

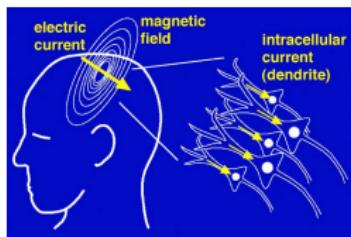
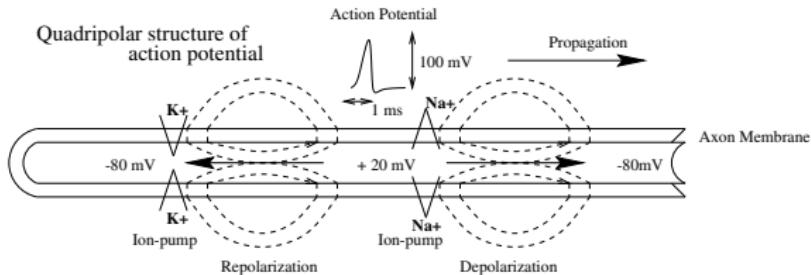
# EEG a MEG

## část 2, matematické metody

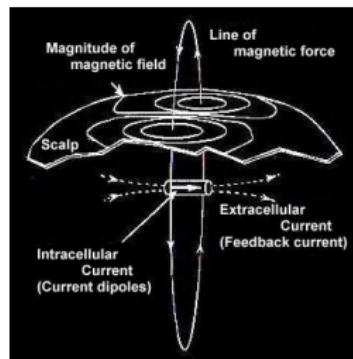
Jan Kybic

March 1, 2006

# Currents in the brain



Primary currents  $\mathbf{J}^p [\text{A}/\text{m}^2]$



# Electroencephalography (EEG)



- ▶ Hans Berger, 1920s
- ▶ 16 ~ 64 electrodes
- ▶  $100 \mu\text{V}$  on the scalp ( $1 \sim 2 \text{ mV}$  on the cortex)

# Magnetoencephalography (MEG)



- ▶ Measuring (very weak) magnetic fields ( $\sim 10^{-14} \text{ T} = 100 \text{ fT}$ )
- ▶ Shielded room, superconductive sensors

# Maxwellovy rovnice (ve vakuu)

## Gaussův zákon

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

$$\oint \mathbf{E} \cdot d\mathbf{S} = \frac{q}{\epsilon_0}$$

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## Gaussův zákon pro magnetismus

$$\nabla \cdot \mathbf{B} = 0$$

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## Faradayův zákon indukce

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\oint \mathbf{E} \cdot d\mathbf{s} = -\frac{d\Phi}{dt}$$

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## Ampérův zákon

$$\nabla \times \mathbf{B} = \mu_0 \left( \mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) \quad \oint \mathbf{B} \cdot d\mathbf{s} = \mu_0 \left( I + \epsilon_0 \frac{dD}{dt} \right)$$

## └ Maxwellovy rovnice (ve vakuu)

Uzavřená plocha:

$$\int_{\Omega} \nabla \cdot \mathbf{f} \, d\mathbf{x} = \oint_{\partial\Omega} \mathbf{f} \cdot d\mathbf{S}$$

Uzavřená křivka:

$$\int_S \nabla \times \mathbf{f} \, d\mathbf{S} = \oint_{\partial S} \mathbf{f} \cdot d\mathbf{s}$$

$$\mu_0 \epsilon_0 = \frac{1}{c^2}$$

Posuvný proud / Elektrická indukce

$$\Psi = \int E \cdot d\mathbf{S}$$

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## Kvazistatická aproximace

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = 0$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

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$$\nabla \times \mathbf{E} = 0 \quad \rightarrow \quad \mathbf{E} = -\nabla V$$

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$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \quad \rightarrow \quad \nabla \cdot \mathbf{J} = 0$$

## └ Kvazistatická approximace

$$\begin{aligned}\nabla \cdot \mathbf{E} &= \frac{\rho}{\epsilon_0} & \nabla \times \mathbf{E} &= 0 \\ \nabla \cdot \mathbf{B} &= 0 & \nabla \times \mathbf{B} &= \mu_0 \mathbf{J} \\ \nabla \times \mathbf{E} = 0 &\quad \Rightarrow \quad \mathbf{E} = -\nabla V \\ \nabla \times \mathbf{B} = \mu_0 \mathbf{J} &\quad \Rightarrow \quad \nabla \cdot \mathbf{J} = 0\end{aligned}$$

## Potenciálové pole

$$\int_{C_1: A \rightarrow B} \mathbf{f} \, d\mathbf{s} = \int_{C_2: A \rightarrow B} \mathbf{f} \, d\mathbf{s} = \phi(B) - \phi(A)$$

## Rotace

$$(\nabla \times \mathbf{f}) \cdot \mathbf{n} = \lim_{S \rightarrow 0} \frac{\oint \mathbf{f} \cdot d\mathbf{s}}{A}$$

$$\nabla \times \mathbf{f} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_x & f_y & f_z \end{vmatrix} = \left[ \frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z}, \quad \frac{\partial f_x}{\partial z} - \frac{\partial f_z}{\partial x}, \quad \frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y} \right]$$

$$\nabla \cdot (\nabla \times \mathbf{f}) = \frac{\partial^2 f_z}{\partial y \partial x} - \frac{\partial^2 f_y}{\partial z \partial x} + \frac{\partial^2 f_x}{\partial z \partial y} - \frac{\partial^2 f_z}{\partial x \partial y} + \frac{\partial^2 f_y}{\partial x \partial z} - \frac{\partial^2 f_x}{\partial y \partial z} = 0$$

$$\nabla \cdot (\nabla \times \mathbf{B}) = 0 = \nabla \cdot \mathbf{J}$$

## Proud ve vodiči

$$\mathbf{J} = \mathbf{J}^p + \sigma \mathbf{E} = \mathbf{J}^p - \sigma \nabla V$$

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$$\mathbf{J} = \mathbf{J}^P + \sigma \mathbf{E} = \mathbf{J}^P - \sigma \nabla V$$

$$\nabla \cdot \mathbf{J} = 0 \quad - \quad \boxed{\nabla \cdot (\sigma \nabla V) = \nabla \cdot \mathbf{J}^P}$$

### 1. Ohmův zákon $\mathbf{J} = \sigma \mathbf{E}$

## Poissonova a Laplaceova rovnice

$$\nabla \cdot (\sigma \nabla V) = \nabla \cdot \mathbf{J}^p$$

Konstantní  $\sigma$

$$\sigma \Delta V = \nabla \cdot \mathbf{J}^p$$

Poissonova / Laplaceova rovnice (homogenní forma)

2006-03-01

## └ Poissonova a Laplaceova rovnice

$$\nabla \cdot (\sigma \nabla V) = \nabla \cdot J^P$$

Konstantní  $\sigma$

$$\sigma \Delta V = \nabla \cdot J^P$$

Poissonova / Laplaceova rovnice (homogenní forma)

$$\nabla \cdot \nabla f = \Delta f$$

# Greenova funkce pro Laplaceovu rovnici

$$\Delta V = f = \nabla \cdot \mathbf{J}^p$$

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{4\pi \|\mathbf{r} - \mathbf{r}'\|} \quad -\Delta_{\mathbf{r}} G(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}') = \delta_{\mathbf{r}'}$$

$$G(\mathbf{r}) = \frac{1}{4\pi \|\mathbf{r}\|}$$

Řešení:

$$V(\mathbf{r}) = f * G(\mathbf{r})$$

## └ Greenova funkce pro Laplaceovu rovnici

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Řešíme  $m \frac{dv}{dt} = -Rv + f(t)$

$v = 0$  v čase  $t = \tau$ , v čase  $\tau, \tau + \Delta\tau$  přijde 'rána' s impulsem  
 $\int_{\tau}^{\tau + \Delta\tau} f(t) dt = I.$

Pro  $t > \tau + \Delta\tau$ ,  $m \frac{dv}{dt} = -Rv$

Příklad  $f(t) = I\delta(t - \tau)$

Řešení  $v(t) = Ae^{-(R/m)t}$ . Jak je velké  $A$ ? Integrujeme původní rovnici:

$$m(v(\tau + \Delta\tau) - v(\tau)) = I - R \int_{\tau}^{\tau + \Delta\tau} v(t) dt$$

Zanedbáme integrál  $v$ , protože  $\Delta\tau$  je malé.

## └ Greenova funkce pro Laplaceovu rovnici

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$$G(\mathbf{r}) = \frac{1}{4\pi \|\mathbf{r}\|}$$

Řešení:

$$V(\mathbf{r}) = f + G(\mathbf{r})$$

Po srovnání s integrálem máme  $m A e^{-(R/m)\tau} = I$  a z toho

$$v(t) = \begin{cases} 0 & t \leq \tau \\ (I/m)e^{-(R/m)(t-\tau)} & t > \tau \end{cases} = I G(t, \tau)$$

Pokud přijdou dva impulzy,  $I_1$  a  $I_2$  v časech  $\tau_1$ ,  $\tau_2$ , je řešením superpozice  $v(t) = I_1 G(t, \tau_1) + I_2 G(t, \tau_2)$ .

Totéž přijde-li sekvence impulzů:

$$f(t) = \sum_i I_i \delta(t - \tau_i)$$

$$v(t) = \sum_i I_i G(t, \tau_i)$$

Máme-li spojitý průběh  $f(t)$ , dostaneme:

$$v(t) = \int f(t) G(t, \tau) d\tau$$

## └ Greenova funkce pro Laplaceovu rovnici

$$\Delta V = f = \nabla \cdot J^p$$

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$$G(\mathbf{r}) = \frac{1}{4\pi \|\mathbf{r}\|}$$

Rušení:

$$V(\mathbf{r}) = f + G(\mathbf{r})$$

Jelikož  $G(t, \tau) = G'(t - \tau)$  lze psát

$$v(t) = \int f(t)G'(t - \tau) dt = f(t) * G'$$

$G$  resp.  $G'$  je impulzní odezva, nebo také Greenova funkce.

Obecněji pro  $\mathcal{L} * u = f$ ,  $\mathcal{L} * G = \delta$ , potom  $u = G * f$ .

## Biot-Savartův zákon

$$\mathbf{B} = \frac{\mu_0}{4\pi} J I \times \frac{\mathbf{r} - \mathbf{r}'}{\|\mathbf{r} - \mathbf{r}'\|^2}$$

$$\mathbf{B} = \frac{\mu_0}{4\pi} \int_{\Omega} \mathbf{J}(\mathbf{r}') \times \nabla_{\mathbf{r}'} \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} d\mathbf{r}'$$

## └ Biot-Savartův zákon

$$\mathbf{B} = \frac{\mu_0}{4\pi} J l \times \frac{\mathbf{r} - \mathbf{r}'}{\|\mathbf{r} - \mathbf{r}'\|^2}$$
$$\mathbf{B} = \frac{\mu_0}{4\pi} \int_{\Omega} \mathbf{J}(\mathbf{r}') \times \nabla' \cdot \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} d\mathbf{r}'$$

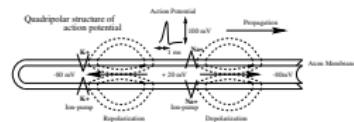
$$\nabla_{\mathbf{r}'} \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} = \frac{\mathbf{r} - \mathbf{r}'}{\|\mathbf{r} - \mathbf{r}'\|^2}$$

# Model zdrojů proudu

- ▶ Current “dipole”  $\mathbf{q}$  [Am]:

$$\mathbf{q} = \mathbf{J} \Delta / \Delta S$$

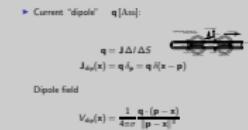
$$\mathbf{J}_{\text{dip}}(\mathbf{x}) = \mathbf{q} \delta_{\mathbf{p}} = \mathbf{q} \delta(\mathbf{x} - \mathbf{p})$$



Dipole field

$$V_{\text{dip}}(\mathbf{x}) = \frac{1}{4\pi\sigma} \frac{\mathbf{q} \cdot (\mathbf{p} - \mathbf{x})}{\|\mathbf{p} - \mathbf{x}\|^3}$$

## └ Model zdrojů proudu



Jak vypočítat  $V_{\text{dip}}$  pomocí Greenovy funkce

$$\Delta V = f = \nabla \mathbf{J} \quad V(\mathbf{r}) = f * G(\mathbf{r})$$

$$\text{Přidáme } \sigma: \sigma \Delta V = f = \nabla \mathbf{J} \quad \sigma V(\mathbf{r}) = f * G(\mathbf{r})$$

Dosadíme za  $f$  a upravíme:  $\sigma V_{\text{dip}}(\mathbf{x}) = \nabla \mathbf{J}_{\text{dip}} * G(\mathbf{r}) = \mathbf{J}_{\text{dip}} * \nabla G(\mathbf{r})$

$$G(\mathbf{r}) = \frac{1}{4\pi \|\mathbf{r}\|}$$

$$\mathbf{J}_{\text{dip}}(\mathbf{x}) = \mathbf{q} \delta_{\mathbf{p}} = \mathbf{q} \delta(\mathbf{x} - \mathbf{p})$$

$$\nabla_{\mathbf{r}} G(\mathbf{r}) = \frac{1}{4\pi} \frac{\mathbf{r}}{\|\mathbf{r}\|^3}$$

$$\mathbf{J}_{\text{dip}} * \nabla G(\mathbf{r}) = \frac{1}{4\pi} \frac{\mathbf{q} \cdot (\mathbf{p} - \mathbf{x})}{\|\mathbf{p} - \mathbf{x}\|^3}$$

## Metody řešení Poissonovy rovnice

$$\nabla(\sigma \nabla V) = f = \nabla J^p \quad \text{pro } V.$$

- ▶ Umíme řešit analyticky pro konstantní  $\sigma$  a okrajové podmínky  
 $0 \vee \infty$
- ▶ V ostatních případech řešíme numericky.

# Surface versus volume methods

Solve  $\nabla(\sigma \nabla V) = f = \nabla \mathbf{J}^p$  for  $V$ .

- ▶ Volume methods
  - ▶ unknowns ( $V(\mathbf{x})$ ) in the volume  $\mathbf{x} \in \Omega \subset \mathbb{R}^3$
  - ▶ differential equations
- ▶ Surface methods
  - ▶ conductivity piecewise constant
  - ▶ unknowns ( $V(\mathbf{x})$ ) on a set of surfaces  $\mathbf{x} \in S_i(\mathbb{R}^2)$
  - ▶ integral equations

## Finite difference method

$$\sigma \Delta V = \sigma \left( \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} \right) = f$$

- ▶ Regular grid:

$$v_{ij} = V(hi, hj)$$

- ▶ Finite difference approximation

for example  $\Delta V \approx \frac{1}{4h^2} \begin{bmatrix} 1 & & \\ & -4 & \\ 1 & & 1 \end{bmatrix} * v$

- ▶ Linear sparse system of equations:  $A\mathbf{v} = \mathbf{b}$
- ▶ Where  $\sigma = \text{const}$   $\rightarrow$  stationarity  $\rightarrow$  FFT solution

## └ Finite difference method

$$\left( \frac{\partial^2 V}{\partial x^2} \right) (h_i, h_j) \approx \frac{v_{i+1,j} - 2v_{i,j} + v_{i-1,j}}{h^2}$$

$$\sigma \left( \frac{v_{i+1,j} - 2v_{i,j} + v_{i-1,j}}{h^2} + \frac{v_{i,j+1} - 2v_{i,j} + v_{i,j-1}}{h^2} \right) = f_{ij}$$

Vector out of the matrix  $w_{i+jM} = v_{ij}$

$$\sigma \Delta V = \sigma \left( \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} \right) = f$$

- Regular grid:  $v_{ij} = V(h_i, h_j)$
- Finite difference approximation  
for example:  $\Delta V \approx \frac{1}{4h^2} \begin{vmatrix} 1 & -4 & 1 \\ -4 & 12 & -4 \\ 1 & -4 & 1 \end{vmatrix} * v$
- Linear sparse system of equations:  $A v = b$
- Where  $\sigma = \text{const}$  → stationary → FFT solution

## Řešení soustavy z metody konečných diferencí

- ▶ Přímé řešení (např. Gaussova eliminace, pro malý počet neznámých)
- ▶ Iterativní řešení (relaxace, konjugované gradienty, MINRES)
- ▶ Rychlé řešení (FFT) – jen pro konstantní koeficienty

## └ Řešení soustavy z metody konečných diferencí

- ▶ Přímé řešení (např. Gaušsova eliminace pro malý počet rovnání)
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### 1. Například Jacobiho iterace

$$v_{i,j}^n = \frac{1}{4} (v_{i+1,j}^n + v_{i-1,j}^n + v_{i,j-1}^n + v_{i,j+1}^n - f_{ij} h^2 \sigma)$$

Je potřeba ukázat, že konverguje.

$$2. \partial_t f \leftrightarrow j\omega \hat{f}$$

$$\Delta f \leftrightarrow -(\omega_x^2 + \omega_y^2) \hat{f}$$

$$\text{Lze proto řešit } -\sigma(\omega_x^2 + \omega_y^2) \hat{V} = \hat{f}$$

Nebo diskretizovanou rovnici

$$\sigma \left( \frac{v_{i+1,j} - 2v_{i,j} + v_{i-1,j}}{h^2} + \frac{v_{i,j+1} - 2v_{i,j} + v_{i,j-1}}{h^2} \right) = f_{ij}$$

# Finite element method

$$\nabla(\sigma \nabla V) = f$$

- ▶ Test functions  $\phi_j$

$$\langle \nabla(\sigma \nabla V), \phi_j \rangle = \langle f, \phi_j \rangle$$

- ▶ Discretize  $V = \sum_i v_i \phi_i$

$$\begin{aligned} \sum_i v_i \langle \nabla(\sigma \nabla \phi_i), \phi_j \rangle &= \langle f, \phi_j \rangle \\ - \sum_i v_i \sigma_i \langle \nabla \phi_i, \nabla \phi_j \rangle &= \langle f, \phi_j \rangle \end{aligned}$$

- ▶ Linear **symmetric** system  $\mathbf{Av} = \mathbf{b}$

$$\nabla(\sigma \nabla V) = f$$

► Test functions  $\phi_j$

$$\langle \nabla(\sigma \nabla V), \phi_j \rangle = \langle f, \phi_j \rangle$$

► Discretize  $V = \sum_i v_i \phi_i$

$$\sum_i v_i \langle \nabla(\sigma \nabla \phi_i), \phi_j \rangle = \langle f, \phi_j \rangle$$

$$-\sum_i v_i \sigma_i \langle \nabla \phi_i, \nabla \phi_j \rangle = \langle f, \phi_j \rangle$$

► Linear **symmetric** system  $Aw = b$

## └ Finite element method

Testovací funkce P0,P1 na trojúhelnících.

Gradient  $\nabla\phi$  je konstantní.

Ukázat integraci per partes:

$$\nabla\phi = \begin{bmatrix} \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{bmatrix}$$

$$\langle \Delta\phi, \psi \rangle = \int \left( \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} \right) \psi \, d\mathbf{z}$$

$$\int \frac{\partial^2\phi}{\partial x^2} \psi \underbrace{\int dx dy dz}_{dx} = \int \left[ \frac{\partial\phi}{\partial x} \psi \right]_{-\infty}^{\infty} dy dz - \int \frac{\partial\phi}{\partial x} \frac{\partial\psi}{\partial x} dx dy dz$$

$$\langle \nabla\phi, \nabla\psi \rangle = \begin{bmatrix} \int \frac{\partial\phi}{\partial x} \frac{\partial\psi}{\partial x} dx \\ \int \frac{\partial\phi}{\partial y} \frac{\partial\psi}{\partial y} dx \\ \int \frac{\partial\phi}{\partial z} \frac{\partial\psi}{\partial z} dx \end{bmatrix}^T$$

## Surface method / Boundary element method

- ▶ Mathematical basis
- ▶ Representation theorem
- ▶ Integral representations
  - ▶ Single layer formulation
  - ▶ Double layer formulation
  - ▶ Symmetric formulation

# The Green connection

Suppose constant  $\sigma$ :  $\boxed{\Delta u = f}$

- ▶ Green function (Nedelec's convention)

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{4\pi \|\mathbf{r} - \mathbf{r}'\|} \quad -\Delta_{\mathbf{r}} G(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}') = \delta_{\mathbf{r}'}$$

- ▶ Stokes theorem

$$\int_{\partial\Omega} \mathbf{g}(\mathbf{r}) \cdot d\mathbf{s}(\mathbf{r}) = \int_{\Omega} \nabla \cdot \mathbf{g}(\mathbf{r}) d\mathbf{r}$$

# The Green connection

- ▶ Stokes theorem

$$\int_{\partial\Omega} \mathbf{g}(\mathbf{r}) \cdot d\mathbf{s}(\mathbf{r}) = \int_{\Omega} \nabla \cdot \mathbf{g}(\mathbf{r}) d\mathbf{r}$$

- ▶ First Green identity

$$\mathbf{g} = u \nabla v \quad \rightarrow \quad \int_{\partial\Omega} u \nabla v \, d\mathbf{s}(\mathbf{r}) = \int_{\Omega} \nabla u \cdot \nabla v + u \Delta v \, d\mathbf{r}$$

# The Green connection

- ▶ First Green identity

$$\mathbf{g} = u \nabla v \quad \rightarrow \quad \int_{\partial\Omega} u \nabla v \, d\mathbf{s}(\mathbf{r}) = \int_{\Omega} \nabla u \nabla v + u \Delta v \, d\mathbf{r}$$

- ▶ Second Green identity

$$\int_{\partial\Omega} u \nabla v - v \nabla u \, d\mathbf{s}(\mathbf{r}) = \int_{\Omega} u \Delta v - v \Delta u \, d\mathbf{r}$$

$$\int_{\partial\Omega} u \partial_{\mathbf{n}'} v - v \partial_{\mathbf{n}'} u \, ds(\mathbf{r}) = \int_{\Omega} u \Delta v - v \Delta u \, d\mathbf{r}$$

# The Green connection

- ▶ Second Green identity

$$\int_{\partial\Omega} u \partial_{\mathbf{n}'} v - v \partial_{\mathbf{n}'} u \, ds(\mathbf{r}) = \int_{\Omega} u \Delta v - v \Delta u \, d\mathbf{r}$$

- ▶ Third Green identity

$$v = -G(\mathbf{r}, \mathbf{r}'), \quad \Delta u = 0 \quad \rightarrow$$

$$\nu u(\mathbf{r}) = \int_{\partial\Omega} G(\mathbf{r}, \mathbf{r}') \partial_{\mathbf{n}'} u(\mathbf{r}') - u \partial_{\mathbf{n}'} G(\mathbf{r}, \mathbf{r}') \, ds(\mathbf{r}')$$

$$\nu = \begin{cases} 1, & \mathbf{r} \in \Omega \\ 1/2, & \mathbf{r} \in \partial\Omega \\ 0, & \mathbf{r} \in \mathbb{R}^3 \setminus \overline{\Omega} \end{cases}$$

# Representation theorem

Third Green identity:

$$\int_{\partial\Omega} G(\mathbf{r}, \mathbf{r}') \partial_{\mathbf{n}'} u(\mathbf{r}') - u(\mathbf{r}') \partial_{\mathbf{n}'} G(\mathbf{r}, \mathbf{r}') d\mathbf{s}(\mathbf{r}') = \begin{cases} u(\mathbf{r}), & \mathbf{r} \in \Omega \\ u(\mathbf{r})/2, & \mathbf{r} \in \partial\Omega \\ 0, & \mathbf{r} \in \mathbb{R}^3 \setminus \overline{\Omega} \end{cases}$$

$$\Delta u^{\text{int}} = 0 \quad \text{in } \Omega, \quad \Delta u^{\text{ext}} = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{\Omega}, \quad u^{\text{ext}} \xrightarrow{\|\mathbf{r}\| \rightarrow \infty} O(\|\mathbf{r}\|^{-1})$$

$$\int_{\partial\Omega} G(\mathbf{r}, \mathbf{r}') [\partial_{\mathbf{n}'} u](\mathbf{r}') - \partial_{\mathbf{n}'} G(\mathbf{r}, \mathbf{r}') [u](\mathbf{r}') d\mathbf{s}(\mathbf{r}') = \begin{cases} u^{\text{int}}(\mathbf{r}), & \mathbf{r} \in \Omega \\ u^{\text{ext}}(\mathbf{r}), & \mathbf{r} \in \mathbb{R}^3 \setminus \overline{\Omega} \\ \frac{u^{\text{int}} + u^{\text{ext}}}{2}(\mathbf{r}), & \mathbf{r} \in \partial\Omega \end{cases}$$

where  $[u] = u^{\text{int}}(\mathbf{r}') - u^{\text{ext}}(\mathbf{r}')$ ,  $[\partial_{\mathbf{n}'} u] = \partial_{\mathbf{n}'}^- u^{\text{int}}(\mathbf{r}') - \partial_{\mathbf{n}'}^+ u^{\text{ext}}(\mathbf{r}')$

## Extended representation theorem

- Regular case,  $\mathbf{r} \in \mathbb{R}^3 \setminus \partial\Omega$ .

$$\mathbf{p} = \sigma \nabla V, \quad \nabla \cdot \mathbf{p} = 0, \quad \text{in } \mathbb{R}^3 \setminus \partial\Omega$$

$$-p(\mathbf{r}) = \int_{\partial\Omega} \sigma \partial_{\mathbf{n}, \mathbf{n}'} G(\mathbf{r}, \mathbf{r}') [V](\mathbf{r}') - \partial_{\mathbf{n}} G(\mathbf{r}, \mathbf{r}') [p](\mathbf{r}') ds(\mathbf{r}')$$

$$V(\mathbf{r}) = \int_{\partial\Omega} -\partial_{\mathbf{n}'} G(\mathbf{r}, \mathbf{r}') [V](\mathbf{r}') + \sigma^{-1} G(\mathbf{r}, \mathbf{r}') [p](\mathbf{r}') ds(\mathbf{r}')$$

where  $p = \mathbf{p} \cdot \mathbf{n}$ ,  $[V] = V^- - V^+$ ,  $[p] = p^- - p^+$

## Extended representation theorem

- ▶ Limit case,  $\mathbf{r} \in \partial\Omega$ .

$$\mathbf{p} = \sigma \nabla V, \quad \nabla \cdot \mathbf{p} = 0, \quad \text{in } \mathbb{R}^3 \setminus \partial\Omega$$

$$-p^\pm(\mathbf{r}) = \pm \frac{[p]}{2} + \int_{\partial\Omega} \sigma \partial_{\mathbf{n}, \mathbf{n}'} G(\mathbf{r}, \mathbf{r}') [V] - \partial_{\mathbf{n}} G(\mathbf{r}, \mathbf{r}') [p] \, ds(\mathbf{r}')$$

$$V^\pm(\mathbf{r}) = \mp \frac{[V]}{2} + \int_{\partial\Omega} -\partial_{\mathbf{n}'} G(\mathbf{r}, \mathbf{r}') [V] + \sigma^{-1} G(\mathbf{r}, \mathbf{r}') [p] \, ds(\mathbf{r}')$$

where  $p = \mathbf{p} \cdot \mathbf{n}$ ,  $[V] = V^- - V^+$ ,  $[p] = p^- - p^+$

## Extended representation theorem

- Operator form,  $\mathbf{r} \in \partial\Omega$ .

$$-\rho^\pm(\mathbf{r}) = \pm \frac{[p]}{2} + \int_{\partial\Omega} \sigma \partial_{\mathbf{n}, \mathbf{n}'} G(\mathbf{r}, \mathbf{r}') [V] - \partial_{\mathbf{n}} G(\mathbf{r}, \mathbf{r}') [p] \, ds(\mathbf{r}')$$

$$V^\pm(\mathbf{r}) = \mp \frac{[V]}{2} + \int_{\partial\Omega} -\partial_{\mathbf{n}'} G(\mathbf{r}, \mathbf{r}') [V] + \sigma^{-1} G(\mathbf{r}, \mathbf{r}') [p] \, ds(\mathbf{r}')$$

$$\begin{aligned}-\rho^\pm(\mathbf{r}) &= \sigma \mathcal{N}[V] + (\pm \frac{\mathcal{J}}{2} - \mathcal{D}^*)[p] \\ V^\pm(\mathbf{r}) &= (\mp \frac{\mathcal{J}}{2} - \mathcal{D})[V] + \sigma^{-1} \mathcal{S}[p]\end{aligned}$$

where  $p = \sigma \partial_{\mathbf{n}} V$

## EEG a MEG část 2, matematické metody

└ Extended representation theorem
► Operator form,  $\mathbf{r} \in \partial\Omega$ .

$$-\rho^\pm(\mathbf{r}) = \pm \frac{[\rho]}{2} + \int_{\partial\Omega} \sigma \partial_{\mathbf{n}, \mathbf{r}'} G(\mathbf{r}, \mathbf{r}') [V] - \partial_\mathbf{n} G(\mathbf{r}, \mathbf{r}') [\rho] d\sigma(\mathbf{r}')$$

$$V^\pm(\mathbf{r}) = \mp \frac{[V]}{2} + \int_{\partial\Omega} \partial_{\mathbf{n}'} G(\mathbf{r}, \mathbf{r}') [V] + \sigma^{-1} G(\mathbf{r}, \mathbf{r}) [\rho] d\sigma(\mathbf{r}')$$

$$\boxed{-\rho^\pm(\mathbf{r}) = \sigma N[V] + (\pm \frac{\rho}{2} - D^\pm)[\rho]$$

$$V^\pm(\mathbf{r}) = (\mp \frac{V}{2} - D)[V] + \sigma^{-1} S[\rho]}$$

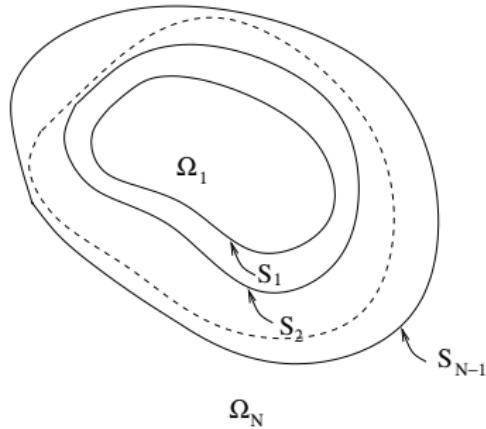
where  $\rho = \sigma \partial_{\mathbf{n}} V$ 

Vysvětlit double layer, single layer potenciál. Napsat vzorec pro  $V, V^\pm$ .

# BEM problem

Solve:

$$\Delta V = f \quad \text{in } \cup \Omega_i$$
$$[\sigma \partial_{\mathbf{n}} V] = [V] = 0 \quad \text{on } S_i$$



$$f = \sum_{i=1}^N f_{\Omega_i}$$

$$v_{\Omega_i}(\mathbf{r}) = -f_{\Omega_i} * G(\mathbf{r})$$

## BEM — Single layer formulation

$$v_s = \sum_{i=1}^N v_{\Omega_i} / \sigma_i \quad \text{verifies} \quad \sigma \Delta v_s = f$$

Consider:  $u_s = V - v_s \rightarrow [u_s]_j = 0$

$$\rightarrow u_s = \sum_{i=1}^N \delta_{ji} \xi_{S_i} \quad \xi_{S_i} = [p]_i$$

From:  $[\sigma \partial_n V] = 0 \rightarrow [\sigma \partial_n u_s] = -[\sigma \partial_n v_s]$

From the representation theorem:

$$\partial_n v_s = \frac{\sigma_j + \sigma_{j+1}}{2(\sigma_{j+1} - \sigma_j)} \xi_{S_j} - \sum_{i=1}^N \mathcal{D}_{ji}^* \xi_{S_i}$$

## └ BEM — Single layer formulation

Consider:

$$\begin{aligned} v_k &= \sum_{i=1}^N v_{ki} / \sigma_i \quad \text{verifies } \sigma \Delta v_k = f \\ u_k &= V - v_k \quad [u_k]_j = 0 \\ u_k &= \sum_{i=1}^N \delta_{ji} \xi_i \quad \xi_i = [\rho]_i \\ \text{From:} \quad [\sigma \partial_h V] &= 0 \quad [\sigma \partial_h u_k] = -[\sigma \partial_h v_k] \end{aligned}$$

From the representation theorem:

$$\partial_h v_k = \frac{\sigma_j + \sigma_{j+1}}{2(\sigma_{j+1} - \sigma_j)} \xi_j - \sum_{l=1}^N D_{jl}^k \xi_l$$

Napsat reprezentační teorém. Napsat hraniční podmínky.

## BEM — Double layer formulation

$$v_d = \sum_{i=1}^N v_{\Omega_i} \quad \text{verifies} \quad \Delta v_d = f$$

Consider:  $u_d = \sigma V - v_d \rightarrow [\partial_n u_d]_j = 0$

$$\rightarrow u_d = \sum_{i=1}^N \mathcal{D}_{ji} \mu_{S_i}$$

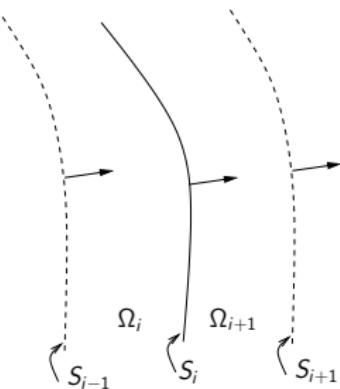
$$\mu_{S_i} = -[u_d]_i = (\sigma_{i+1} - \sigma_i) V_{S_i}$$

From:  $[V] = 0 \rightarrow \sigma_{j+1}(u_d + v_d)^- = \sigma_j(u_d + v_d)^+$

From the representation theorem:

$$v_d = \frac{\sigma_j + \sigma_{j+1}}{2} V_{S_j} - \sum_{i=1}^N (\sigma_{i+1} - \sigma_i) \mathcal{D}_{ji} V_{S_i}$$

## BEM – Symmetric formulation



Consider:

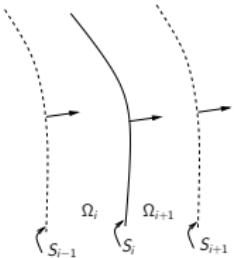
$$u_{\Omega_i} = \begin{cases} V - v_{\Omega_i}/\sigma_i & \text{in } \Omega_i \\ -v_{\Omega_i}/\sigma_i & \text{in } \mathbb{R}^3 \setminus \overline{\Omega}_i \end{cases}$$

$$[u_{\Omega_i}]_i = V_{S_i}, \quad [u_{\Omega_i}]_{i-1} = -V_{S_{i-1}}$$

Define:

$$p_{S_i} = \sigma_i [\partial_n u_{\Omega_i}]_i = \sigma_i (\partial_n V)_{S_i}^- = \sigma_{i+1} (\partial_n V)_{S_i}^+$$

# BEM – Symmetric formulation



From the extended representation theorem:

$$\begin{aligned}(u_{\Omega_i})_{S_i}^- &= (V - v_{\Omega_i}/\sigma_i)_{S_i}^- \\&= \frac{V_{S_i}}{2} + \mathcal{D}_{i,i-1} V_{S_{i-1}} - \mathcal{D}_{ii} V_{S_i} - \sigma_i^{-1} \mathcal{S}_{i,i-1} p_{S_{i-1}} + \sigma_i^{-1} \mathcal{S}_{ii} p_{S_i} \\(u_{\Omega_{i+1}})_{S_i}^+ &= (V - v_{\Omega_{i+1}}/\sigma_{i+1})_{S_i}^+ \\&= \frac{V_{S_i}}{2} + \mathcal{D}_{ii} V_{S_i} - \mathcal{D}_{i,i+1} V_{S_{i+1}} - \sigma_{i+1}^{-1} \mathcal{S}_{ii} p_{S_i} + \sigma_{i+1}^{-1} \mathcal{S}_{i,i+1} p_{S_{i+1}}\end{aligned}$$

## BEM – Symmetric formulation

Subtract:

$$\begin{aligned}\sigma_{i+1}^{-1}(v_{\Omega_{i+1}})_{S_i} - \sigma_i^{-1}(v_{\Omega_i})_{S_i} = \\ \mathcal{D}_{i,i-1} V_{S_{i-1}} - 2\mathcal{D}_{ii} V_{S_i} + \mathcal{D}_{i,i+1} V_{S_{i+1}} \\ - \sigma_i^{-1} \mathcal{S}_{i,i-1} p_{S_{i-1}} + (\sigma_i^{-1} + \sigma_{i+1}^{-1}) \mathcal{S}_{ii} p_{S_i} - \sigma_{i+1}^{-1} \mathcal{S}_{i,i+1} p_{S_{i+1}}\end{aligned}$$

and also:

$$\begin{aligned}(\partial_n v_{\Omega_{i+1}})_{S_i} - (\partial_n v_{\Omega_i})_{S_i} = \\ \sigma_i \mathcal{N}_{i,i-1} V_{S_{i-1}} - (\sigma_i + \sigma_{i+1}) \mathcal{N}_{ii} V_{S_i} + \sigma_{i+1} \mathcal{N}_{i,i+1} V_{S_{i+1}} \\ - \mathcal{D}_{i,i-1}^* p_{S_{i-1}} + 2\mathcal{D}_{ii}^* p_{S_i} - \mathcal{D}_{i,i+1}^* p_{S_{i+1}}\end{aligned}$$

# BEM – Symmetric formulation

$$\begin{bmatrix}
 (\sigma_1 + \sigma_2)N_{11} & -2\mathcal{D}_{11}^* & -\sigma_2 N_{12} & \mathcal{D}_{12}^* \\
 -2\mathcal{D}_{11} & (\sigma_1^{-1} + \sigma_2^{-1})\mathcal{S}_{11} & \mathcal{D}_{12} & -\sigma_2^{-1}\mathcal{S}_{12} \\
 -\sigma_2 N_{21} & \mathcal{D}_{21}^* & (\sigma_2 + \sigma_3)N_{22} & -2\mathcal{D}_{22}^* \\
 \mathcal{D}_{21} & -\sigma_2^{-1}\mathcal{S}_{21} & -2\mathcal{D}_{22} & (\sigma_2^{-1} + \sigma_3^{-1})\mathcal{S}_{22} \\
 & & -\sigma_3 N_{32} & \mathcal{D}_{32}^* \\
 & & \mathcal{D}_{32} & (\sigma_3 + \sigma_4)N_{33} \\
 & & \mathcal{D}_{32} & -2\mathcal{D}_{33} \\
 & & \mathcal{D}_{32} & (\sigma_3^{-1} + \sigma_4^{-1})\mathcal{S}_{33} \\
 & & & \vdots \\
 & & & \vdots \\
 & & & (\sigma_{N-1} + \sigma_N)N_{N-1,N-1} & -2\mathcal{D}_{N-1,N-1}^* & -\sigma_N N_{N-1,N} \\
 & & & -2\mathcal{D}_{N-1,N-1} & (\sigma_{N-1}^{-1} + \sigma_N^{-1})\mathcal{S}_{N-1,N-1} & \mathcal{D}_{N-1,N} \\
 & & & -\sigma_N N_{N,N-1} & \mathcal{D}_{N,N-1} & \sigma_N N_{N,N}
 \end{bmatrix}$$

$$\begin{bmatrix}
 V_{S_1} \\
 p_{S_1} \\
 V_{S_2} \\
 p_{S_2} \\
 V_{S_3} \\
 p_{S_3} \\
 \vdots \\
 V_{S_N}
 \end{bmatrix} = \begin{bmatrix}
 (\partial_n v_{\Omega_1})_{S_1} - (\partial_n v_{\Omega_2})_{S_1} \\
 \sigma_2^{-1}(v_{\Omega_2})_{S_1} - \sigma_1^{-1}(v_{\Omega_1})_{S_1} \\
 (\partial_n v_{\Omega_2})_{S_2} - (\partial_n v_{\Omega_3})_{S_2} \\
 \sigma_3^{-1}(v_{\Omega_3})_{S_2} - \sigma_2^{-1}(v_{\Omega_2})_{S_2} \\
 (\partial_n v_{\Omega_3})_{S_3} - (\partial_n v_{\Omega_4})_{S_3} \\
 \sigma_4^{-1}(v_{\Omega_4})_{S_3} - \sigma_3^{-1}(v_{\Omega_3})_{S_3} \\
 \vdots \\
 (\partial_n v_{\Omega_N})_{S_N}
 \end{bmatrix}$$

# Discretization in BEM

- Discretize unknowns ( $\varphi_i = \text{P0,P1}$ )

$$\xi(\mathbf{r}) = \sum_i \xi_i \varphi_i(\mathbf{r})$$

- Test functions ( $\psi_j = \delta, \text{P0}, \text{P1}$ )



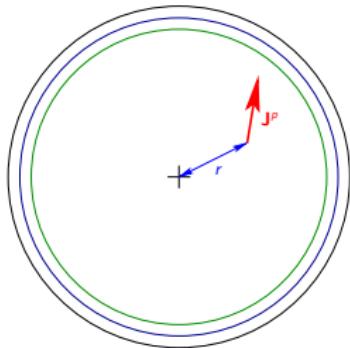
$$\left\langle \frac{\xi}{2}(\mathbf{r}) - \int_{\partial\Omega} \xi(\mathbf{r}') \partial_{\mathbf{n}} G(\mathbf{r}, \mathbf{r}') \, ds(\mathbf{r}'), \psi_j \right\rangle = \langle \partial_{\mathbf{n}} V_0, \psi_j \rangle$$

$$\sum_i \xi_i \left( \tfrac{1}{2} \langle \phi_i, \psi_j \rangle - \int_{\partial\Omega \times \partial\Omega} \varphi_i(\mathbf{r}') \partial_{\mathbf{n}} G(\mathbf{r}, \mathbf{r}') \psi_j(\mathbf{r}) \, ds^2(\mathbf{r}', \mathbf{r}') \right) = \langle \partial_{\mathbf{n}} V_0, \psi_j \rangle$$

- Linear system of equations:  $\mathbf{A}\boldsymbol{\xi} = \mathbf{b}$

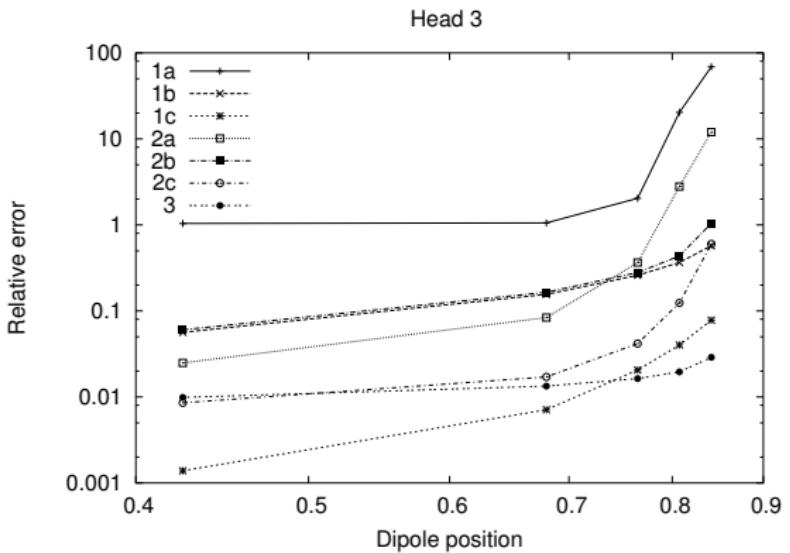
## BEM accuracy

- ▶ Analytical solution



- ▶ Relative error  $\|V - V_{\text{anal}}\|_{\ell_2} / \|V_{\text{anal}}\|_{\ell_2}$
- ▶ Dipoles at  $0.50R, 0.80R, 0.90R, 0.95R, 0.98R$
- ▶ Spherical head phantoms with  
 $N_V = 3 \times 42, 3 \times 162, 3 \times 642$

# BEM accuracy



# Fast Multipole Method – Motivation

$$\Gamma \mathbf{x} = \mathbf{y}$$

- ▶ Solution methods
  - ▶ **Direct**, e.g. LU decomposition.  
Complexity  $O(P^3)$ , memory  $O(P^2)$
  - ▶ **Iterative**, e.g. GMRES, uses products  $\Gamma \mathbf{v}$ .  
Complexity  $O(MP^2)$ , memory  $O(P)$

Number of elements  $P$ , number of iterations  $M$ .

- ▶ Fast Multipole Method
  - ▶ Calculate  $\mathbf{y} = \Gamma \mathbf{v}$  in  $O(P \log P)$  time.
  - ▶ Approximative, hierarchical

# Multipole expansion

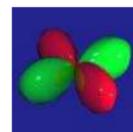
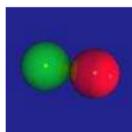
Typical element:

$$\Gamma_{i,j} = \iint_{\substack{\mathbf{r} \in \text{supp } \psi_i \\ \mathbf{r}' \in \text{supp } \varphi_j}} \nabla' \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} \cdot \mathbf{n}_j \varphi_j(\mathbf{r}') \psi_i(\mathbf{r}) d\mathbf{s}^2(\mathbf{r}', \mathbf{r})$$

$$\boxed{\nabla' \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} = - \sum_{\substack{n=0 \dots \infty \\ m=-n \dots n}} \nabla' I_n^{-m}(\mathbf{M}_p - \mathbf{r}') O_n^m(\mathbf{r} - \mathbf{M}_p)}$$

Spherical harmonics

$$I_n^{-m}, O_n^m:$$



## Multipole expansion

Typical element:

$$\Gamma_{i,j} = \iint_{\substack{\mathbb{R}^3 \text{ except } \mathbf{r}_j \\ \mathbf{r}' \text{ close to } \mathbf{r}_j}} \nabla' \frac{1}{|\mathbf{r} - \mathbf{r}'|} \cdot \mathbf{n}_j \varphi_j(\mathbf{r}') \psi_i(\mathbf{r}) d\omega^2(\mathbf{r}', \mathbf{r})$$

$$\nabla' \frac{1}{|\mathbf{r} - \mathbf{r}'|} = - \sum_{\substack{0 \leq m \leq n \\ |m| \leq n \\ 0 \leq l \leq n-m}} \nabla' I_n^{-m}(\mathbf{M}_p - \mathbf{r}') O_n^{lm}(\mathbf{r} - \mathbf{M}_p)$$

Spherical harmonics



Greengard, Rokhlin, 1987  
1D příklad

$$\frac{1}{r - r'} \approx \sum_{n=0}^L \frac{(M - r)^n}{(M - r')^{n+1}}$$

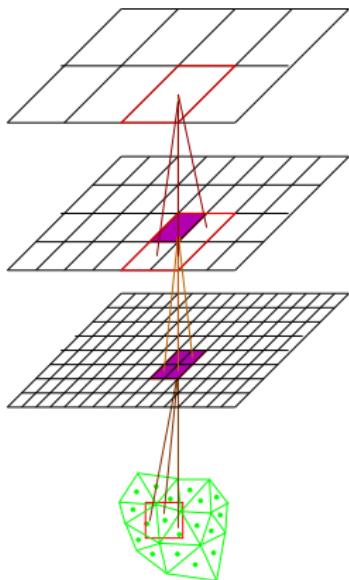
$$\frac{1}{\|\mathbf{r} - \mathbf{r}'\|} = - \sum_{\substack{n=0 \dots \infty \\ m=-n \dots n}} I_n^{-m}(-\mathbf{r}') O_n^m(\mathbf{r})$$

$$\frac{1}{\|\mathbf{r} - \mathbf{r}'\|} = - \sum_{\substack{n=0 \dots \infty \\ m=-n \dots n}} I_n^{-m}(\mathbf{M}_p - \mathbf{r}') O_n^m(\mathbf{r} - \mathbf{M}_p)$$

Aproximace funguje jen pro dostatečně vzdálené elementy.

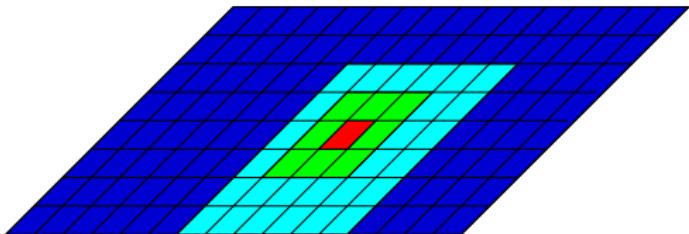
# FMM – Algorithm

Build an oct-tree:



# FMM – Algorithm

Levels involved:



Level 2

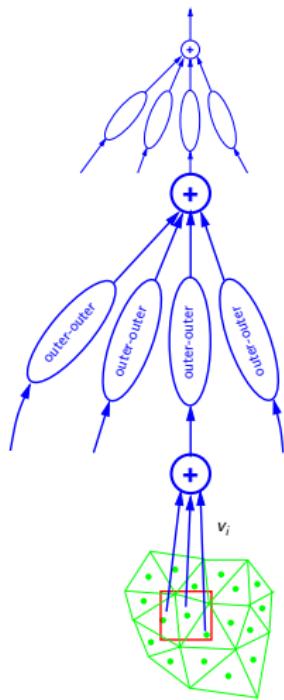
Level 1 – suburb

Treated locally

Cell considered

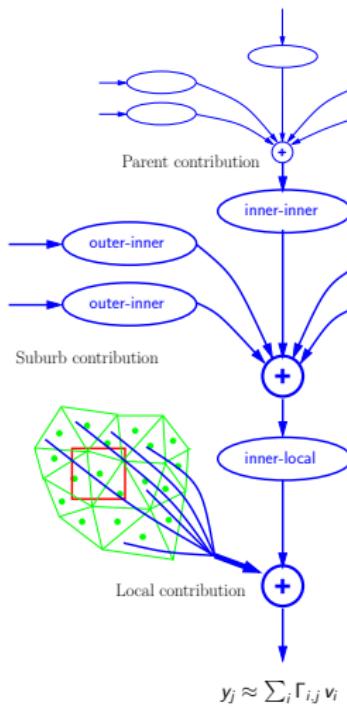
# FMM – Algorithm

Sweep-up → outer fields

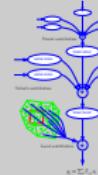


# FMM – Algorithm

Sweep-down → result



## └ FMM – Algorithm



# The full story (FMM)

Two sets of coefficients:

$${}^i \mathbf{a}_n^m(\mathbf{M}) = \langle I_n^m(\mathbf{M} - \mathbf{r}), \psi_i(\mathbf{r}) \rangle_{\mathbf{r}}$$

$${}^{j'} \mathbf{b}_n^m(\mathbf{M}) = \langle \nabla I_n^m(\mathbf{M} - \mathbf{r}) \cdot \mathbf{n}_{j'}, \varphi'_{j'}(\mathbf{r}) \rangle_{\mathbf{r}}$$

To approximate the four operators:

$$4\pi \langle \mathcal{S}\psi_i, \psi_j \rangle = \sum_{n,m} {}^i \mathbf{a}^{-m,n}(\mathbf{M}) {}^i \tilde{\mathbf{a}}^{m,n}(\mathbf{M})$$

$$4\pi \langle \mathcal{D}\varphi'_{i'}, \psi_j \rangle = \sum_{n,m} {}^{i'} \mathbf{b}^{-m,n}(\mathbf{M}) {}^j \tilde{\mathbf{a}}^{m,n}(\mathbf{M}) =$$

$$= 4\pi \langle \mathcal{D}^*\psi_j, \varphi'_{i'} \rangle = \sum_{n,m} {}^j \mathbf{a}^{-m,n}(\mathbf{M}) {}^i \tilde{\mathbf{b}}^{m,n}(\mathbf{M})$$

$$4\pi \langle \mathcal{N}\varphi'_{i'}, \varphi'_{j'} \rangle = \sum_{n,m} {}^{i'} \mathbf{b}^{-m,n}(\mathbf{M}) {}^{j'} \tilde{\mathbf{b}}^{m,n}(\mathbf{M})$$

# The full story (FMM)

Three translation operators:

$$x_{n'}^{-m'}(\mathbf{M}) = (R_{\mathbf{NM}} x)_{n'}^{-m'} = \sum_{n=0 \dots n', m=-n \dots n} I_{n'-n}^{m-m'}(\mathbf{N} - \mathbf{M}) x_n^{-m}(\mathbf{N})$$

$$\tilde{x}_{n'}^{-m'}(\mathbf{M}) = (S_{\mathbf{NM}} \tilde{x})_{n'}^{-m'} = \sum_{n=n' \dots L, m=-n \dots n} I_{n-n'}^{m'-m'}(\mathbf{N} - \mathbf{M}) \tilde{x}_n^{-m}(\mathbf{N})$$

$$\widetilde{x}_{n'}^{m'}(\mathbf{M}) = (T_{\mathbf{NM}} x)_{n'}^{m'} = \sum_{n=0 \dots L, m=-n \dots n} O_{n+n'}^{m+m'}(\mathbf{N} - \mathbf{M}) x_n^{-m}(\mathbf{N})$$

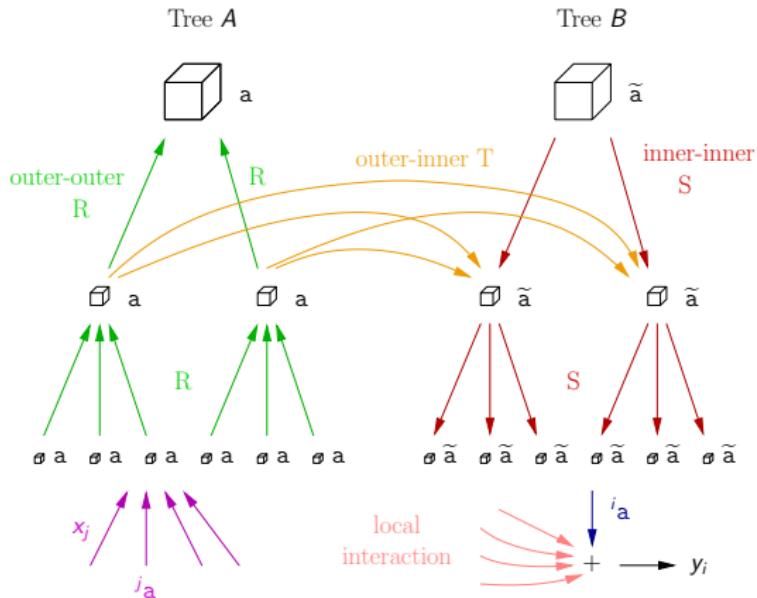
Error estimate (preliminary)

$$C \left( \|\mathbf{M} - \mathbf{r}'\| / \|\mathbf{M} - \mathbf{r}\| \right)^{L+1}$$

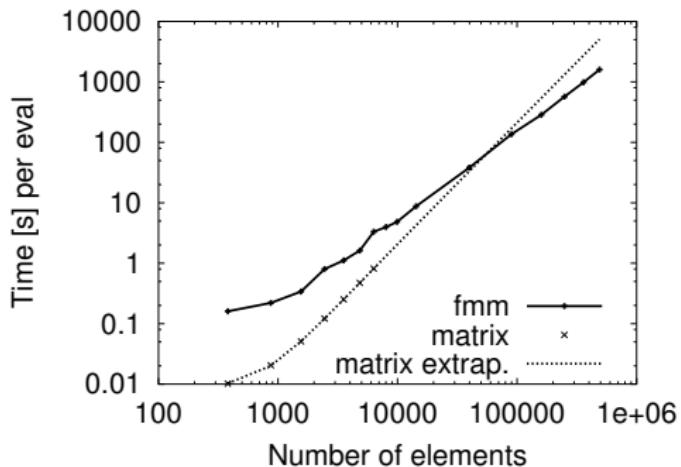
$$C \left( \max \{ \|\mathbf{M} - \mathbf{r}'\|, \|\mathbf{N} - \mathbf{r}\| \} / \|\mathbf{M} - \mathbf{N}\| \right)^{L+1}$$

# FMM tree structure

Calculate all  $y_i = \sum_j x_j \langle S\psi_i, \psi_j \rangle$  for  $\psi_j \in A$ ,  $\psi_i \in B$ .



# FMM – Speed-Up



Single-level FMM,  $O(P^{4/3})$ , faster for  $P \gtrapprox 70000$  triangles.

## Inverzní problém

Najděte zdroje  $\mathbf{u}$

$$\mathbf{A}\mathbf{x} = \mathbf{B}\mathbf{u}$$

Měření napětí a magnetického pole

$$\mathbf{M}\mathbf{x} = \mathbf{m}$$

Regularizace pro zdroje  $\mathbf{u}$

$$\|\mathbf{D}\mathbf{u}\| = \min$$

## └ Inverzní problém

Inverzní problém

Najděte zdroje  $\mathbf{u}$ 

$$\mathbf{Ax} = \mathbf{Bu}$$

Měření napětí a magnetického pole

$$\mathbf{Mx} = \mathbf{m}$$

Regularizace pro zdroje  $\mathbf{u}$ 

$$\|\mathbf{D}\mathbf{u}\| = \min$$

## Lagrangian

$$\mathbf{u}^T \mathbf{D}^T \mathbf{D} \mathbf{u} + \lambda^T (\mathbf{A} \mathbf{x} - \mathbf{B} \mathbf{u}) + \mu^T (\mathbf{M} \mathbf{x} - \mathbf{m})$$

Derivujeme podle  $u, x, \lambda, \mu$ , vznikne symetrická matic.

## MEG/EEG, Conclusions

- ▶ MEG, EEG, brain function analysis
- ▶ Excellent time-resolution, bad spatial resolution
- ▶ Standard diagnostic use
- ▶ Combination with other methods (fMRI, PET) desirable
- ▶ Localization is a hard inverse problem
- ▶ Solution methods — FDM, FEM, BEM methods
- ▶ BEM formulations, single/double/symmetric
- ▶ Implementation, discretization
- ▶ Fast Multipole Method for acceleration