathrm{X})$. Height difference (balance factor) for node X is labeled as $\Delta x = height(left(\mathrm{X})) - height(right(\mathrm{X}))$.

Tree Balancing Case 1

Figure 4-1. Tree balancing case 1



Before operation



After operation

$$\begin{aligned} \Delta s{=}n - b &\quad \geq 2 \\ \Delta n{=}a - p &\quad \geq 0 \end{aligned}$$

The height of subtree S is defined by highest branch, which is branch grovinggrowing from node A . This leads to next equations.

$$\begin{aligned} n{=}a + 1 \\ s{=}a + 2 \end{aligned}$$

$$p{=}a - \Delta n$$
$$b{=}a + 1 - \Delta s$$

The height of branches N and S is marked as $n1$ and $s1$ after balancing.

$$\Delta s1{=}p - b$$
$$\Delta n1{=}a - s1$$

$$s1{=}\max(p, b) + 1$$
$$s1{=}\max(a - \Delta n, a + 1 - \Delta s) + 1$$
$$s1{=}a + 1 + \max(-\Delta n, 1 - \Delta s)$$
$$s1{=}a + 1 - \min(\Delta n, \Delta s - 1)$$

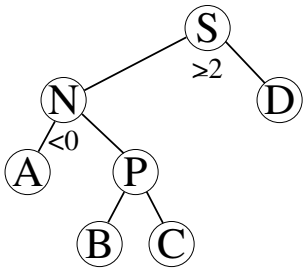
$$\Delta s1{=}a - \Delta n - a - 1 + \Delta s$$
$$\Delta s1{=}\Delta s - \Delta n - 1$$
$$\Delta n1{=}a - a - 1 + \min(\Delta n, \Delta s - 1)$$
$$\Delta n1{=}\min(\Delta n - 1, \Delta s - 2)$$
$$\Delta n1{=}\begin{cases} \Delta n - 1 & \Delta s \geq \Delta n + 1 \\ \Delta s - 2 & \Delta s \leq \Delta n + 1 \end{cases}$$

Because balancing in case 1 does not guarantee that new tree has lower height than original tree, it is necessary to compute tree height change (tree height lowering).

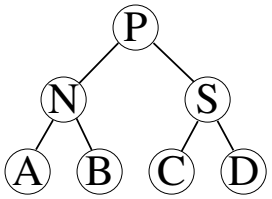
$$\begin{aligned} s - n1 &{=} s - (\max(a, s1) + 1) \\ &{=} a + 2 - (\max(a, a + 1 + \max(-\Delta n, 1 - \Delta s)) + 1) \\ &{=} 1 - \max(0, \max(1 - \Delta n, 2 - \Delta s)) \\ &{=} \min(1, \Delta n, \Delta s - 1) \qquad \Delta n \geq 0, \Delta s \geq 2 \\ &{=} \min(1, \Delta n) \end{aligned}$$
$$s - n1{=}\begin{cases} 1 & \Delta n \neq 0 \\ 0 & \Delta n = 1 \end{cases}$$

Tree Balancing Case 2

Figure 4-2. Tree balancing case 2



Before operation



After operation

This case has advantage, that it reduces tree height for all node height combinations. Tree height is given by branch P B or P C .

$$\begin{aligned} \Delta s{=}n - d \quad & \geq 2 \\ \Delta n{=}a - p \quad & \leq -1 \end{aligned}$$

$$\begin{aligned} n{=}p + 1 \\ s{=}p + 2 \\ d{=}p + 1 - \Delta s \\ a{=}p + \Delta n \end{aligned}$$

$$\begin{aligned} \text{for } b \geq c \quad & c{=}b - \Delta p \quad & p = b + 1 \\ \text{for } c \geq b \quad & b{=}c + \Delta p \quad & p = c + 1 \end{aligned}$$

When $b \geq c$ then the height differences $\Delta s1$, $\Delta n1$ and $\Delta p1$ for nodes S , N and P of the balanced tree can be computed from next equations.

$$
\begin{aligned}
&b \geq c\ ,\ \Delta p \geq 0 \\
&\quad n1{=}b+1 \\
&\quad p1{=}b+2 \\
&\quad s1{=}\max(c,d)+1 = \max(b-\Delta p, b+2-\Delta s)+1 \\
&\Delta s1{=}c-d = b-\Delta p-b-2+\Delta s \\
&\Delta s1{=}\Delta s-2-\Delta p \\
&\Delta n1{=}a-b = b+1+\Delta n-b \\
&\Delta n1{=}\Delta n+1 \\
&\Delta p1{=}n1-s1 = b+1-\max(b-\Delta p, b+2-\Delta s)-1 \\
&\Delta p1{=}\min(\Delta p, \Delta s-2) \qquad \Delta p \geq 0, \Delta s \geq 2
\end{aligned}
$$

When $c \geq b$ then the height differences $\Delta s1$, $\Delta n1$ and $\Delta p1$ for nodes S , N and P of the balanced tree can be computed from next equations.

$$
\begin{aligned}
&c \geq b\ ,\ \Delta p \leq 0 \\
&\quad s1{=}c+1 \\
&\quad p1{=}c+2 \\
&\quad n1{=}\max(a,b)+1 = \max(c+1+\Delta n, c+\Delta p)+1 \\
&\Delta s1{=}c-d = c-c-2+\Delta s \\
&\Delta s1{=}\Delta s-2 \\
&\Delta n1{=}a-b = c+1+\Delta n-c-\Delta p \\
&\Delta n1{=}\Delta n+1-\Delta p \\
&\Delta p1{=}n1-s1 = \max(c+1+\Delta n, c+\Delta p)+1-c-1 \\
&\Delta p1{=}\max(\Delta n, \Delta s-1) \qquad \Delta n \leq -1, \Delta p \leq 0
\end{aligned}
$$

Height of the AVL Tree

Minimal number of nodes T can be computed recursively.

$$T(h) = T(h-1) + T(h-2) + 1,\ T(0) = 0,\ T(1) = 1$$

1, 2, 4, 7, 12, 20, 33, 54, 88, 143, 232, 376, 609, 986

$$T(h) = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^{(h+2)} - 1$$

$$T(h) = \frac{1}{\sqrt{5}} 2^{(h+2)\, log_2\left(\frac{1+\sqrt{5}}{2}\right)} - 1$$

$$h = \left(log_2(T+1) - log_2\left(\frac{1}{\sqrt{5}}\right) \right) log_2^{-1}\left(\frac{1+\sqrt{5}}{2}\right) - 2$$

$$log_2^{-1}\left(\frac{1+\sqrt{5}}{2}\right) = 1.44042$$

An AVL tree with n nodes has height between log2 (n + 1) and 1.44 * log2 (n + 2) - 0.328. An AVL tree with height h has between pow (2, (h + .328) / 1.44) - 1 and pow (2, h) - 1 nodes.

A red-black tree with n nodes has height at least log2 (n + 1) but no more than 2 * log2 (n + 1). A red-black tree with height h has at least pow (2, h / 2) - 1 nodes but no more than pow (2, h) - 1.