Dyadic Wavelets

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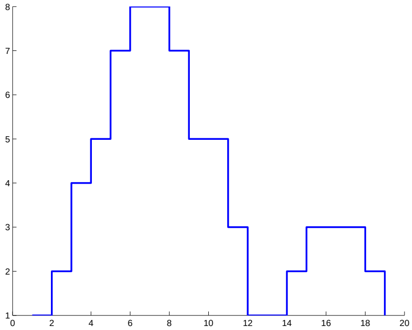
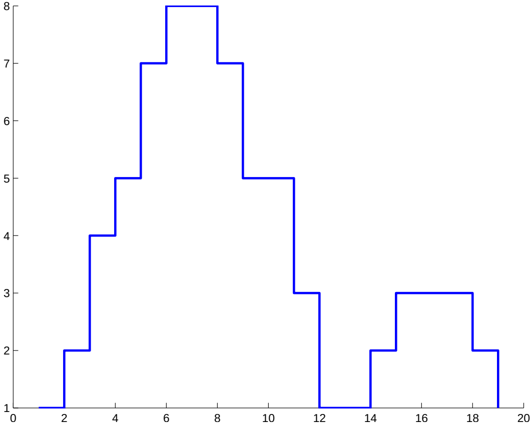
winter semester 2007

Input 1D signal

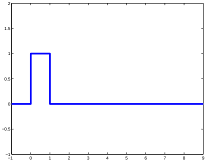


$$a^{(0)} = [1\ 2\ 4\ 5\ 7\ 8\ 8\ 7\ 5\ 5\ 3\ 1\ 1\ 2\ 3\ 3\ 3\ 2\ 1]$$

Approximation coefficients

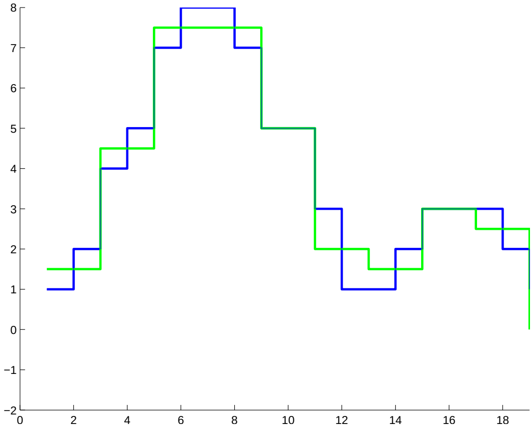


$$f(t) = \sum_i a_k^{(0)} \phi(t-k)$$
$$f \in V_0$$



$$\phi(t)$$

Approximation coefficients

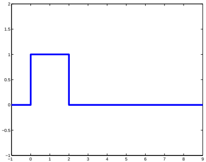


$$f(t) \approx \sum_k a_k^{(-1)} \phi(2^{-1}t - k)$$

$$f \in V_{-1}$$

Analysis: $a^{(-1)} = \left(a^{(0)} * h\right)_{\downarrow 2}$

Synthesis: $a^{(0)} \approx \left(a^{(-1)}\right)_{\uparrow 2} * \tilde{h}$

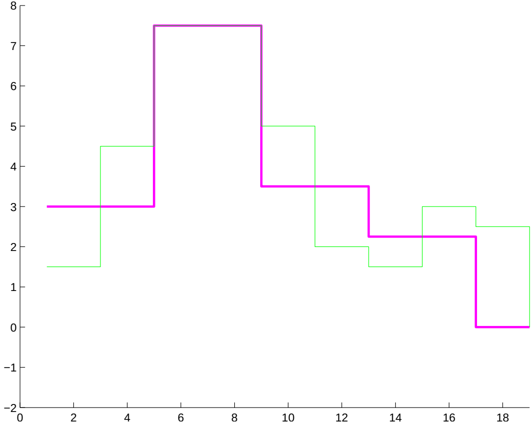


$$\phi(2^{-1}t)$$

$$h = \frac{1}{2}[1\ 1]$$

$$\tilde{h} = [1\ 1]$$

Approximation coefficients

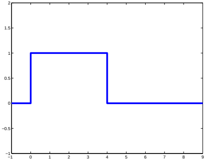


$$f(t) \approx \sum_k a_k \phi(2^{-2}t - k)$$

$$f \in V_{-2}$$

Analysis: $a^{(-2)} = \left(a^{(-1)} * h\right)_{\downarrow 2}$

Synthesis: $a^{(-1)} \approx \left(a^{(-2)}\right)_{\uparrow 2} * \tilde{h}$

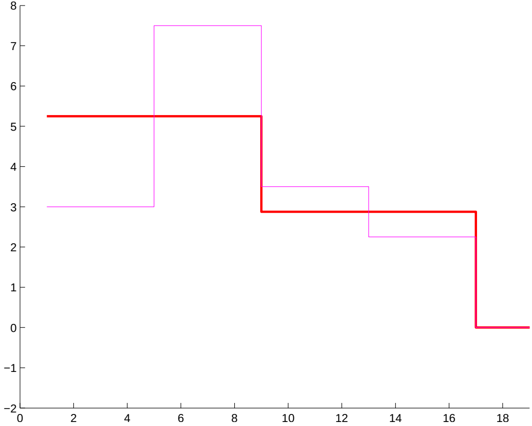


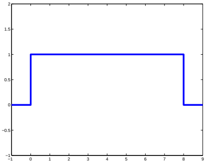
$$\phi(2^{-2}t)$$

$$h = \frac{1}{2}[1\ 1]$$

$$\tilde{h} = [1\ 1]$$

Approximation coefficients





$$f(t) \approx \sum_k a_k^{(-3)} \phi(2^{-3}t - k)$$

$$f \in V_{-3}$$

$$\phi(2^{-3}t)$$

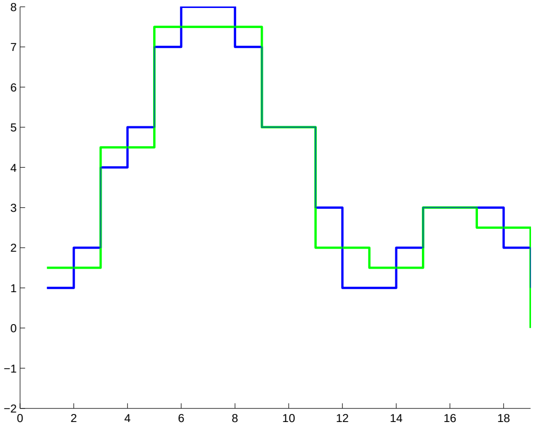
$$a^{(-3)} = \left(a^{(-2)} * h\right)_{\downarrow 2}$$

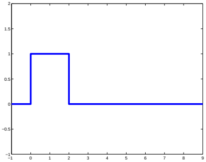
$$h = \tfrac{1}{2}[1\ 1]$$

Synthesis: $a^{(-2)} \approx \left(a^{(-3)}\right)_{\uparrow 2} * \tilde{h}$

$$\tilde{h} = [1\ 1]$$

Detail coefficients





$$f(t) \approx \sum_k a_k^{(-1)} \phi(2^{-1}t - k)$$

$$\phi(2^{-1}t)$$

$$f \in V_{-1}$$

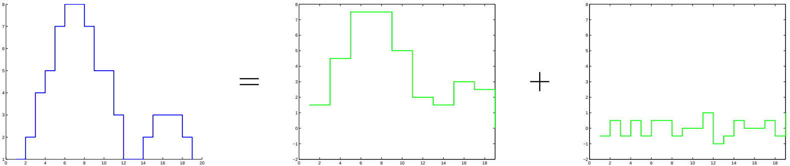
$$a^{(-1)} = \left(a^{(0)} * h\right)_{\downarrow 2}$$

$$h = \tfrac{1}{2}[1\ 1]$$

$$a^{(0)} \approx \left(a^{(-1)}\right)_{\uparrow 2} * \tilde{h}$$

$$\tilde{h} = [1\ 1]$$

Detail coefficients



$$f(t) = \underbrace{\sum_k a_k^{(-1)} \phi(2^{-1}t - k)}_{V_{-1}} + \underbrace{\sum_k b_k^{(-1)} \psi(2^{-1}t - k)}_{W_{-1}}$$

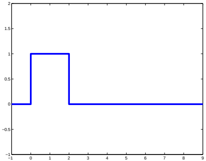
Analysis: $a^{(-1)} = \left(a^{(0)} * h\right)_{\downarrow 2}$ $\qquad h = \frac{1}{2}[1\ 1]$

$b^{(-1)} = \left(a^{(0)} * g\right)_{\downarrow 2}$ $\qquad g = \frac{1}{2}[1\ -1]$

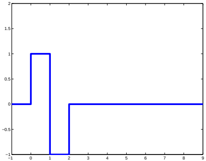
Synthesis: $a^{(0)} = \left(a^{(-1)}\right)_{\uparrow 2} * \tilde{h} + \left(b^{(-1)}\right)_{\uparrow 2} * \tilde{g}$ $\qquad \tilde{h} = [1\ 1]$

$\tilde{g} = [1\ -1]$

Detail coefficients



$\phi(2^{-1}t)$



$\psi(2^{-1}t)$

$$f(t) = \underbrace{\sum_k a_k^{(-1)} \phi(2^{-1}t - k)}_{V_{-1}} + \underbrace{\sum_k b_k^{(-1)} \psi(2^{-1}t - k)}_{W_{-1}}$$

Analysis: $a^{(-1)} = \left(a^{(0)} * h\right)_{\downarrow 2}$ $\qquad h = \frac{1}{2}[1\ 1]$

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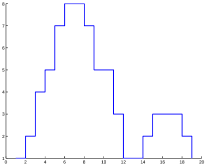
Synthesis: $a^{(0)} = \left(a^{(-1)}\right)_{\uparrow 2} * \tilde{h} + \left(b^{(-1)}\right)_{\uparrow 2} * \tilde{g}$ $\qquad \tilde{h} = [1\ 1]$

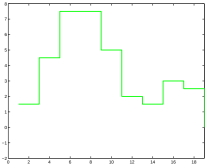
$\tilde{g} = [1\ \ -1]$

Hierarchical decomposition



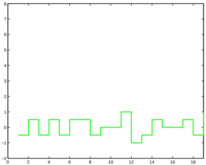
$a^{(0)}$

$=$



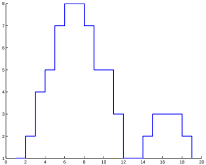
$a^{(-1)}$

$+$



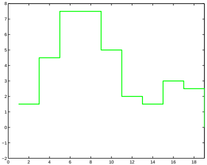
$b^{(-1)}$

Hierarchical decomposition



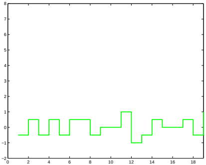
$a^{(0)}$

$=$

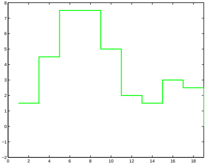


$a^{(-1)}$

$+$

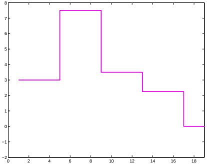


$b^{(-1)}$



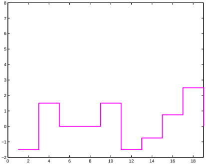
$a^{(-1)}$

$=$



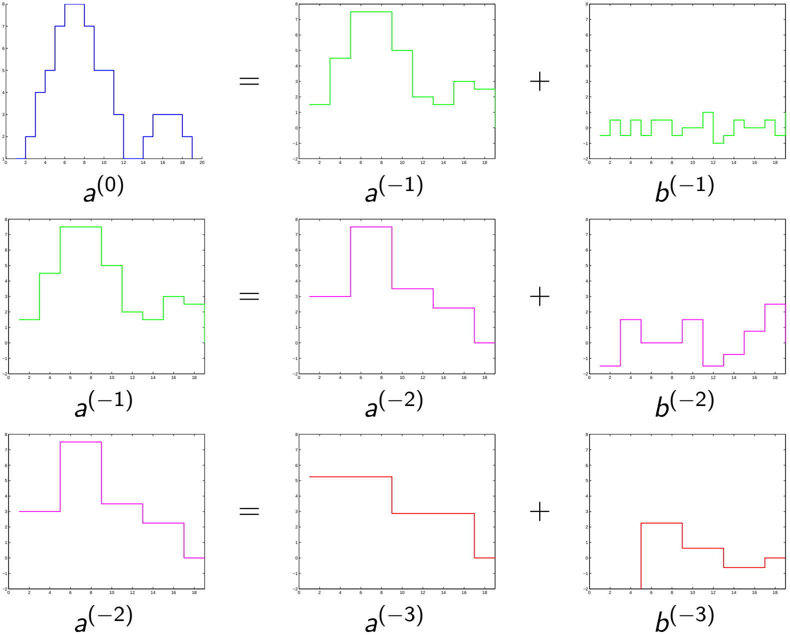
$a^{(-2)}$

$+$

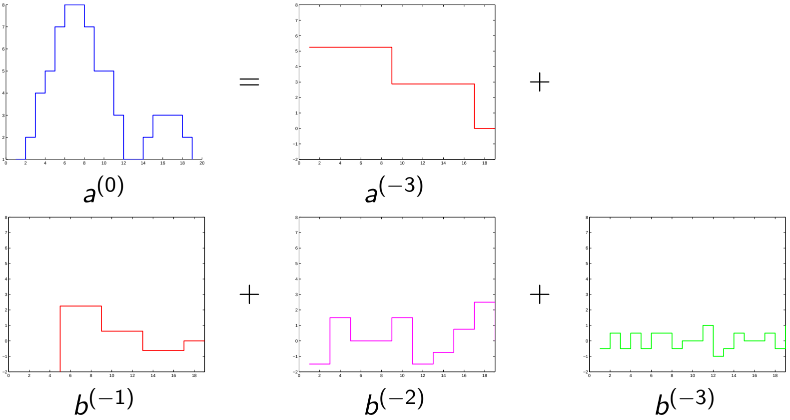


$b^{(-2)}$

Hierarchical decomposition



Hierarchical decomposition



Analysis/synthesis equations

$$\underbrace{f(t)}_{V_0} = \sum_k a_k^{(0)} \phi(t-k)$$

$$= \underbrace{\sum_k a_k^{(-L)} \phi(2^{-1}t-k)}_{V_{-L}} + \sum_{j=L}^{-1} \underbrace{\sum_k b_k^{(-1)} \psi(2^{-1}t-k)}_{W_{-j}}$$

Analysis: $a^{(j-1)} = \left(a^{(j)} * h\right)_{\downarrow 2}$ $\qquad h = \frac{1}{2}[1\ 1]$

$b^{(j-1)} = \left(a^{(j)} * g\right)_{\downarrow 2}$ $\qquad g = \frac{1}{2}[1\ \ -1]$

Synthesis: $a^{(j)} = \left(a^{(j-1)}\right)_{\uparrow 2} * \tilde{h} + \left(b^{(j-1)}\right)_{\uparrow 2} * \tilde{g}$ $\qquad \tilde{h} = [1\ 1]$

$\tilde{g} = [1\ \ -1]$

Analysis/synthesis numerical example

Analysis:

$a^{(0)} = [1\ 2\ 4\ 6]$

$a^{(-1)} = [1.5\ 5]$ $b^{(-1)} = [0.5\ 1]$

$a^{(-2)} = [3.25]$ $b^{(-2)} = [1.75]$

Synthesis:

$$\big(\underbrace{[3.25]}_{a^{(-2)}}\big)_{\uparrow 2} * [1\ 1] = [3.25\ 3.25]$$

$$\big(\underbrace{[1.75]}_{b^{(-2)}}\big)_{\uparrow 2} * [1\ \ -1] = [-1.75\ 1.75]$$

$$[3.25\ 3.25] + [-1.75\ 1.75] = [1.5\ 5] = a^{(-1)}$$

$$\big(\underbrace{[1.5\ 5]}_{a^{(-1)}}\big)_{\uparrow 2} * [11] = [1.5\ 1.5\ 5\ 5]$$

$$\big(\underbrace{[0.5\ 1]}_{b^{(-1)}}\big)_{\uparrow 2} * [1 - 1] = [-0.5\ 0.5\ \ -1\ 1]$$

$$[1.5\ 1.5\ 5\ 5] + [-0.5\ 0.5\ \ -1\ 1] = [1\ 2\ 4\ 6] = a^{(0)}$$

Iterated filterbank (Mallat's algorithm, DWT):

Analysis:

$$\underbrace{a^{(0)}}_{N} \longrightarrow \big[\underbrace{a^{(-1)}}_{N/2}\ \underbrace{b^{(-1)}}_{N/2}\big] \longrightarrow \big[\underbrace{a^{(-2)}}_{N/4}\ \underbrace{b^{(-2)}}_{N/4}\ \underbrace{b^{(-1)}}_{N/2}\big] \longrightarrow \cdots \longrightarrow$$

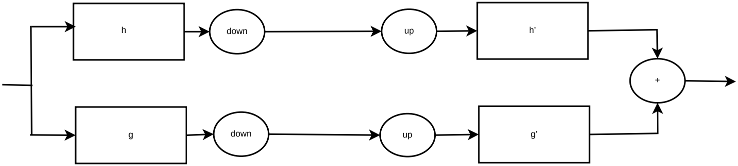
$$\big[\underbrace{a^{(-L)}}_{1}\ \underbrace{b^{(-L)}}_{1} \ldots \underbrace{b^{(-2)}}_{N/4}\underbrace{b^{(-1)}}_{N/2}\big]$$

Synthesis:

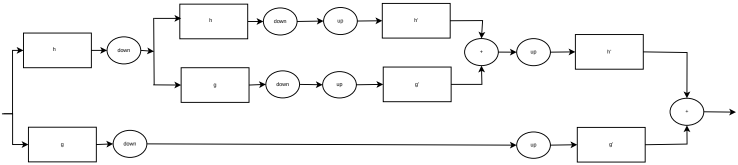
$$\big[\underbrace{a^{(-L)}}_{1}\ \underbrace{b^{(-L)}}_{1}\underbrace{b^{(-L+1)}}_{2} \ldots \underbrace{b^{(-2)}}_{N/4}\underbrace{b^{(-1)}}_{N/2}\big] \longrightarrow$$

$$\big[\underbrace{a^{(-L+1)}}_{2}\ \underbrace{b^{(-L+1)}}_{2} \ldots \underbrace{b^{(-2)}}_{N/4}\underbrace{b^{(-1)}}_{N/2}\big] \longrightarrow \cdots \longrightarrow \underbrace{a^{(0)}}_{N}$$

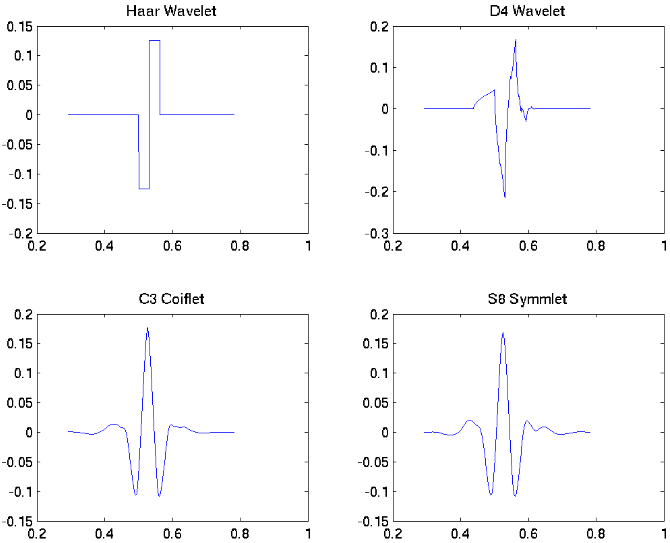
Analysis/synthesis flowchart



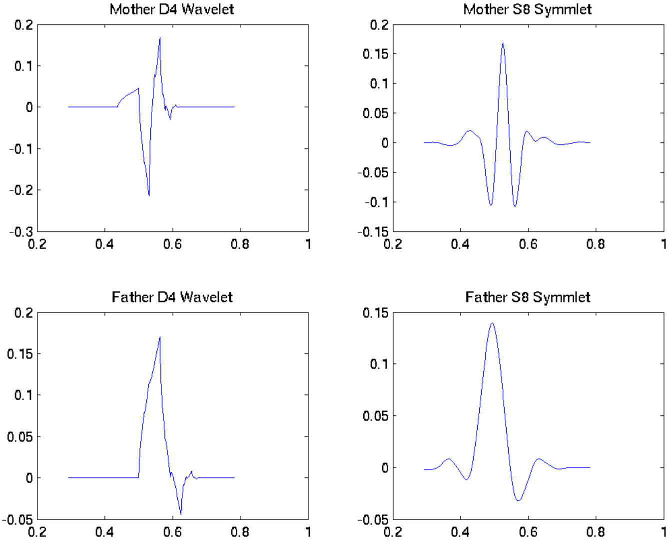
Analysis/synthesis flowchart



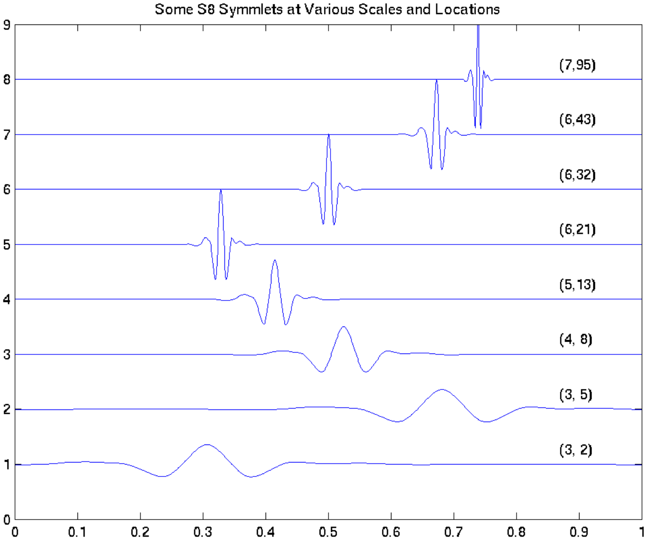
Wavelet examples



Wavelet examples



Wavelet examples



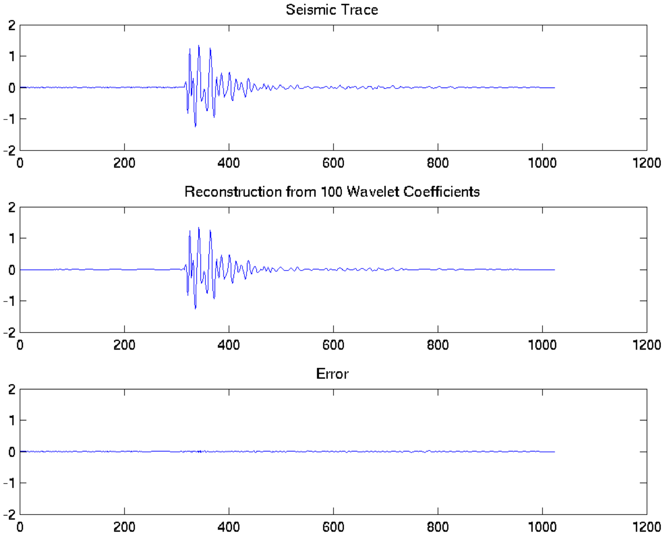
Wavelet compression

- Wavelet transform (analysis)
- Order coefficients by magnitude
- Only use M largest (set the rest to zero)
- Inverse wavelet transform (synthesis)

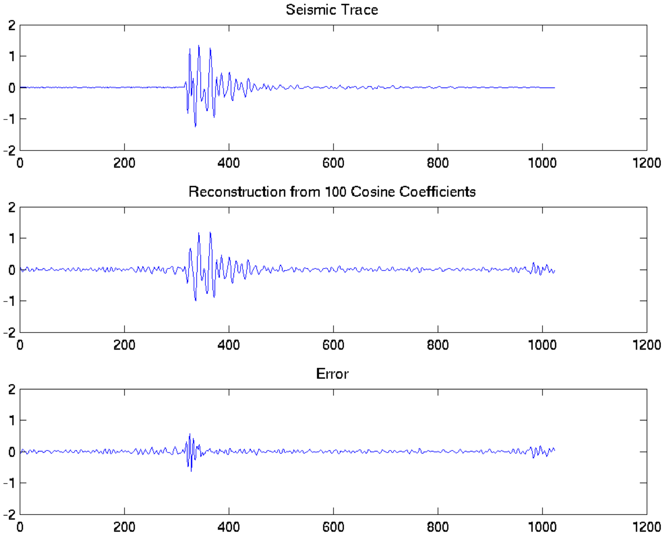
Wavelet denoising

- Wavelet transform (analysis)
- Threshold coefficients (smaller than $T \longrightarrow$ set to zero)
- Inverse wavelet transform (synthesis)

Wavelet compression example

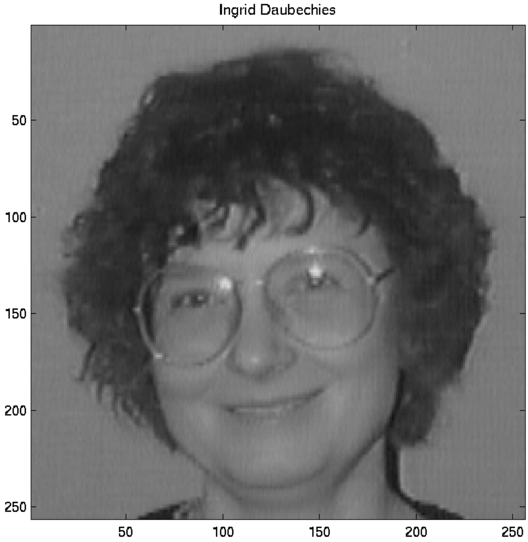


Wavelet compression example



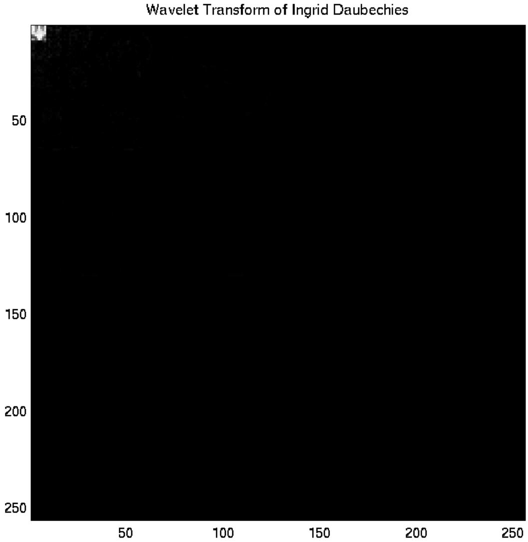
2D Wavelet compression example

Separable decomposition, alternate x and y.



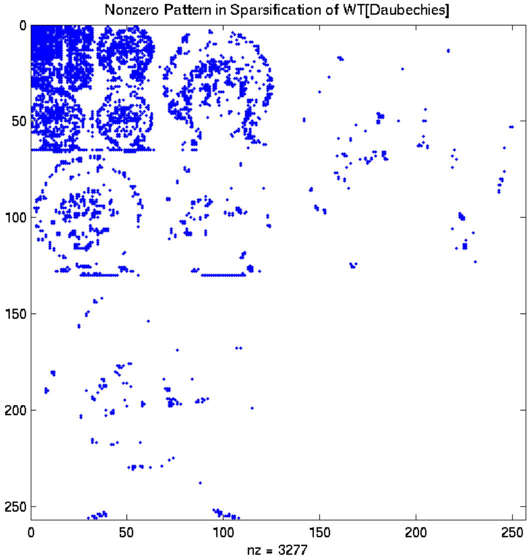
2D Wavelet compression example

Separable decomposition, alternate x and y.



2D Wavelet compression example

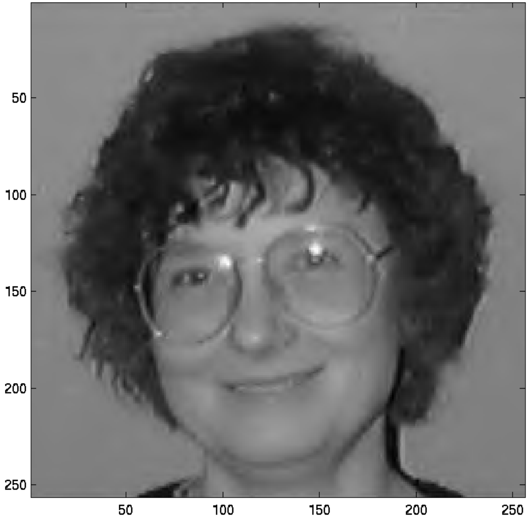
Separable decomposition, alternate x and y.



2D Wavelet compression example

Separable decomposition, alternate x and y.

95% Wavelet Co/Dec of Daubechies



2D Wavelet compression example

Separable decomposition, alternate x and y.

Dyadic wavelets — conclusions

- Good basis for piecewise signals
- Tunable
- Fast algoprithms