Maximum Persistency in Energy Minimization

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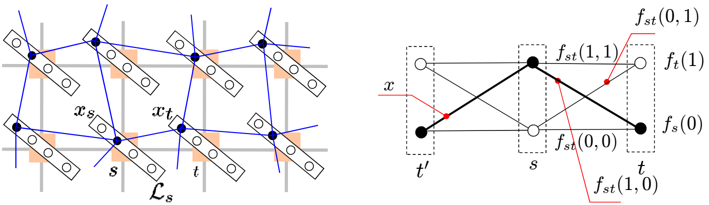
Discrete Energy Minimization



- Minimize partially separable function

$$E_f(x) = f_0 + \sum_{s\in\mathcal{V}} f_s(x_s) + \sum_{st\in\mathcal{E}} f_{st}(x_s, x_t),$$

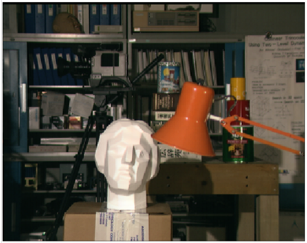
over assignments (labelings): $x = (x_s \in \mathcal{L}_s \mid s \in \mathcal{V})$

- studied as MAP MRF/CRF, WCSP
- NP-hard to approximate (*e.g.* Orponen 1990 for TSP)
- This work: reduce domains (sets of labels $\mathcal{L}_s$) while retaining some/all optimal solutions, in polynomial time

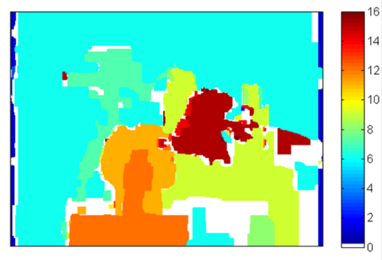
Partial Optimality

Example to illustrate what is the hope here:

Stereo Reconstruction
(Model of Alahari *et al*. 2010)



Partial Optimality
(Method of Kovtun, 2010)



- Can find a partial assignment that holds for any global optimum, which is unknown

Several Different Methods

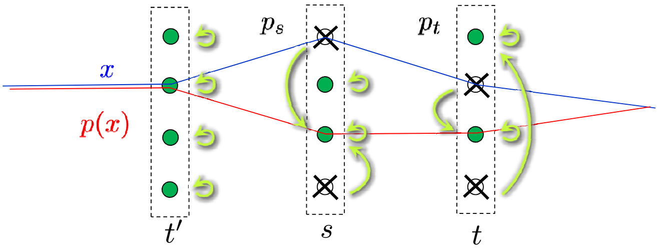
There were proposed several substantially different methods:

- Dead End Elimination (DEE)
- Persistency in Quadratic Pseudo-Boolean Optimization (QPBO)
- MQPBO
- Methods of Kovtun 04, 10
- Methods of Swoboda *et al*. 13 (14)

What do they have in common?

Improving Mappings

Improving Mapping



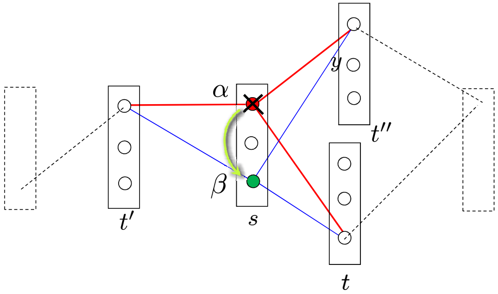
Definition

Mapping $p\colon \mathcal{L} \to \mathcal{L}$ is *improving* if $(\forall x \in \mathcal{L})\ E_f(p(x)) \leq E_f(x)$
strictly improving if $x \neq p(x) \Rightarrow E_f(p(x)) < E_f(x)$

- If x is optimal then $p(x)$ is optimal
- For strictly improving all optimal solutions are in $p(\mathcal{L})$
- Composition: if p, q are improving $\Rightarrow p \circ q$ is improving: $E_f(p(q(x))) \leq E_f(q(x)) \leq f(x)$

Dead End Elimination (DEE)

Family of methods by Desmet *et al*. 1992, Goldstein 1994, *etc*.



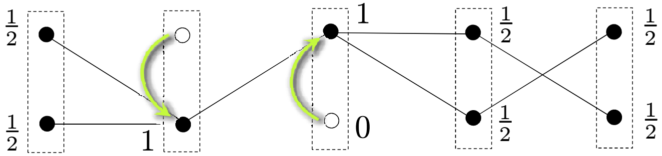
- Apply mapping in a single pixel s
- Improving iff

  $$f_s(\alpha) - f_s(\beta) + \sum_{t \in \mathcal{N}(s)} \min_{x_t \in \mathcal{L}_t} [f_{st}(\alpha, x_t) - f_{st}(\beta, x_t)] \geq 0$$

  (worst case energy change over neighbours assignment)
- Compose many such mappings

Quadratic Pseudo-Boolean Optimization (QPBO)

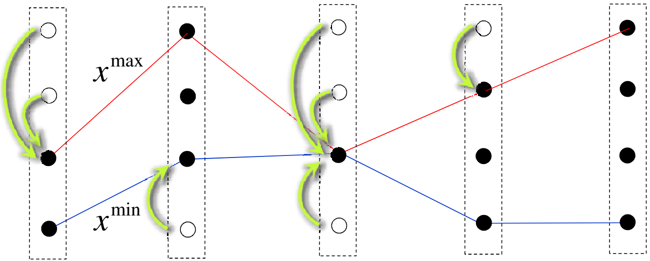
Nemhauser and Trotter 75, Hammer *et al.* 84, Boros *et al.* 02, Rother*et al.* 07



- Integral part of the LP relaxation is globally optimal
  $\mathcal{A} \subset \mathcal{V}$, $y = (y_s \mid s \in \mathcal{A})$
- "Autarky":
  replace x with y on $\mathcal{A}$ ($x[\mathcal{A} \leftarrow y]$) is guaranteed not to increase the energy
  mapping $x \mapsto x[\mathcal{A} \leftarrow y]$ is improving

Multilabel QPBO (MQPBO)

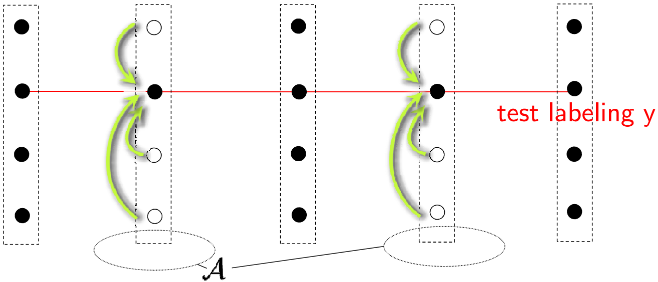
Kohli *et al.* 08, Windheuser *et al.* 12



- Fixed linear ordering
- Reduction to pseudo-Boolean + QPBO guarantees
- "Autarky": mapping $x \mapsto (x \vee x^{\min}) \wedge x^{\max}$ is improving

Kovtun one vs. all Method

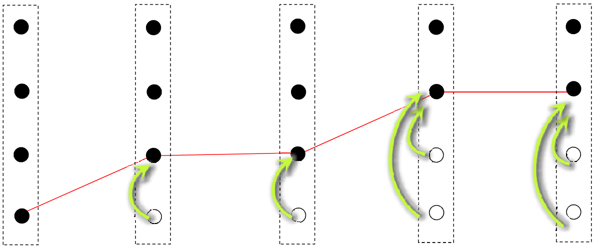
Kovtun 2004, 2010



- Builds auxiliary submodular 2-label energy for given y
- "Autarky": mapping $x \mapsto x[\mathcal{A} \leftarrow y]$ is improving

Kovtun general Method

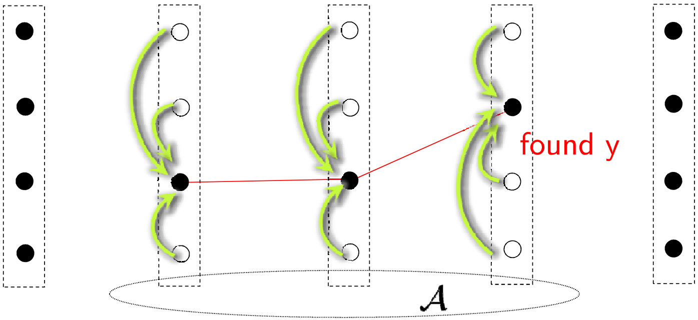
Kovtun 2004, 2010



- Builds auxiliary submodular multilabel energy and y
- Mapping $x \mapsto (x \vee y)$ is improving

Iterative Pruning

Swoboda *et al.* 2013, 2014



- Iteratively builds auxiliary energy and solves its LP relaxation
- "Autarky": mapping $x \mapsto x[\mathcal{A} \leftarrow y]$ is improving

Verification Problem

- Verifying whether $p\colon \mathcal{L} \to \mathcal{L}$ is improving is NP-hard

_e.g._, Boros _et al._ 2006

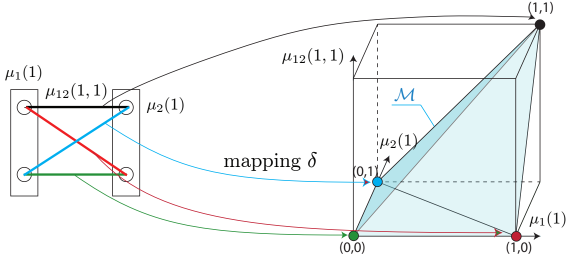
Determining whether a partial assignment is an autarky is NP-hard

- How do these methods find one? – Finer sufficient conditions.

Generalized Sufficient Condition

LP Relaxation

Schlesinger 76, Koster *et al*. 98, 99, Chekuri *et al*. 01, Wainwright *et al*. 02, Werner 08.

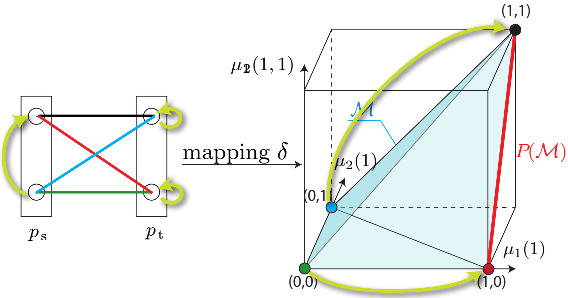


- Embedding: $\delta(x) \in \mathbb{R}^{\mathcal{I}}$
- $E_f(x) = \langle f, \delta(x) \rangle$
- Relaxation:

$$\min_{x \in \mathcal{L}} \langle f, \delta(x) \rangle \geq \min_{\mu \in \Lambda} \langle f, \mu \rangle$$

- $\Lambda \supset \operatorname{conv}(\delta(\mathcal{L}))$

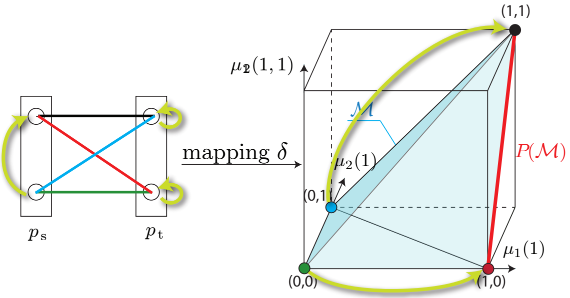
Relaxed Improving Mapping



Linear Extension

$$(\forall x \in \mathcal{L})\ \delta(p(x)) = P\delta(x)$$

Relaxed Improving Mapping



Linear Extension

$$(\forall x \in \mathcal{L})\ \delta(p(x)) = P\delta(x)$$

Definition

Mapping $p\colon \mathcal{L} \to \mathcal{L}$ is Relaxed-improving if

$$(\forall \mu \in \Lambda)\ \ \langle f, P\mu\rangle \leq \langle f, \mu\rangle$$

Relaxed Improving Mapping

Improving	Relaxed-Improving
$(\forall x) \quad E_f(p(x)) \leq E_f(x)$	$(\forall \mu \in \Lambda) \quad \langle f, P\mu \rangle \leq \langle f, \mu \rangle$
	$\Lambda \supset \mathrm{conv}(\delta(\mathcal{L}))$

- Sufficient condition

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- Sufficient condition
- Can be verified via LP: $\min\limits_{\mu \in \Lambda} \langle f, (I - P)\mu \rangle \leq 0$

Relaxed Improving Mapping

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$(\forall x) \quad E_f(p(x)) \leq E_f(x)$	$(\forall \mu \in \Lambda) \quad \langle f, P\mu \rangle \leq \langle f, \mu \rangle$
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- Sufficient condition
- Can be verified via LP: $\min\limits_{\mu \in \Lambda} \langle f, (I - P)\mu \rangle \leq 0$

Theorem

Relaxed-improving condition is satisfied for all methods:

- Goldstein's General DEE
- QPBO
- MQPBO (prev. work, Shekhovtsov *et al*. 07)
- Methods of Kovtun (prev. work, Shekhovtsov *et al*. 12)
- Methods of Swoboda *et al*. 13 (14*)

Maximum Persistency

Maximum Persistency

- Given that verification problem is polynomially solvable,
- Which method is better?

Proposition

Pose "the best partial optimality" as optimization problem

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- Which method is better?

Proposition

Pose "the best partial optimality" as optimization problem

Find the mapping $p\colon \mathcal{L} \to \mathcal{L}$ that delivers the maximum problem reduction:

$$\min_{p \in \mathcal{P}} \sum_s |p(\mathcal{L}_s)| \quad \text{s.t. } p \text{ is relaxed improving for } f,$$

$\mathcal{P}$ - class of mappings.

Maximum Persistency

- For pseudo-Boolean case is solved by QPBO (strong and weak persistency)
- Polynomial for further cases

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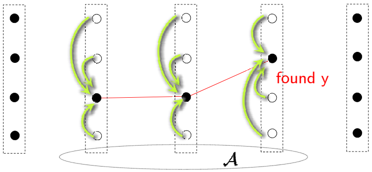
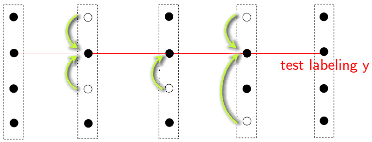
All-to-one maps, strictly improving



Covers:

- Swoboda *et al.* 13 (14*)
- QPBO
- one vs. all Kovtun 04

Subset-to-one maps



- General method of Kovtun
- DEE if applied K times
- Other case, if y selected by LP

Method

Maximum Persistency

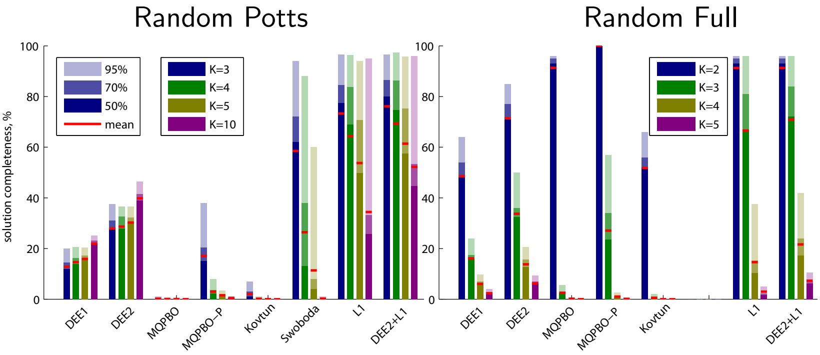
$$\min_{p \in \mathcal{P}} \sum_s |p(\mathcal{L}_s)|$$

$$\text{s.t.} \ \min_{\mu \in \Lambda} \langle f, (I-P)\mu \rangle \leq 0 \quad \Leftrightarrow \quad (\exists \varphi) \ \ (I-P)^{\mathsf{T}} f - \varphi A^{\mathsf{T}} \geq 0$$

Reformulate as a linear program, **L1**

- Optimizes over relaxed mapping and reparametrization jointly
- Optimal solution recovers optimal discrete mapping

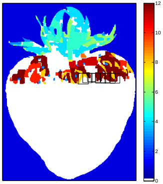
Experimental Validation



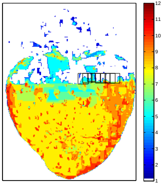
- 10 x 10 Grid graph, random weights
- All test problems have integrality gap (not LP-tight)
- Verified correctness by solving LP

Windowing

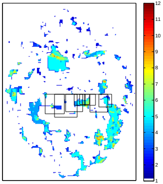
image



partial labeling



$|\mathcal{L}_s|$



$|\mathcal{L}_s|$ at the end

- Can restrict mapping to a window - global correctness guarantees (generalization of DEE)

Conclusion

- $+$ Generalized sufficient condition
- $+$ Direct formulation of the maximum persistency
- $+$ Optimal method in a range of cases
- $-$ Requires to solve LP

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