Maximum Persistency via Iterative Relaxed Inference with Graphical Models

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Outline

Introduction

Persistency

ILP	LP
$\min c^\mathsf{T} x$	$\min c^\mathsf{T} x$
$Ax \leq b$	$Ax \leq b$
$x \in \{0, 1\}^n$	$x \in [0, 1]^n$

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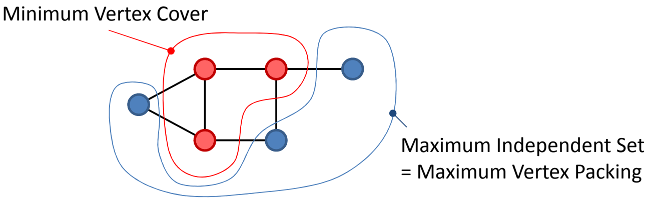
4. Sufficient conditions for a part of an optimal solution to LP to be optimal for ILP?
5. Find the largest part of LP solution satisfying 4.

Vertex Packing / Maximum Independent Set



Maximum Weighted Vertex Packing

- $(\mathcal{V}, \mathcal{E})$ – an undirected graph;
- *Vertex Packing* is a subset $P \subset \mathcal{V}$ for which $u, v \in P \Rightarrow (u, v) \notin \mathcal{E}$;
- Weights $c \colon \mathcal{V} \to \mathbb{R}$;
- Problem:

$$\max_x \sum_{v \in \mathcal{V}} c_v x_v \tag{VP}$$

$$(\forall uv \in \mathcal{E})\ x_u + x_v \leq 1,$$

$$(\forall v \in \mathcal{V})\ x_v \in \{0, 1\}.$$

Vertex Packing / Maximum Independent Set

Relaxing the integrality constraints:

$$\max_{\mu} \sum_{v \in \mathcal{V}} c_v \mu_v \qquad \text{(VPL)}$$
$$(\forall uv \in \mathcal{E})\ \mu_u + \mu_v \leq 1,$$
$$(\forall v \in \mathcal{V})\ \mu_v \geq 0.$$

Theorems

- (Balinski, 1965; Lorentzen, 1966): Any basic feasible solution to (VLP) is $\{0, \frac{1}{2}, 1\}$-valued.
- (Edmonds and Pulleyblank) (VLP) reduces to a maxflow problem on a related symmetric bipartite graph;
- (Nemhauser and Trotter, 1975): Variables which assume binary values in an optimum (VLP) solution retain the same values in an optimum (VP) solution.
- (Picard and Queyranne, 1977): There exists a unique maximum set of variables that are integer valued in an optimal solution to (VLP).

QPBO

Quadratic pseudo-Boolean Optimization (QPBO)

- $(\mathcal{V}, \mathcal{E})$ – an undirected graph;
- Weights $a\colon \mathcal{V} \cup \mathcal{E} \to \mathbb{R}$;
- Problem:

$$\min_x \sum_{v \in \mathcal{V}} a_v x_v + \sum_{uv \in \mathcal{E}} a_{uv} x_u x_v$$

$$(\forall v \in \mathcal{V})\ x_v \in \{0, 1\}.$$

- Generalizes Vertex Packing (let $a_{uv} = B$, a big number; $a = -c_v$).

QPBO

Natural linear relaxation: $x_s \to \mu_s \in [0, 1]$, $x_s x_t \to \mu_{st} \in [0, 1]$;

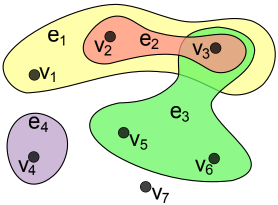
$$\min_{\mu\colon \mathcal{V}\cup\mathcal{E}\to[0,1]} \sum_{v\in\mathcal{V}} a_v \mu_v + \sum_{uv\in\mathcal{E}} a_{uv}\mu_{uv} \quad \text{s.t.} \tag{LP}$$

$$(\forall uv \in \mathcal{E})\ \mu_u + \mu_v - 1 \le \mu_{uv} \le \min(\mu_u, \mu_v).$$

Theorems

- (?): Each extreme point of the feasible set is $\{0, \frac{1}{2}, 1\}$-valued.
- (Hammer et al., 1984; Boros et al., 1991): LP reduces to a maxflow problem;
- **Weak Persistency** (Hammer et al., 1984): Variables μ_v which assume binary values in an optimum (LP) solution retain the same values in an ILP solution.
- **Strong Persistency** (Hammer et al., 1984): Variables μ_v which assume binary values in **all** optimal (LP) solutions retain the same values in **all** optimal ILP solutions.

0-1 Polynomial Programming



A hypergraph (courtesy of wikipedia).

0-1 Polynomial Programming / pseudo-Boolean Optimization

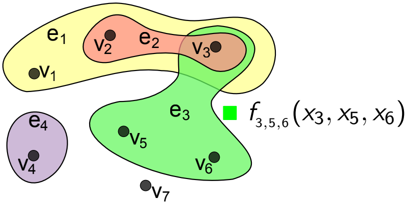
- $(\mathcal{V}, \mathcal{E})$ – a hypergraph, $\mathcal{E} \subset 2^{\mathcal{V}}$;
- Weights $f\colon \mathcal{E} \to \mathbb{R}$;
- Problem: $$\min_{x \in \{0,1\}^{\mathcal{V}}} \sum_{C \in \mathcal{E}} f_C \prod_{v \in C} x_v. \qquad \text{(PP)}$$

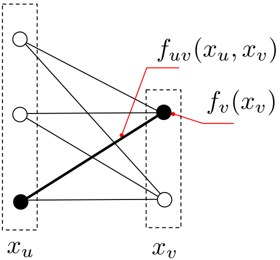
- Any pseudo-Boolean function can be represented as a multilinear polynomial.

0-1 Polynomial Programming

- Relaxation of Sherali and Adams (1990)
  - – optimal solutions are not half-integral in general;
  - – no combinatorial method to solve;
  - – does not possess persistency in general;
  - + Tightest "local" relaxation
- (Bi)submodular relaxations (Kolmogorov, 2012)
  - + extreme feasible solutions are half-integral;
  - + reduces to sum of (bi)submodular functions minimization;
  - + possess persistency;
  - – Weaker than relaxation of Sherali and Adams (1990);

Energy Minimization





Energy Minimization / Weighted Constraint Satisfaction

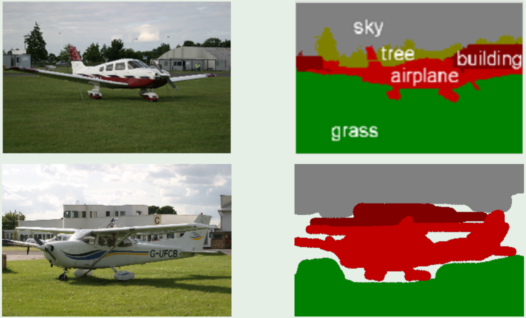
- $(\mathcal{V}, \mathcal{E})$ - a hypergraph;
- $\mathcal{X}_v$ - a finite set of *labels*, $v \in \mathcal{V}$;
- Costs $f_{\mathrm{C}}\colon \prod_{v \in \mathrm{C}} \mathcal{X}_v \to \mathbb{R}$; Cost vector $f \in \mathbb{R}^{\mathcal{I}}$.
- Energy: $E_f(x) = \sum_{\mathrm{C} \in \mathcal{E}} f_{\mathrm{C}}(x_{\mathrm{C}})$;
- Probloem: $\min_{x \in \mathcal{X}} E_f(x)$;

Energy Minimization

Example: Potts Model for Object Class Segmentation

- $\mathcal{V}$ - set of pixels; $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ neighboring pixels;
- $\mathcal{X}_s = \{1, \dots K\}$ – class label;
- $E_f(x) = \sum_{s\in\mathcal{V}} f_s(x_s) + \sum_{st\in\mathcal{E}} \lambda_{st}[\![x_s \neq x_t]\!]$.

Image Ground Truth



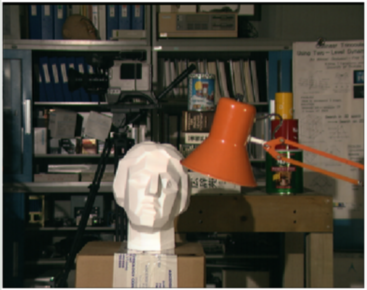
(MSRC object class segmentation)

Persistncy / Partial Optimality

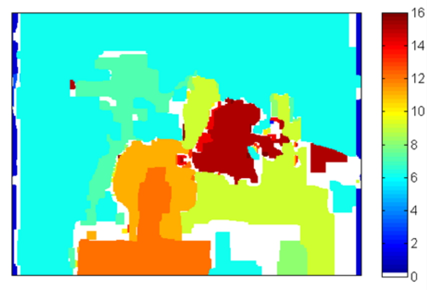
Example: Potts Model for Stereo

- $\mathcal{V}$ - set of pixels; $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ neighboring pixels;
- $\mathcal{X}_s = \{1, \dots K\}$ – disparity value;
- $E_f(x) = \sum_{s\in\mathcal{V}} f_s(x_s) + \sum_{st\in\mathcal{E}} \lambda_{st}[\![x_s \neq x_t]\!]$.

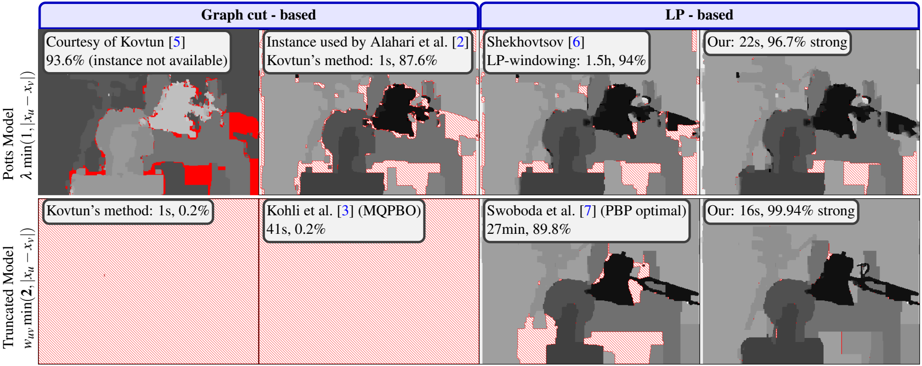
Stereo Reconstruction



Partial Optimality (Method of Kovtun (2003))



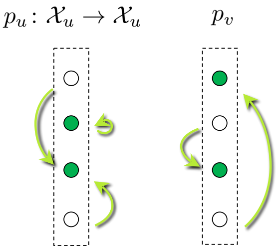
Development of Partial Optimality Methods



- Model 1 (Kovtun'03, Alahari et al.'10): Potts, strong unaries with window aggregation
- Model 2 (Szeliski et al., 2008): Nearly Potts, per-pixel unaries

Maximum Persistency Problem

Improving Mapping (substitution of labels)

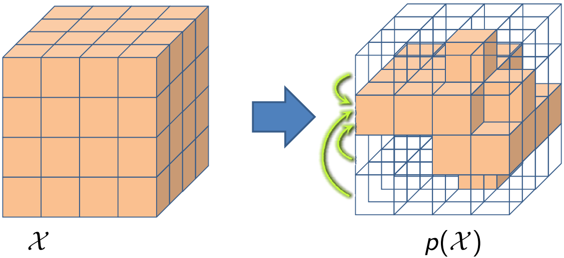


$p\colon \mathcal{X} \to \mathcal{X}$ node-wise

Definition

Mapping p is improving if $\forall x \; E_f(p(x)) \leq E_f(x)$.

- If x is optimal then $p(x)$ is optimal. Search space can be reduced.



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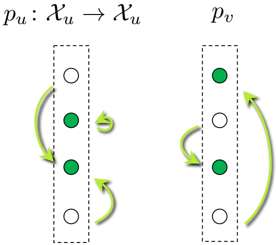

$p \colon \mathcal{X} \to \mathcal{X}$ node-wise

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Mapping p is improving if $\forall x \ E_f(p(x)) \leq E_f(x)$.

Remark: no distinction between strict/non-strict in this talk!

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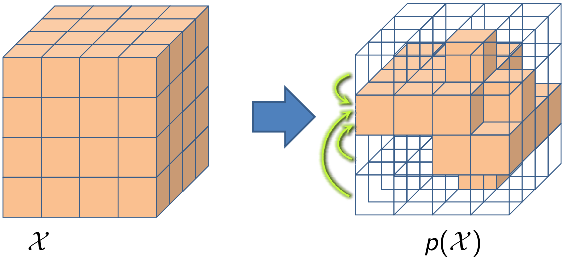
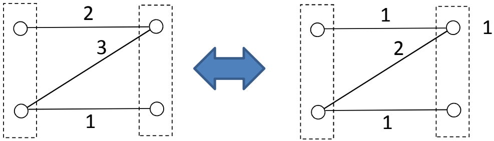


Equivalent Transformations

$E_f(x) = \sum_{\mathrm{C}\in\mathcal{E}} f_{\mathrm{C}}(x_{\mathrm{C}})$. Cost vectors $f, g \in \mathbb{R}^{\mathcal{I}}$ equivalent when $E_f = E_g$.



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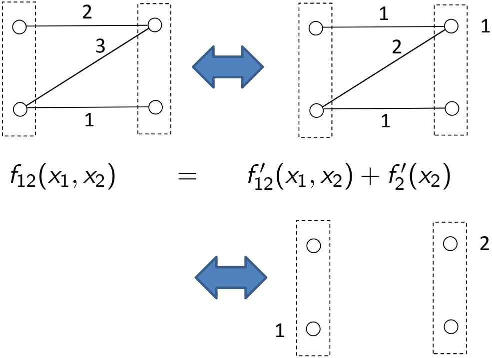

$$f_{12}(x_1, x_2) \quad = \quad f'_{12}(x_1, x_2) + f'_2(x_2)$$

$$= \quad f''_1(x_1) + 0 + f''_2(x_2)$$

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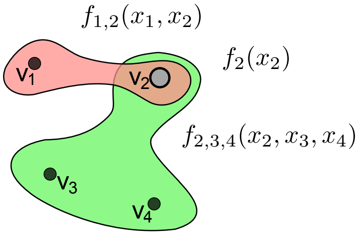
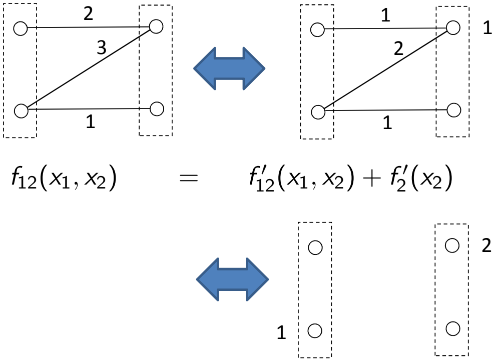
$E_f(x) = \sum_{c \in \mathcal{E}} f_c(x_c)$. Cost vectors $f, g \in \mathbb{R}^{\mathcal{I}}$ equivalent when $E_f = E_g$.





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- Affine space: $g \sim h \sim f$
  $\Rightarrow \quad E_{\alpha g + (1-\alpha)h} = E_f$.
- $f \sim g \quad \Leftrightarrow \quad (\exists \varphi)\ g = f - A^{\mathsf{T}} \varphi$.

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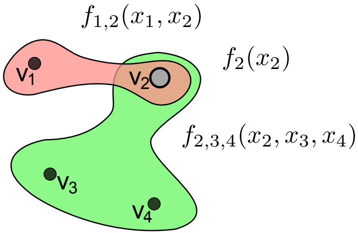

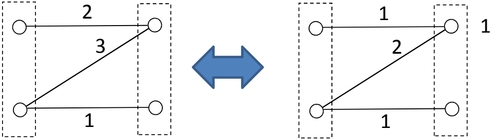

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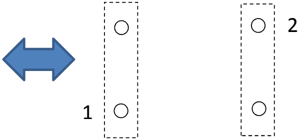

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- Equivalent Transformations, Shlezinger (1976)
- Reparametrization, Wainwright et al. (2003)
- Equivalence Preserving Transformations in WCSP

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A system of linear inequalities in φ.

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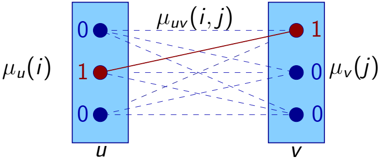
A system of linear inequalities in φ.
In matrix form: $(\exists\varphi)\ P^\mathsf{T} f^\varphi \leq f^\varphi$

$$(\exists\varphi)\ (I - P^\mathsf{T})(f - A^\mathsf{T}\varphi) \geq 0 \text{ – Linear program in } \varphi.$$

P^T: $(P^\mathsf{T} f)_\mathrm{C}(x_\mathrm{C}) = f_\mathrm{C}(p_\mathrm{C}(x_\mathrm{C}))$.

LP Relaxation

- Energy: $E_f(x) = f_\varnothing + \sum_{v\in\mathcal{V}} f_v(x_v) + \dots$
- Linearize: $E_f(x) = \langle f, \mu \rangle$
  $(f_\varnothing \mu_\varnothing + \sum_v \sum_{i\in\mathcal{X}_u} f_v(i)\mu_v(i) + \dots)$
- $\mu = \delta(x) \in \mathbb{R}^{\mathcal{I}}$ - indicator vector of costs selected by x.



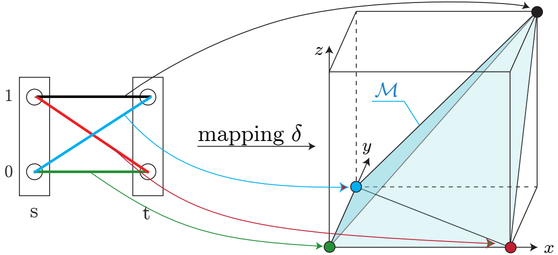
- Constraints:
  $\mu \in \{0,1\}^{\mathcal{I}}$
  $\mu_\varnothing = 1$
  $\sum_i \mu_u(i) = 1$ – normalization
  $\sum_i \mu_{uv}(i,j) = \mu_v(j)$ – marginalization
- LP relaxation: $\min \langle f, \mu \rangle$

$$A\mu = 0$$
$$\mu_\varnothing = 1$$
$$\mu \geq 0$$

LP Relaxation

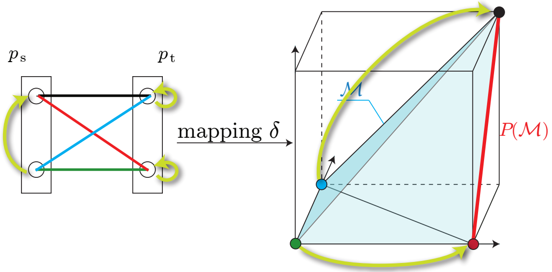
On a higher level:

- Embedding: $\delta\colon \mathcal{X} \to \mathbb{R}^{\mathcal{I}}$;
- Linearization: $E_f(x) = \langle f, \delta(x) \rangle$;
- $\min_x E_f(x) \geq \min_{\mu \in \Lambda} \langle f, \mu \rangle$;
- Polytope: $\Lambda = \{\mu \mid A\mu = 0,\ \mu_\varnothing = 1,\ \mu \geq 0\} \supset \delta(\mathcal{X})$;



Relaxed Improving Mapping

- Mapping $p\colon \mathcal{X} \to \mathcal{X}$ can be represented in the space $\mathbb{R}^{\mathcal{I}}$:



Definition

$P\colon \mathbb{R}^{\mathcal{I}} \to \mathbb{R}^{\mathcal{I}}$ is a *linear extension* of $p\colon \mathcal{X} \to \mathcal{X}$ if
$$(\forall x \in \mathcal{X})\ \delta(p(x)) = P\delta(x)$$

- An oblique projection in $\mathbb{R}^{\mathcal{I}}$.

Sufficient Conditions for Persistency (Primal)

- Recall: $p\colon \mathcal{X} \to \mathcal{X}$ is improving if $(\forall x \in \mathcal{X}) \;\; E_f(p(x)) \leq E(x)$
- In the embedding: $(\forall \mu \in \delta(\mathcal{X})) \;\; \langle f, P\mu \rangle \leq \langle f, \mu \rangle$;

Definition

Mapping $p\colon \mathcal{X} \to \mathcal{X}$ is *relaxed-improving* if

$$(\forall \mu \in \Lambda) \;\; \langle f, P\mu \rangle \leq \langle f, \mu \rangle;$$

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Mapping $p\colon \mathcal{X} \to \mathcal{X}$ is *relaxed-improving* if

$$(\forall \mu \in \Lambda)\ \ \langle f, P\mu \rangle \leq \langle f, \mu \rangle;$$

- A sufficient condition since $\Lambda \supset \delta(\mathcal{X})$

Sufficient Conditions for Persistency (Primal)

- Recall: $p \colon \mathcal{X} \to \mathcal{X}$ is improving if $(\forall x \in \mathcal{X}) \;\; E_f(p(x)) \leq E(x)$
- In the embedding: $(\forall \mu \in \delta(\mathcal{X})) \;\; \langle f, P\mu \rangle \leq \langle f, \mu \rangle$;

Definition

Mapping $p \colon \mathcal{X} \to \mathcal{X}$ is *relaxed-improving* if

$$(\forall \mu \in \Lambda) \;\; \langle f, P\mu \rangle \leq \langle f, \mu \rangle;$$

- A sufficient condition since $\Lambda \supset \delta(\mathcal{X})$
- Linear program in μ: $(\forall \mu \in \Lambda) \; \langle f, (I - P)\mu \rangle \geq 0$;

$$\min_{\mu \in \Lambda} \langle f, (I - P)\mu \rangle \geq 0.$$

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Theorem

Primal and Dual sufficient conditions are equivalent.

(by LP strong duality and substitution)

Generality of Sufficient Conditions

Theorems (Shekhovtsov (2014, 2015))

Relaxed-improving condition with natural (local) relaxations are satisfied for a.o.f.:

pairwise multilabel	Simple DEE (Goldstein, 1994)	✓
	MQPBO (Kohli et al., 2008)	✓
	Kovtun (2003) one-agains-all	✓
	Kovtun (2011) iterative	✓
	Swoboda et al. (2014)*	✓
	Roof dual / QPBO Hammer et al. (1984)	✓
higher order pseudo-Boolean	Reductions: HOCR (Ishikawa, 2011), (Fix et al., 2011)	FLP
	Bisubmodular relaxations (Kolmogorov, 2010)**	BLP
	Generalized Roof Dualilty (Kahl and Strandmark, 2011)	FLP
	Persistency by Adams et al. (1998)	FLP

BLP = Basic LP Relaxation Werner (2007); Thapper and Živný (2013);
FLP = Full Local LP Relaxation, equivalent to Sherali and Adams (1990);

*Swoboda et al. (2014) is higher order but the comparison proof is for pairwise case. **Result holds for sum of bisubmodular functions over the same hypergraph as the BLP relaxation.

Maximum Persistency

- Given that verification problem is polynomially solvable,
- which method is better?

Proposition

Pose "the best partial optimality" as optimization problem

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Maximum Persistency Problem

Find the mapping $p\colon \mathcal{X} \to \mathcal{X}$ that delivers the maximum problem reduction:

$$\min_{p \in \mathcal{P}} \sum_{u \in \mathcal{V}} |p(\mathcal{X}_u)| \quad \text{s.t. } p \text{ is relaxed-improving,}$$

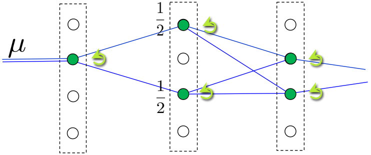
$\mathcal{P}$ - class of mappings.

The Algorithm

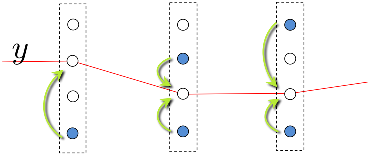
Restricted Class of Mappings

Theorem

Let μ be a solution to LP-relaxation: $\mu \in \mathrm{argmin}_{\mu\in\Lambda}\langle f, \mu\rangle$ and $p\colon \mathcal{X} \to \mathcal{X}$ be (strictly) relaxed-improving. Then $P\mu = \mu$.



- Fix a *test* labeling y from μ and try substitute other labels with it.



Restricted Class of Mappings

Still Covers

pairwise multilabel	Simple DEE (Goldstein, 1994)	
	MQPBO (Kohli et al., 2008)	
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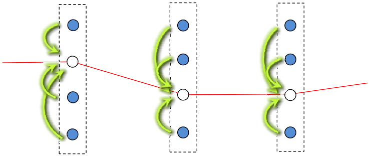
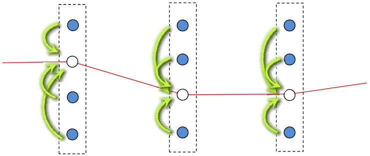
Discrete Cutting Plane



Algorithm

- Start with a mapping p that substitutes *everything* with y

Discrete Cutting Plane



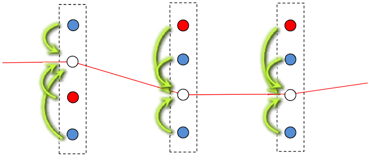
Algorithm

- Start with a mapping p that substitutes *everything* with y
- Auxiliary problem $g = (I - P^{\mathsf{T}})f$
- Check relaxed-improving conditions by solving LP dual:

$$\max_{\varphi} g^{\varphi}_{\varnothing} \quad \overset{?}{\geq} \quad 0$$

$$(\forall \mathrm{C} \in \mathcal{E},\ \forall x_{\mathrm{C}})\ g^{\varphi}_{\mathrm{C}}(x_{\mathrm{C}}) \geq 0$$

Discrete Cutting Plane



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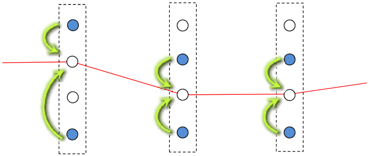

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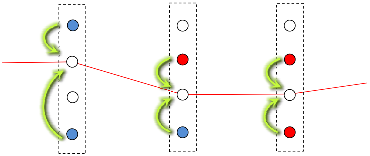
- Release blocking constraints by pruning some maps

Discrete Cutting Plane



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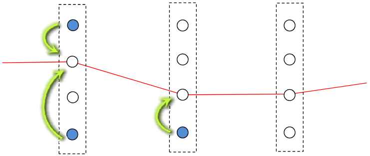

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Correctness and Optimality

Properties:

- Runs in polynomial time;
- Solves the maximum persistency problem exactly when the dual is solved exactly and the solution is a strict relative interior optimal (e.g. interior point method);
- Returns an improving mapping even when the dual is solved sub-optimally – can use fast dual solvers, we used TRW-S by Kolmogorov (2006).
- Generalizes LP approach by Shekhovtsov (2014) and pruning Swoboda et al. (2014) (CVPR'14).

Efficiency

- solving relaxed inference approximately even once is slow
- TRW-S is not finitely converging

How can we iterate such relaxed inference?

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- TRW-S is not finitely converging

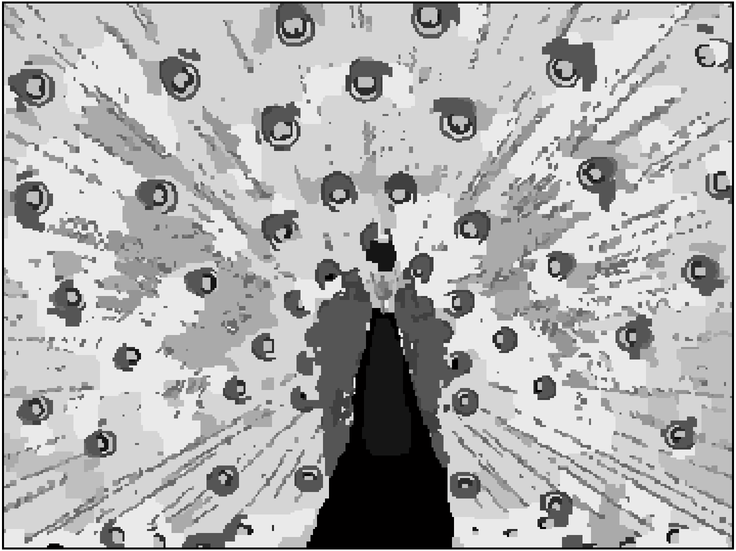
How can we iterate such relaxed inference?

Fast implementation with TRW-S

- Incremental: reuse reparametrizations φ
- Guaranteed to prune something even after 1 iteration of TRW-S (there is a blocking constraint not yet pruned)
- An optimal pruning is often possible before the dual is solved (cuts)
- Problem reductions preserving the sufficient condition
- Fast message passing for $(I - P^{\mathsf{T}})f$ with reductions

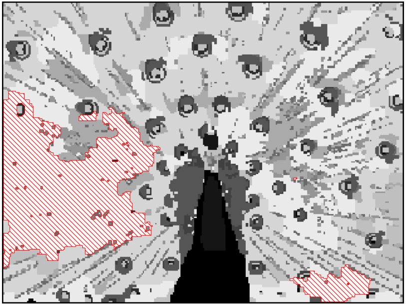
Instance	Initialization (1000 it.)	Extra time for persistency				
		no speedups	+reduction	+node pruning	+labeling pruning	+fast msgs
Protein folding `1CKK`	8.5s	268s (26.53%)	168s (26.53%)	2.0s (26.53%)	2.0s (26.53%)	2.0s (26.53%)
colorseg-n4 `pfau-small`	9.3s	439s (88.59%)	230s (93.41%)	85s (93.41%)	76s (93.41%)	19s (93.41%)

Algorithm Video

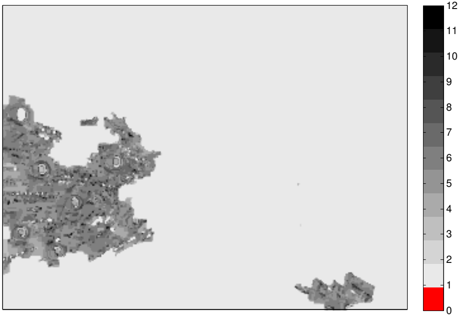


Algorithm Output

Proved optimal part



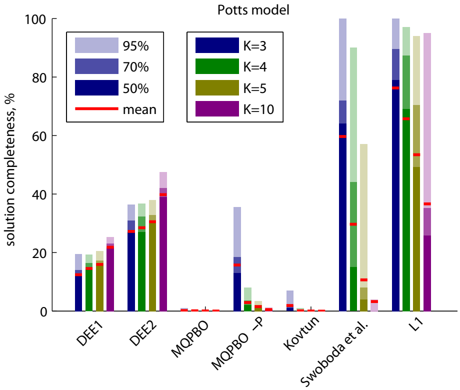
Reminder



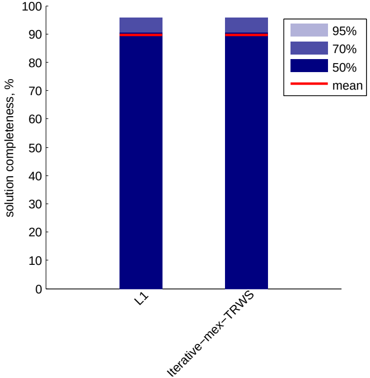
number of labels remained

Evaluation: Random Problems

CVPR'14 Comparison

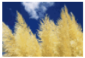


L1 vs. new algorithm using TRW-S



- Algorithm using TRW-S performs closely to maximum persistency

OpenGM Benchmark



ColorSegmentation (N8)
J. Lellmann et.al.
converted by J. Lellmann and J.H. Kappes

Object Segmentation
K. Alahari et.al.
converted by J.H. Kappes

MRF Stereo
R. Szeliski et.al.
converted by J.H. Kappes

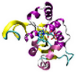
ColorSegmentation
K. Alahari et.al.
converted by J.H. Kappes

MRF Photomontage
R. Szeliski et.al.
converted by J.H. Kappes

MRF Inpainting
R. Szeliski et.al.
converted by J.H. Kappes

Chinese Characters
S. Nowozin et.al.
converted by S. Nowozin and J. H. Kappes

Scene Decomposition
Gould et.al.
converted by S. Nowozin and J. H. Kappes



Protein Folding
Yanover et. al.
converted by Joerg Kappes

Brain 3mm
J. H. Kappes et.al.
converted by J. H. Kappes

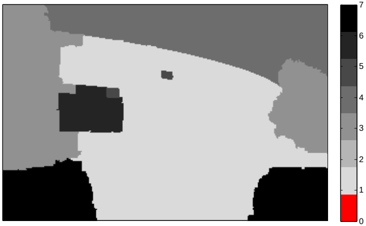
Geometric Surface Labeling (3)
Gallagher et.al.
converted by D. Batra and J. H. Kappes

Problem family	#I	#L	#V	MQPBO		MQPBO-10		Kovtun		[29]-TRWS		Our-TRWS	
mrf-stereo	3	16-60	> 100000	†		†		†		2.5h	13%	117s	**73.56%**
mrf-photomontage	2	5-7	≤ 514080	93s	22%	866s	16%	†		3.7h	16%	483s	**41.98%**
color-seg	3	3-4	≤ 424720	22s	11%	87s	16%	**0.3s**	98%	1.3h	>**99%**	61.8s	**99.95%**
color-seg-n4	9	3-12	≤ 86400	22s	8%	398s	14%	**0.2s**	67%	321s	90%	4.9s	**99.26%**
ProteinFolding	21	≤ 483	≤ 1972	685s	2%	2705s	2%	†		48s	18%	9.2s	**55.70%**
object-seg	5	4-8	68160	3.2s	0.01%	†		**0.1s**	93.86%	138s	98.19%	2.2s	**100%**

OpenGM Benchmark: Easy Examples

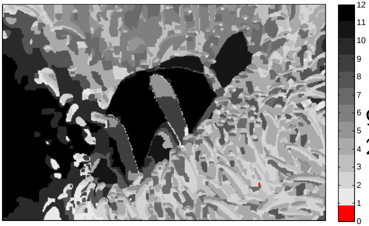
- Some problems are easy (TRWS finds optimal solution or near)

Object Segmentation



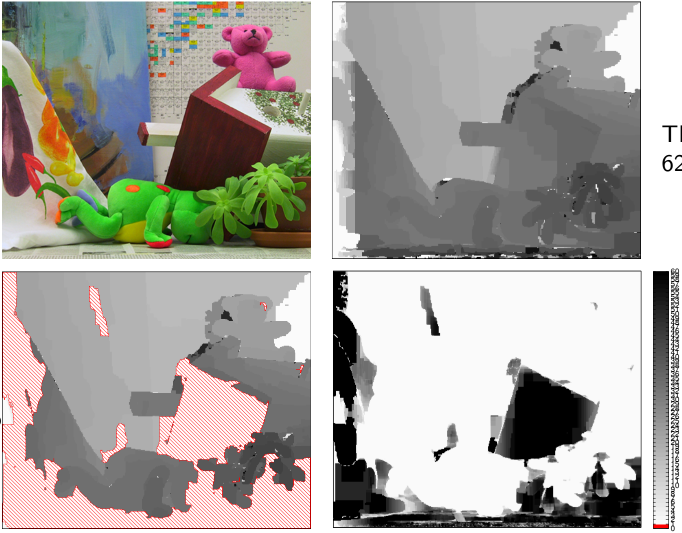
100%
1.7s

Color Segmentation



99.99%
2.8s

OpenGM Benchmark: Hard Examples

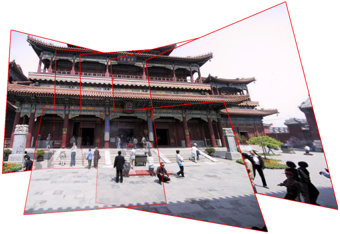


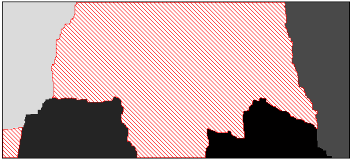
TRW-S
62s

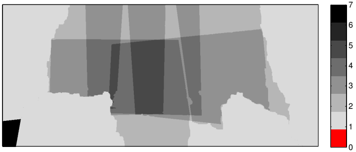
+180s
75%

OpenGM Benchmark: Very Hard Examples

Panorama Stitching







Panorama Stitching with Constraints

Conclusion

- New general sufficient condition (local + equivalent transforms)
- Covers many methods in the literature (! does not imply it is very powerful)
- Developed an efficient algorithm (implementation available, matlab interface)
- Algorithm can be understood as converting a method without guarantees (TRW-S) into a method with guarantees at a reasonable overhead
- What are these guarantees useful for further? Model verification? Learning?

Thank You!

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