Binary Mathematical Morphology

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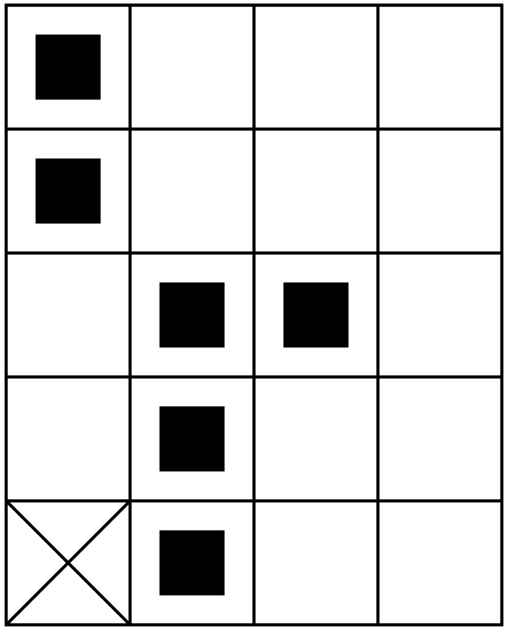
Point set

- Images can be modeled by point sets of arbitrary dimensions.
- 2D Euclidean space $\mathbb{E}^2$ with its subsets is a natural domain for description of planar figures.
- Digital counterpart of Euclidean space
- set of integer pairs ($\in \mathbb{Z}^2$) for binary morphology.

Point set — example



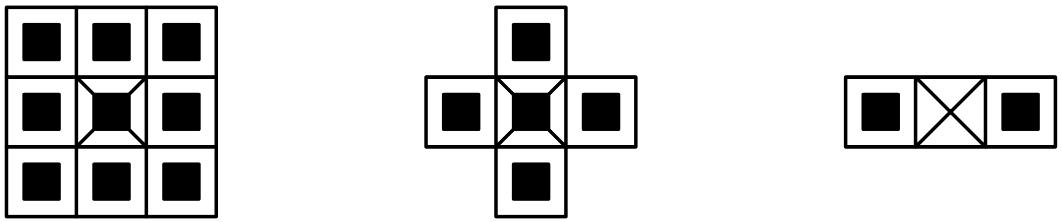
$$X = \{(1,0),(1,1),(1,2),(2,2),(0,3),(0,4)\}$$

Set operators

- inclusion: $X \subseteq Y$, X is subset of Y, and $Y \supseteq X$, Y is superset of X.
- intersection $X \cap Y$
- union $X \cup Y$
- empty set $\varnothing$
- complement X^c
- set difference $X \setminus Y = X \cap Y^c$

Morphological transformation Ψ

- Ψ is given by the relation of the image (point set X) with another small point set B called a structuring element.
- B is expressed with respect to a local origin



(a) (b) (c)

- Application of the morphological transformation $\Psi(X)$ to the image X means that the structuring element B is moved systematically across the entire image.
- The result of the relation (0 or 1) is stored in the output image in the current image pixel position.

Duality of $\Psi(X)$ with respect to X^c

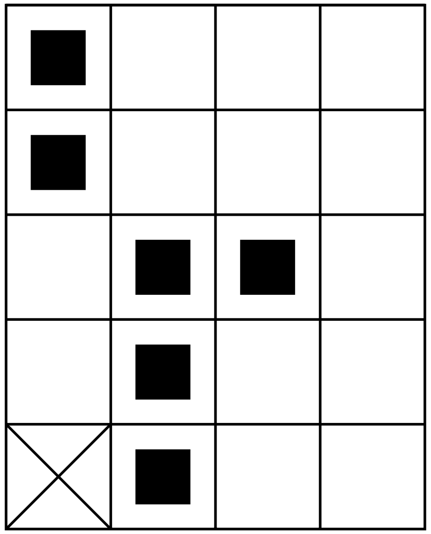
To each morphological transformation $\Psi(X)$ there exists a dual transformation $\Psi^*(X)$,

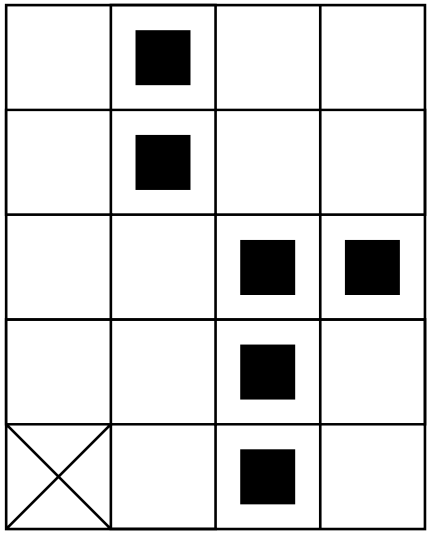
$$\Psi(X) = [\Psi^*(X^c)]^c$$

Translation

Translation X_h of point set X by a vector h

$$X_h = \{p \in \mathbb{E}^2,\ p = x + h \ \text{ for some } x \in X\}$$





Symmetrical point set

- with respect to a representative point $\mathcal{O}$.
- sometimes called the transpose or rational set
- Definition: $\breve{B} = \{-b : \; b \in B\}$.
- Example: $B = \{(1,2),(2,3)\}$, $\quad \breve{B} = \{(-1,-2)(-2,-3)\}$.

Minkowski set addition, subtraction

Minkowski set addition (Hermann Minkowski 1864-1909)

$$X \oplus B = \bigcup_{b \in B} X_b$$

Minkowski set subtraction (introduced by H. Hadwiger 1957)

$$X \ominus B = \bigcap_{b \in B} X_{-b}$$

Dilation $\oplus$

Sums two point sets.

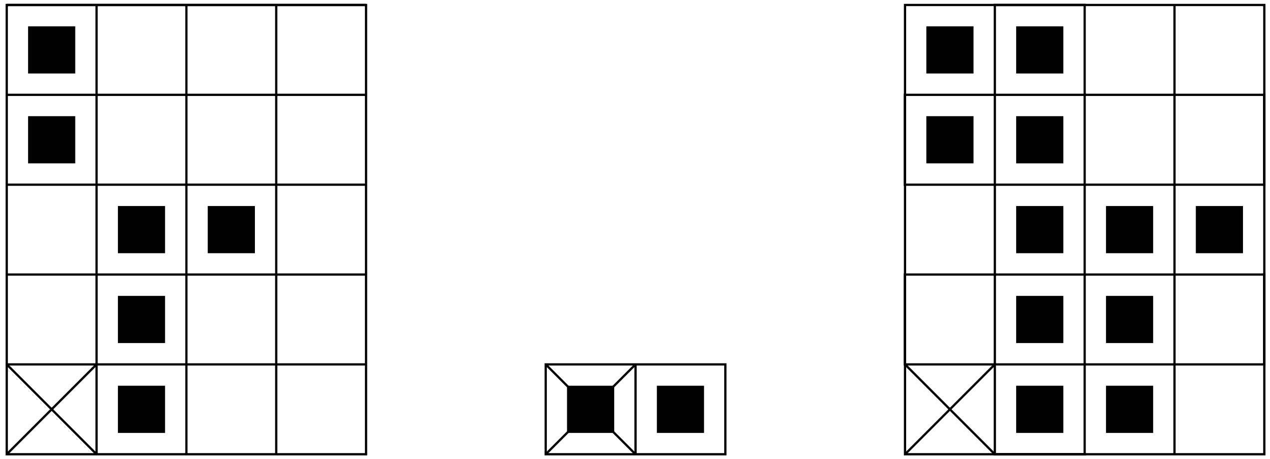
$$X \oplus B = \{p \in \mathbb{E}^2 \colon p = x + b,\ \ x \in X \text{ and } b \in B\}$$

Dilation can be expressed as a union of translated point sets.

$$X \oplus B = \bigcup_{b \in B} X_b \,.$$

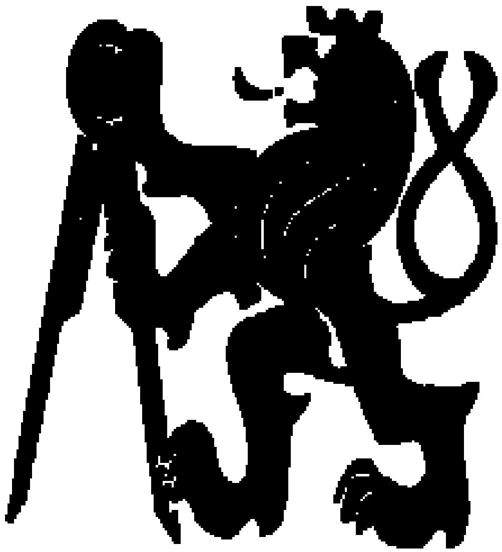
Dilation — example

$$\begin{aligned} X &= \{(1,0),(1,1),(1,2),(2,2),(0,3),(0,4)\} \\ B &= \{(0,0),(1,0)\} \\ X \oplus B &= \{(1,0),(1,1),(1,2),(2,2),(0,3),(0,4), \\ &\quad (2,0),(2,1),(2,2),(3,2),(1,3),(1,4)\} \end{aligned}$$



Dilation by isotropic structural element 3×3

original



dilated

Dilation fills small holes and narrow gulfs in objects. It increases the object size. If we need to preserve the size the dilation is followed by erosion.

Properties of the dilation

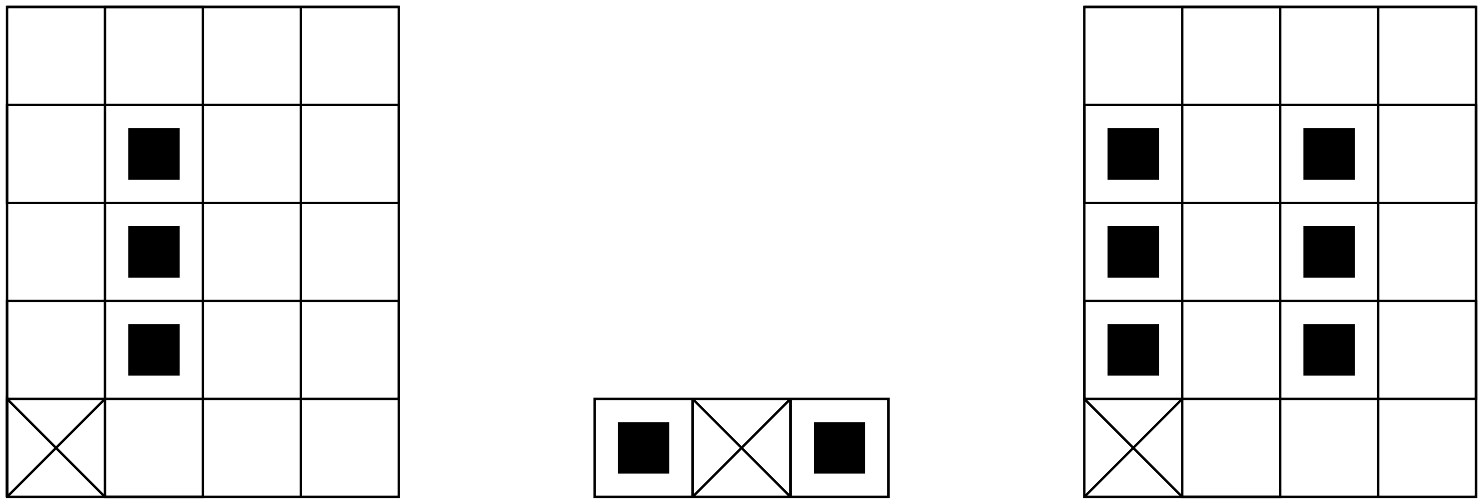
Commutative: $X \oplus B = B \oplus X$.

Associative: $X \oplus (B \oplus D) = (X \oplus B) \oplus D$.

Invariant to translation: $X_h \oplus B = (X \oplus B)_h$.

Increasing transform: if $X \subseteq Y$ and B has non-empty representative point, then $X \oplus B \subseteq Y \oplus B$.

What happens with empty representative point?



Erosion $\ominus$

Combines two point sets by Minkowski subtraction. It is a dual operator of dilation.

$$X \ominus B = \{p \in \mathbb{E}^2 : \; p + b \in X \text{for each} b \in B\} \, .$$

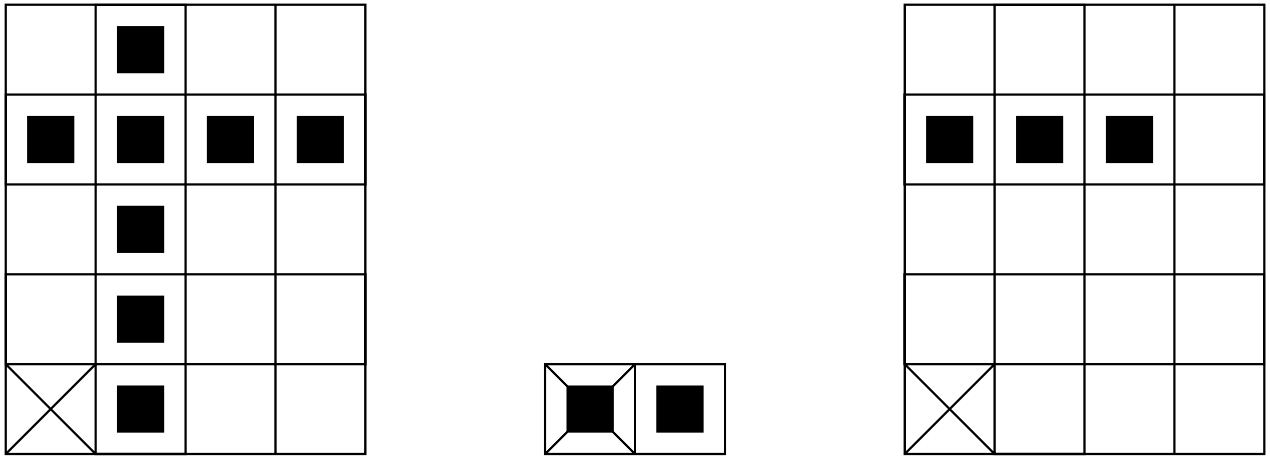
every point p from the image is tested; the result of the erosion is given by those points p for which all possible $p + b$ are in X.

Erosion can be expressed as an intersection of all translations of the image X by the vectors $-b \in B$

$$X \ominus B = \bigcap_{b \in B} X_{-b} \, .$$

Erosion — example

$$
\begin{aligned}
X &= \{(1,0),(1,1),(1,2),(0,3),(1,3),\\
&\quad (2,3),(3,3),(1,4)\}\\
B &= \{(0,0),(1,0)\}\\
X \ominus B &= \{(0,3),(1,3),(2,3)\}
\end{aligned}
$$



Erosion by isotropic structural element 3×3

original

eroded

Single-pixel-wide lines disappear. Erosion with an isotropic structuring element is sometimes called shrink or reduce.

Erosion is used to simplify the structure of an object. It decomposes complicated object into several simpler ones.

Object contour by erosion

Contour ∂X (region border X, thickness 1).

$$\partial X = X - (X \ominus B)\,.$$

original X

contour ∂X

Properties of erosion

Antiextensive: If $(0,0) \in B$, then $X \ominus B \subseteq X$.

Translation invariant: $X_h \ominus B = (X \ominus B)_h$, $X \ominus B_h = (X \ominus B)_{-h}$.

Increasing transform: If $X \subseteq Y$, then $X \ominus B \subseteq Y \ominus B$.

Duality between erosion and dilation: $(X \ominus Y)^C = X^C \oplus \breve{Y}$.

Erosion is not commutative: $X \ominus B \neq B \ominus X$

Combination of erosion and intersection:

$$(X \cap Y) \ominus B = (X \ominus B) \cap (Y \ominus B),$$
$$B \ominus (X \cap Y) \supseteq (B \ominus X) \cup (B \ominus Y).$$

Properties of erosion and dilation

Order of erosion and intersection:

$(X \cap Y) \oplus B = B \oplus (X \cap Y) \subseteq (X \oplus B) \cap (Y \oplus B)$. The dilation of the intersection of two images is contained in the intersection of their dilations.

The order of erosion may be interchanged with set union which enables the structuring element to be decomposed into a union of simpler structuring elements:

$$\begin{aligned} B \oplus (X \cup Y) = (X \cup Y) \oplus B &= (X \oplus B) \cup (Y \oplus B)\,, \\ (X \cup Y) \ominus B &\supseteq (X \ominus B) \cup (Y \ominus B)\,, \\ B \ominus (X \cup Y) &= (X \ominus B) \cap (Y \ominus B)\,. \end{aligned}$$

Properties of erosion and dilation II

Successive dilation (respectively erosion) of the image X first by the structuring element B and then by the structuring element D is equivalent to the dilation (erosion) of the image X by $B \oplus D$

$$
\begin{aligned}
(X \oplus B) \oplus D &= X \oplus (B \oplus D)\,, \\
(X \ominus B) \ominus D &= X \ominus (B \oplus D)\,.
\end{aligned}
$$

Hit-or-miss transformation $\otimes$

- uses a composite structuring element $B = (B_1, B_2)$, $B_1 \cap B2 = \emptyset$.

$$X \otimes B = \{x : \; B_1 \subset X \text{ and} B_2 \subset X^c\} \,.$$

- finding local patterns in image. B_1 tests objects, B_2 background (complement). Useful for finding corners, for instance.

- it can be expressed by using erosions and dilations

$$X \otimes B = (X \ominus B_1) \cap (X^c \ominus B_2) = (X \ominus B_1) \setminus (X \oplus \breve{B}_2) \,.$$

Hit-or-miss — Matlab example, finding corners

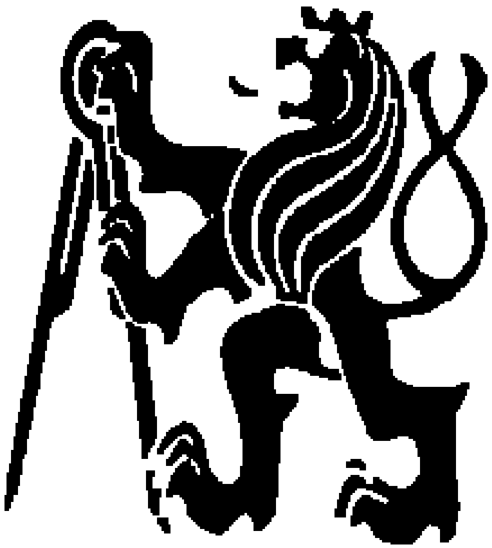

```
bw = [0 0 0 0 0 0
      0 0 1 1 0 0
      0 1 1 1 1 0
      0 1 1 1 1 0
      0 0 1 1 0 0
      0 0 1 0 0 0]
b1 =  [0 0 0   b2  = [0 1 1
       1 1 0          0 0 1
       0 1 0];        0 0 0];
bw2 = bwhitmiss(bw,b1,b2)
bw2 =
     0     0     0     0     0     0
     0     0     0     1     0     0
     0     0     0     0     1     0
     0     0     0     0     0     0
     0     0     0     0     0     0
     0     0     0     0     0     0
```

Opening $\circ$

Erosion followed by dilation

$$X \circ B = (X \ominus B) \oplus B$$

If an image X is unchanged by opening with the structuring element B, it is called open with respect to B

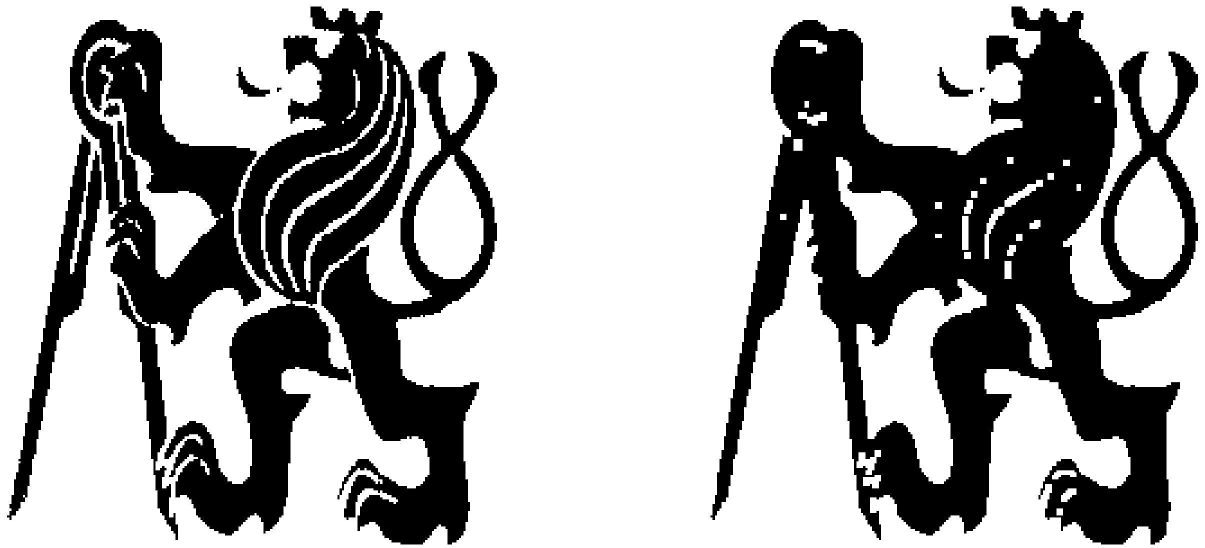


Closing $\bullet$

Dilation followed by erosion

$$X \bullet B = (X \oplus B) \ominus B$$

If an image X is unchanged by closing with the structuring element B, it is called closed with respect to B



Properties of opening and closing

Opening and closing are dual transformations

$$(X \bullet B)^C = X^C \circ \breve{B}$$

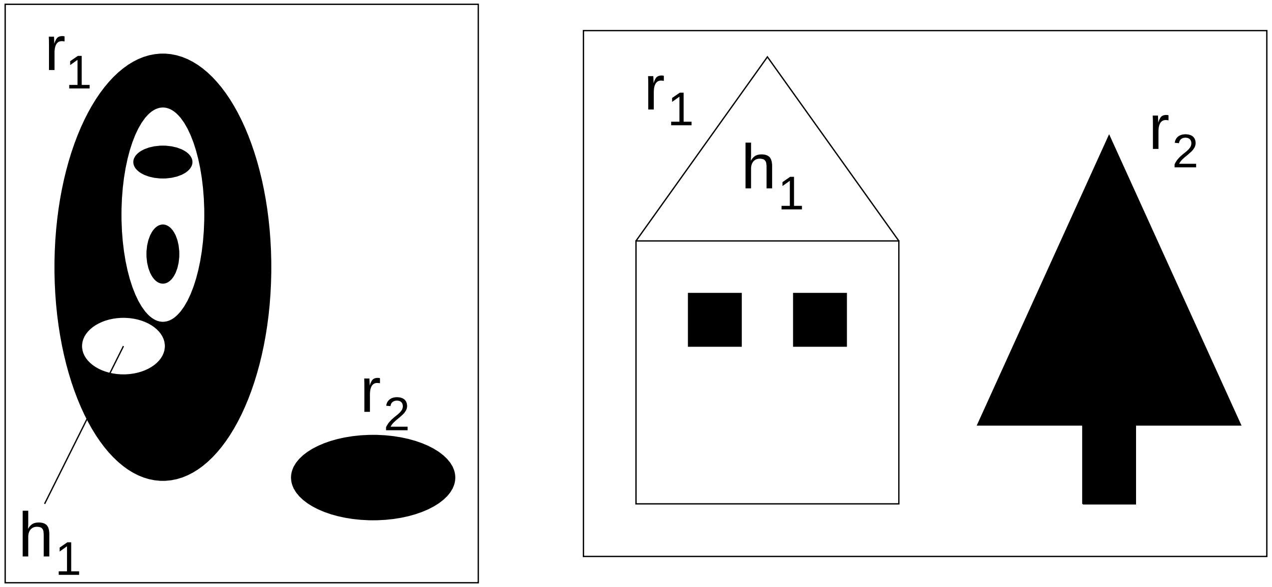
Iteratively used opening and closing are idempotent

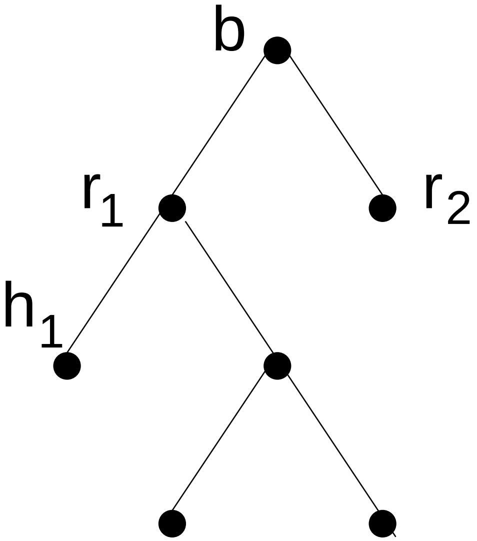
$$X \circ B = (X \circ B) \circ B$$

$$X \bullet B = (X \bullet B) \bullet B$$

Homotopic transformations

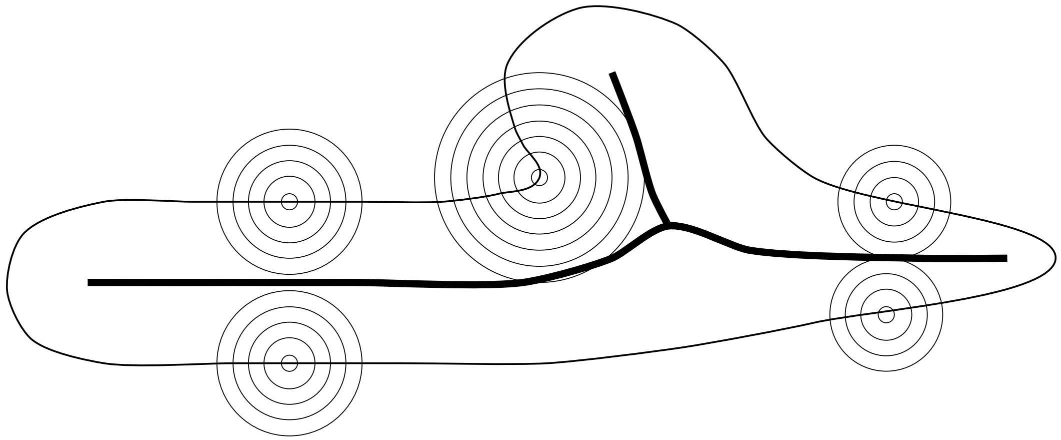
Associated with continuity. Homotopic transformations do not change homotopic tree.





Skeleton

- It is sometimes advantageous to represent elongated objects by their skeleton.
- Blum in 1964 suggested “Medial axis transformation” (grassfire scenario).

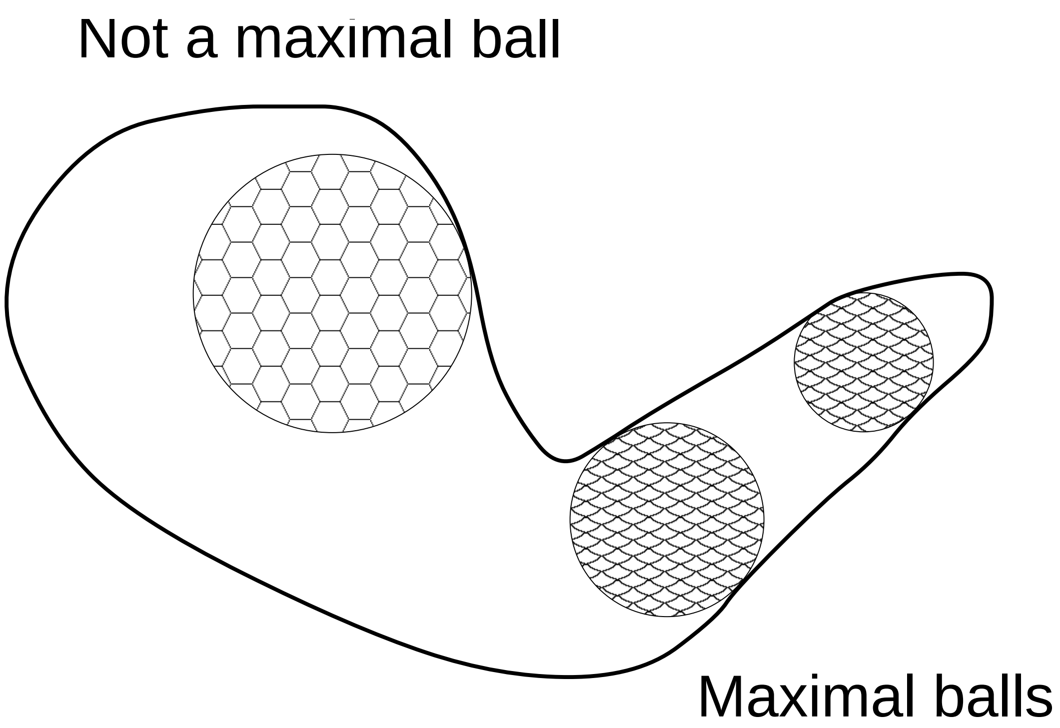


- More formal definition of the skeleton is based on maximal balls

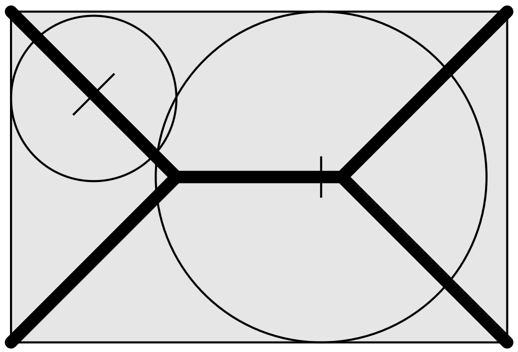
Skeleton by maximal balls

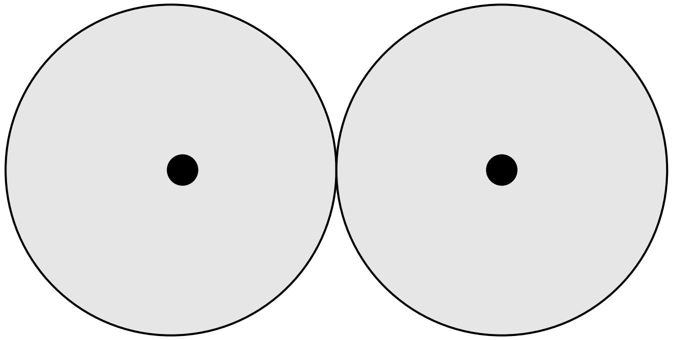
A ball $B(p, r)$ with center p and radius r, $r \geq 0$, is a set of points with distances $d \leq r$.

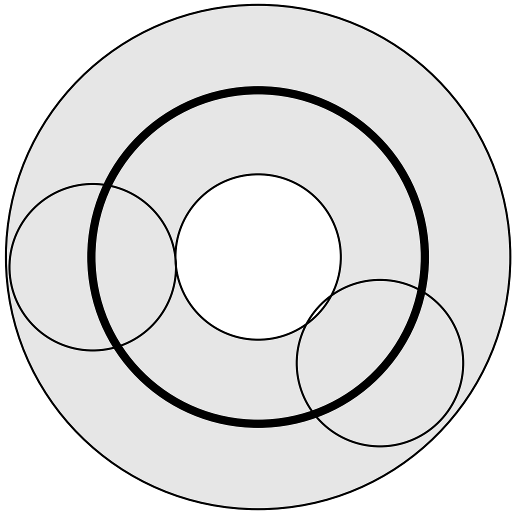
The maximal ball B included X touches the border ∂X in two and more points.



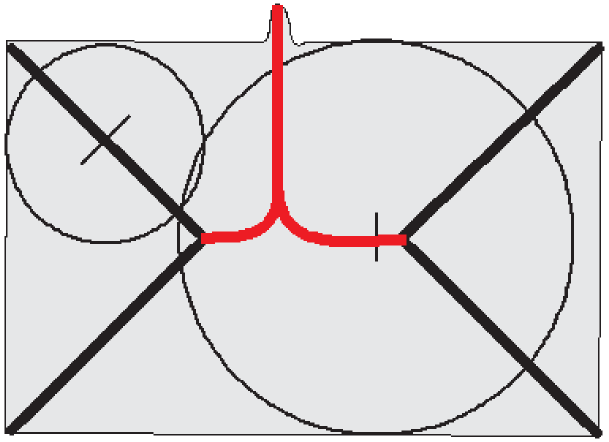
Examples

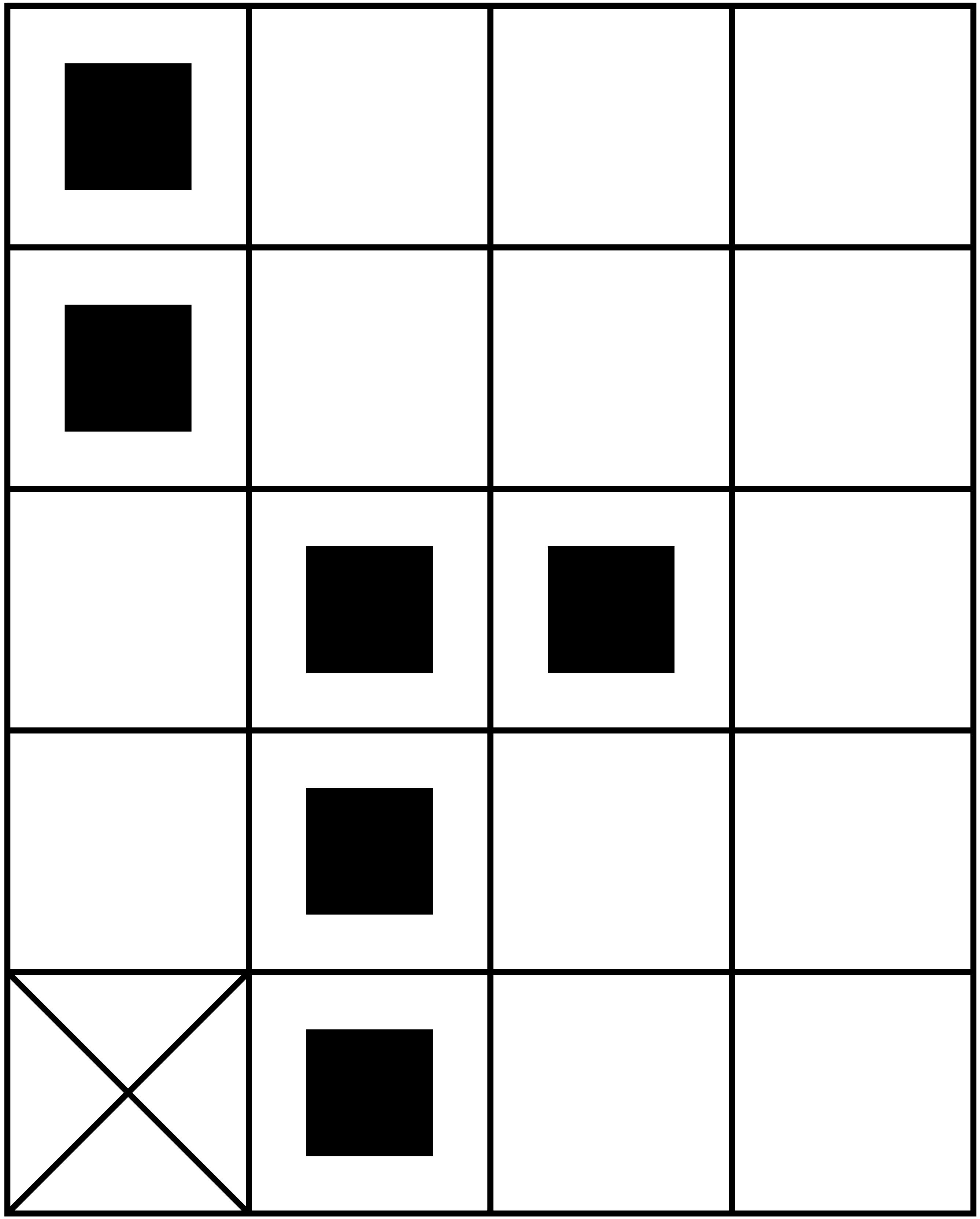


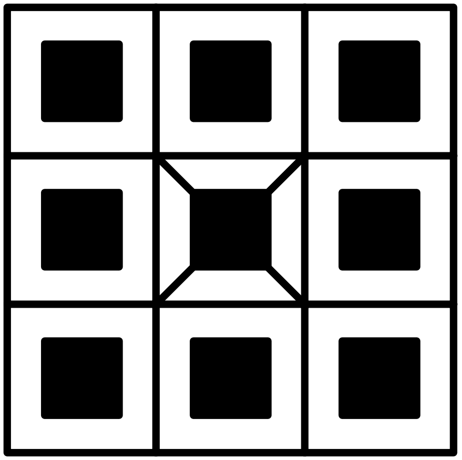




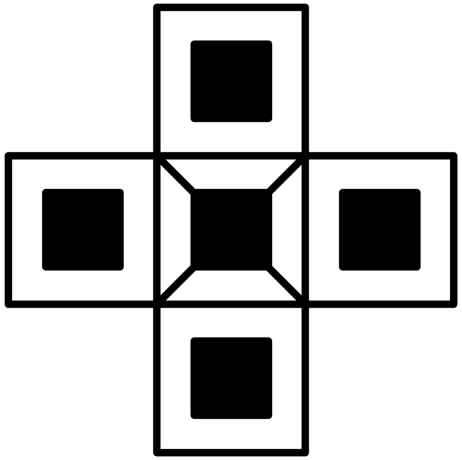
Problems with noise



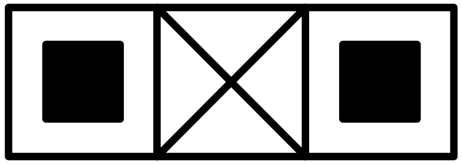




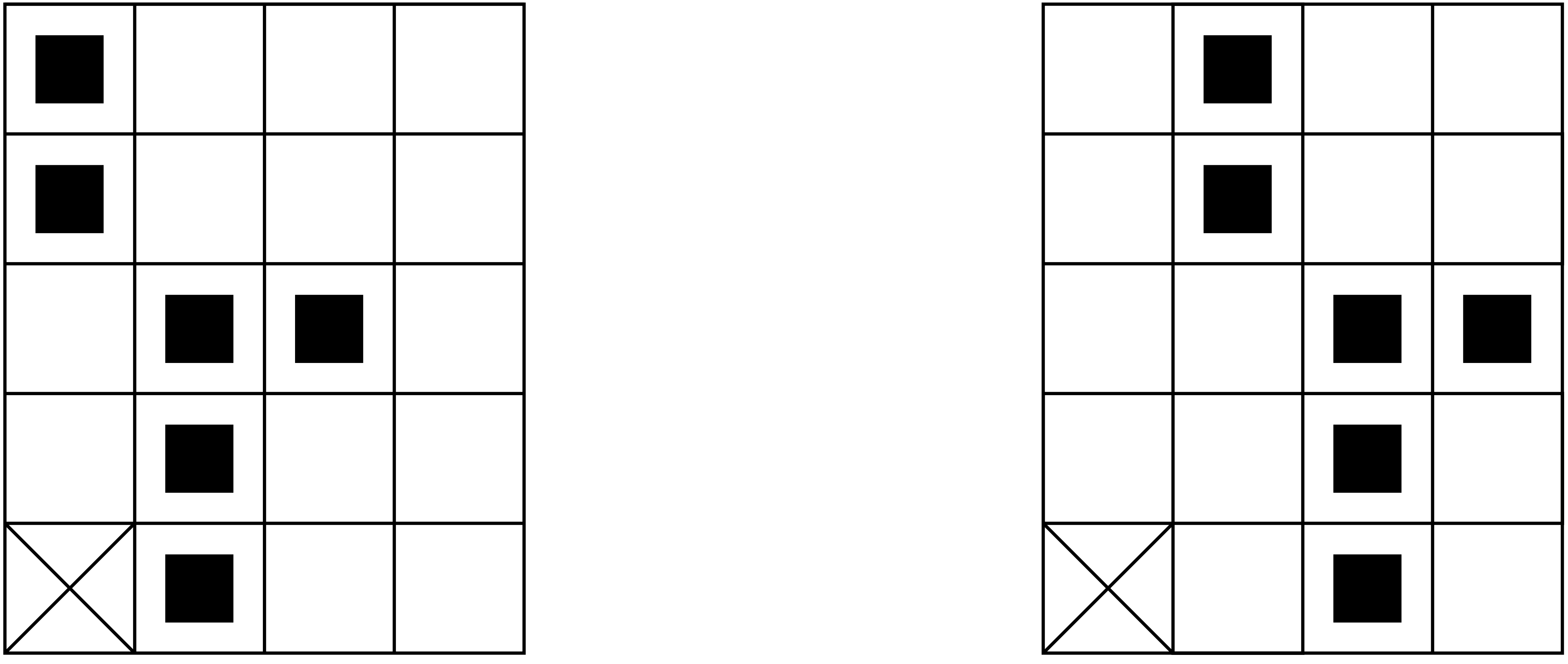
(a)

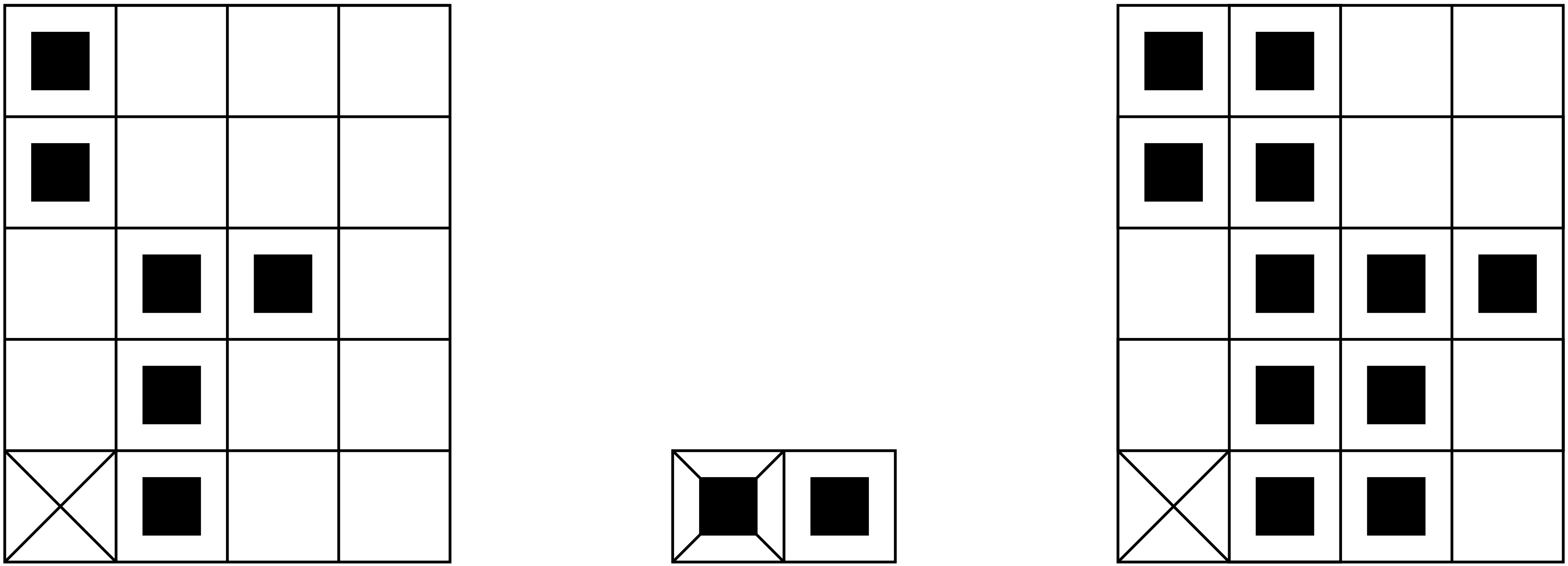


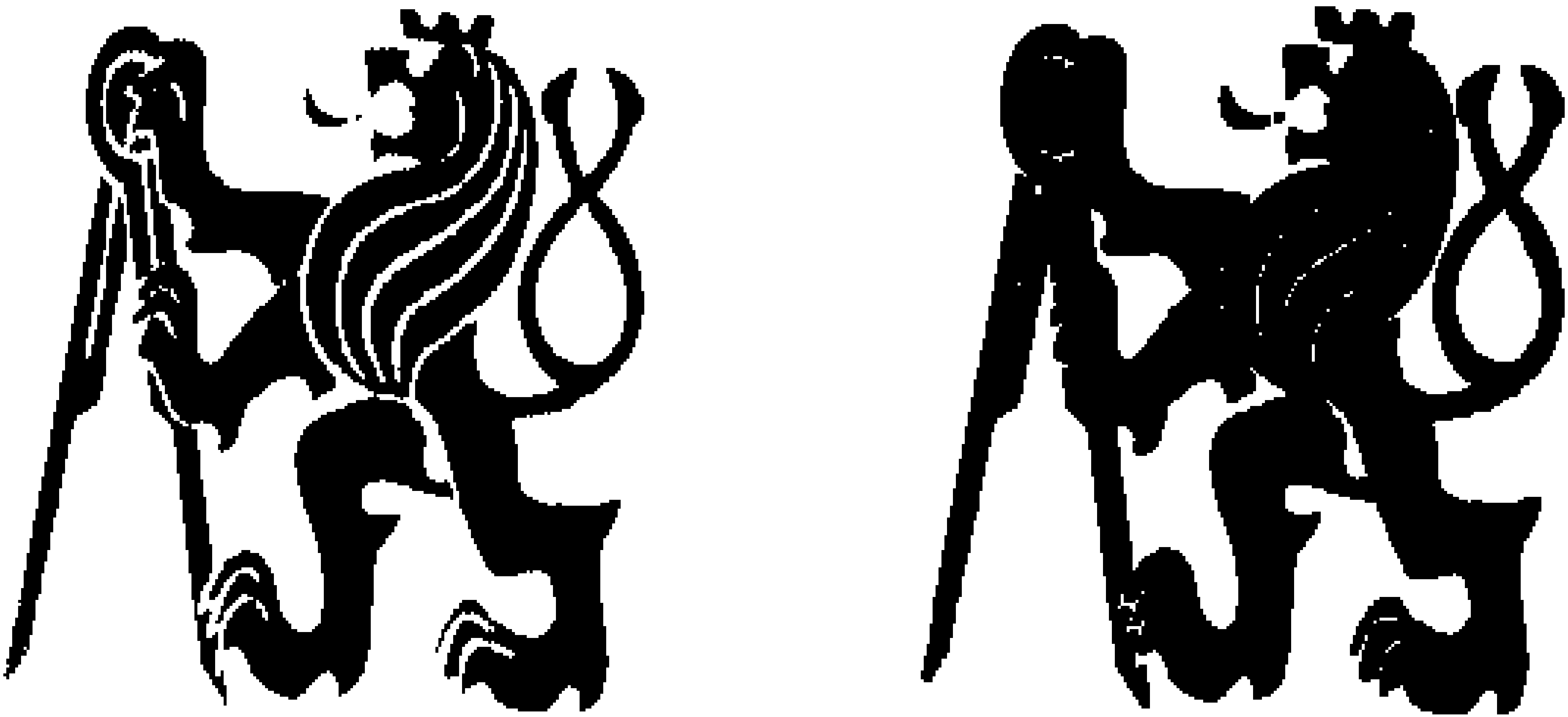
(b)

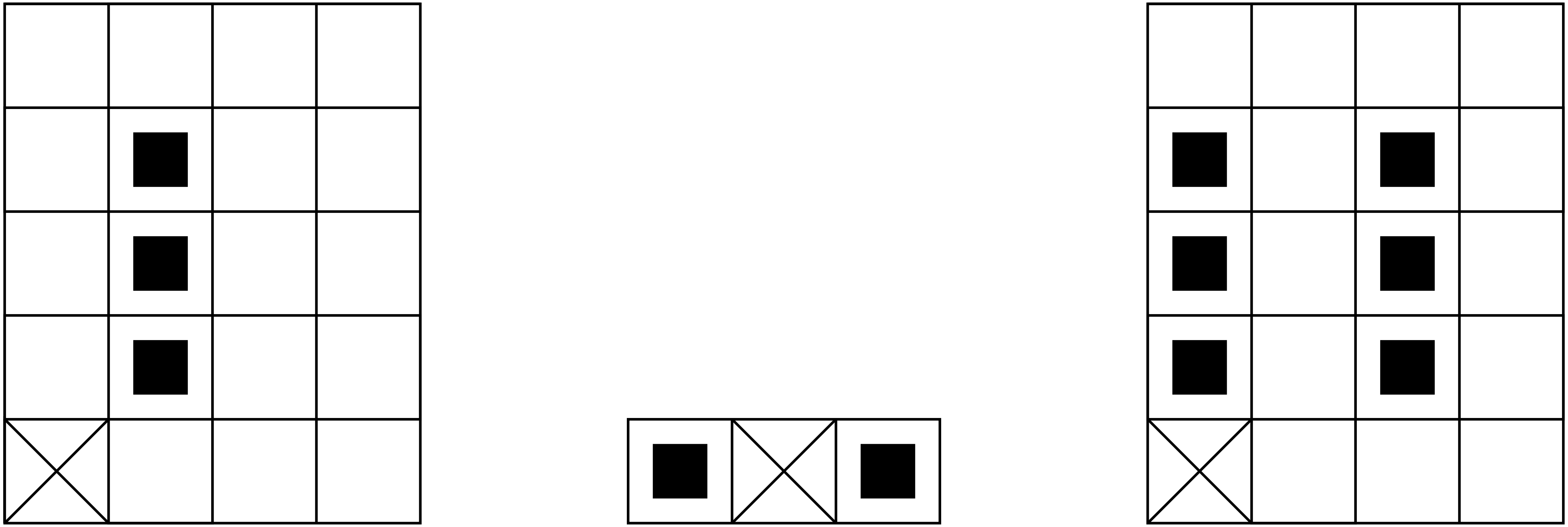


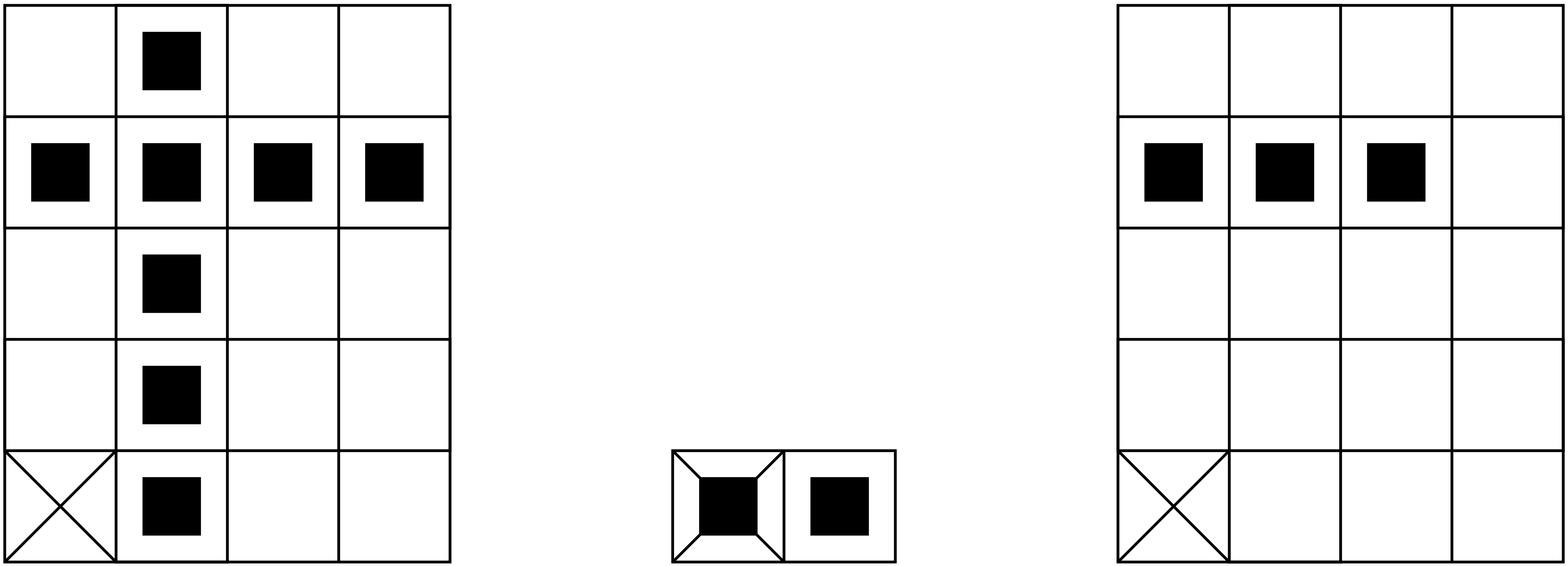
(c)

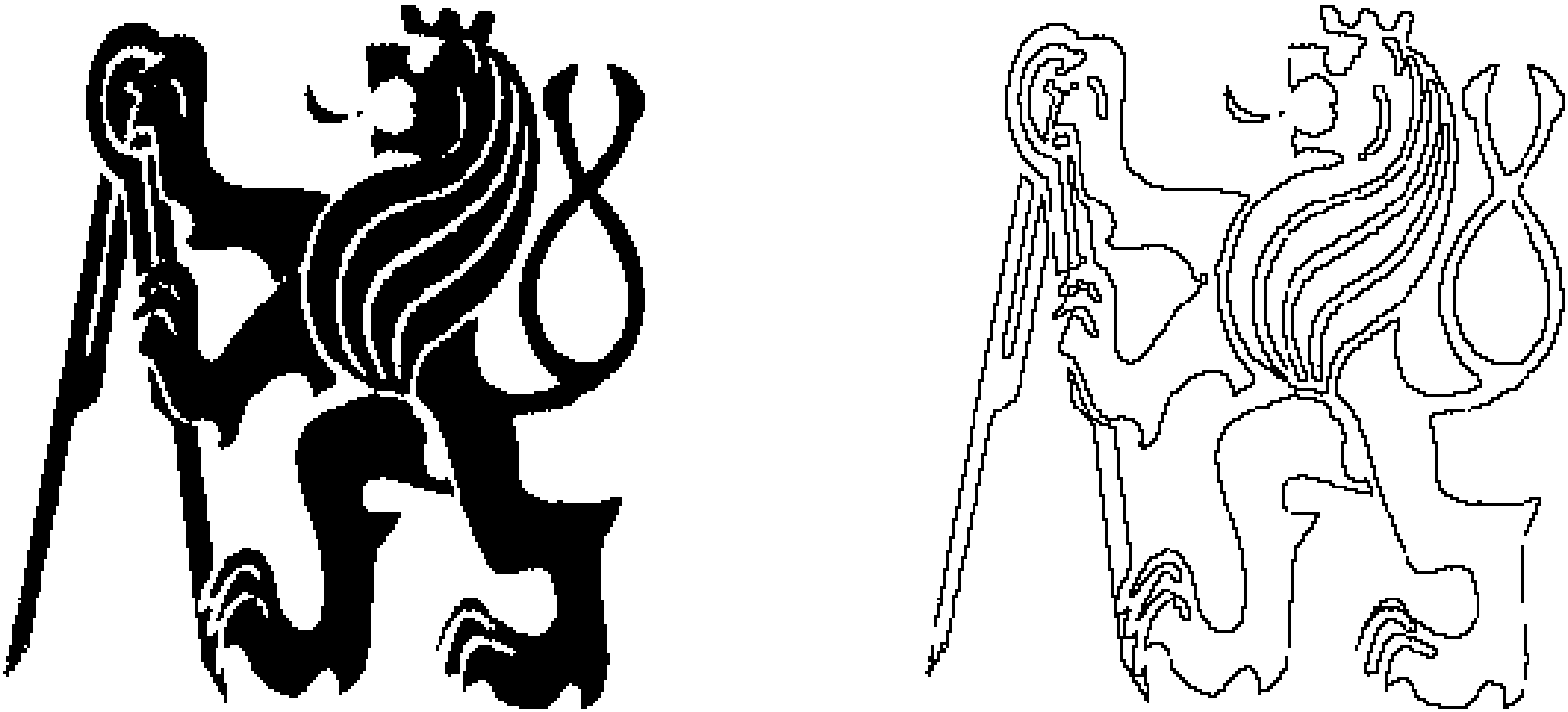


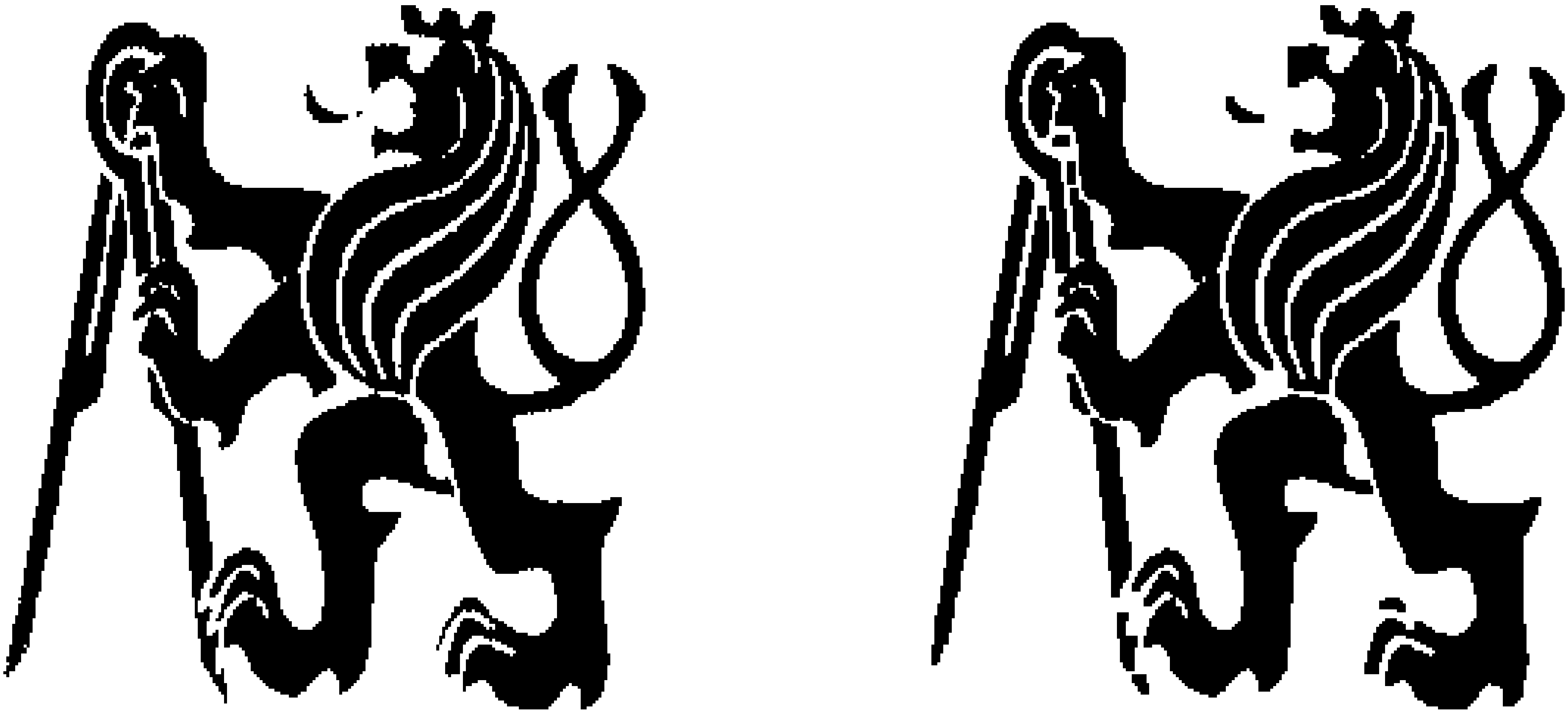


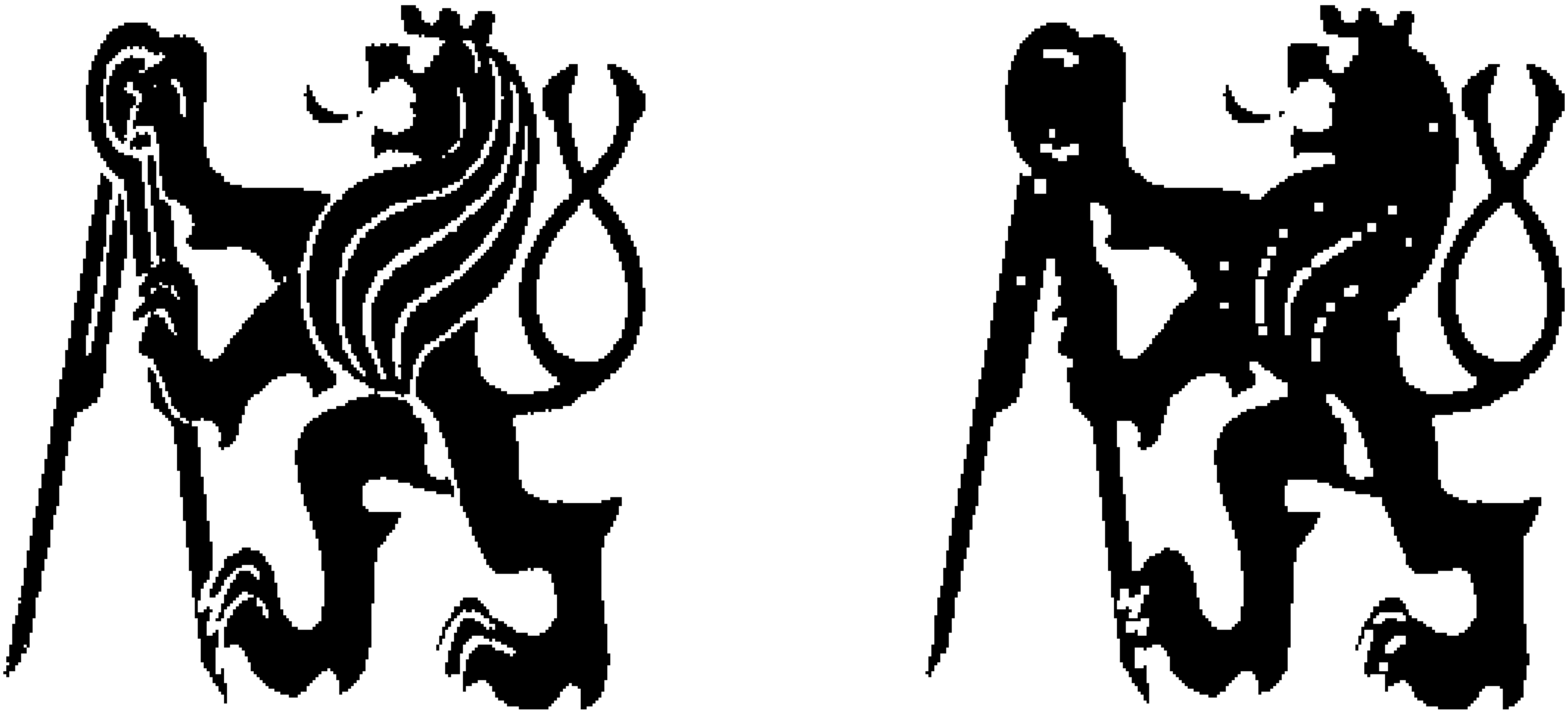


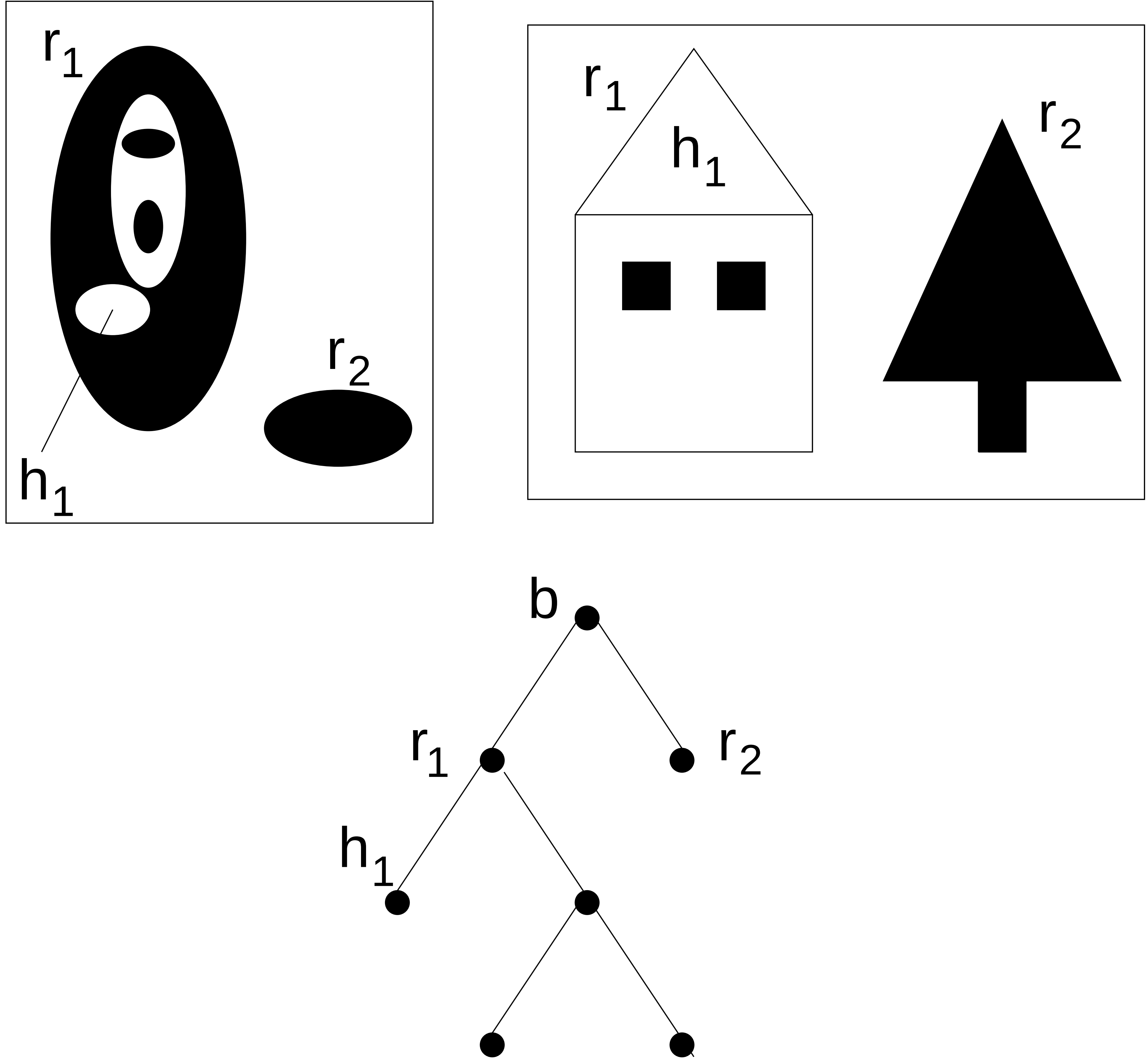


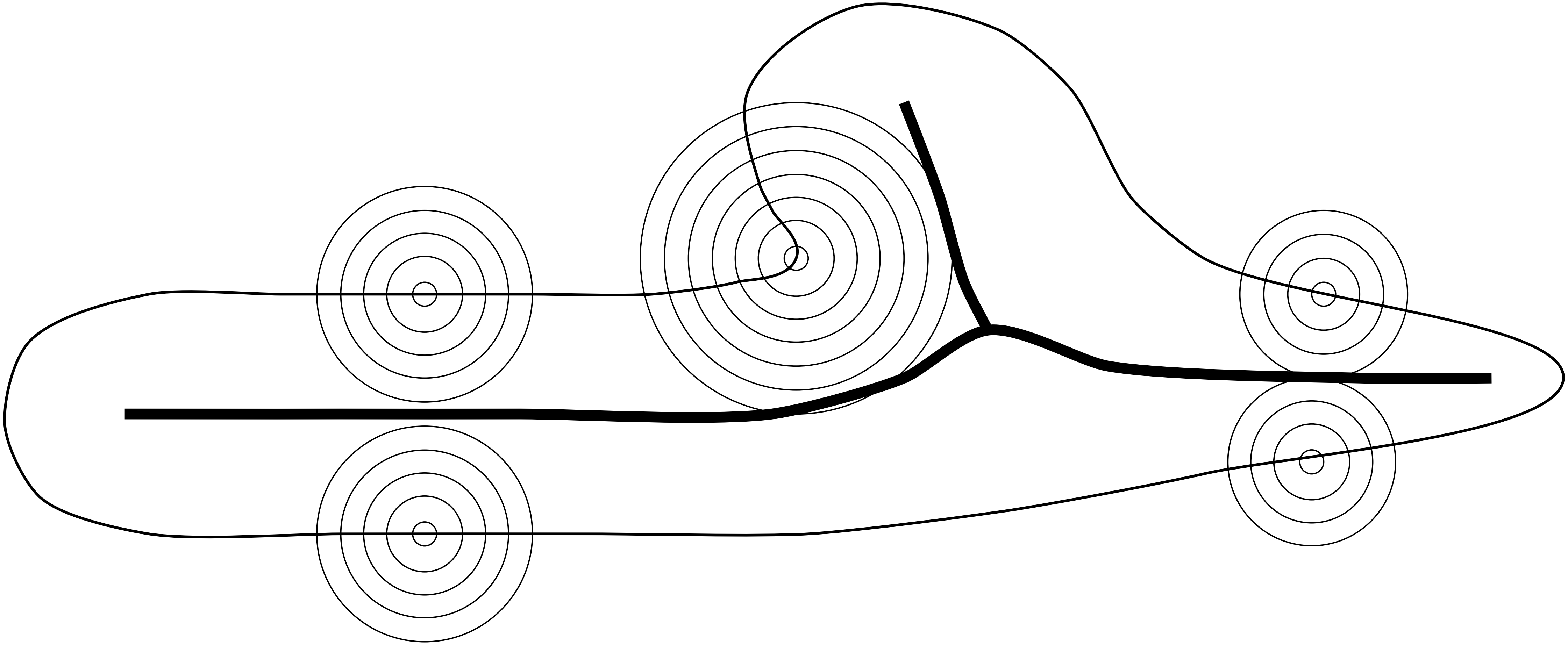


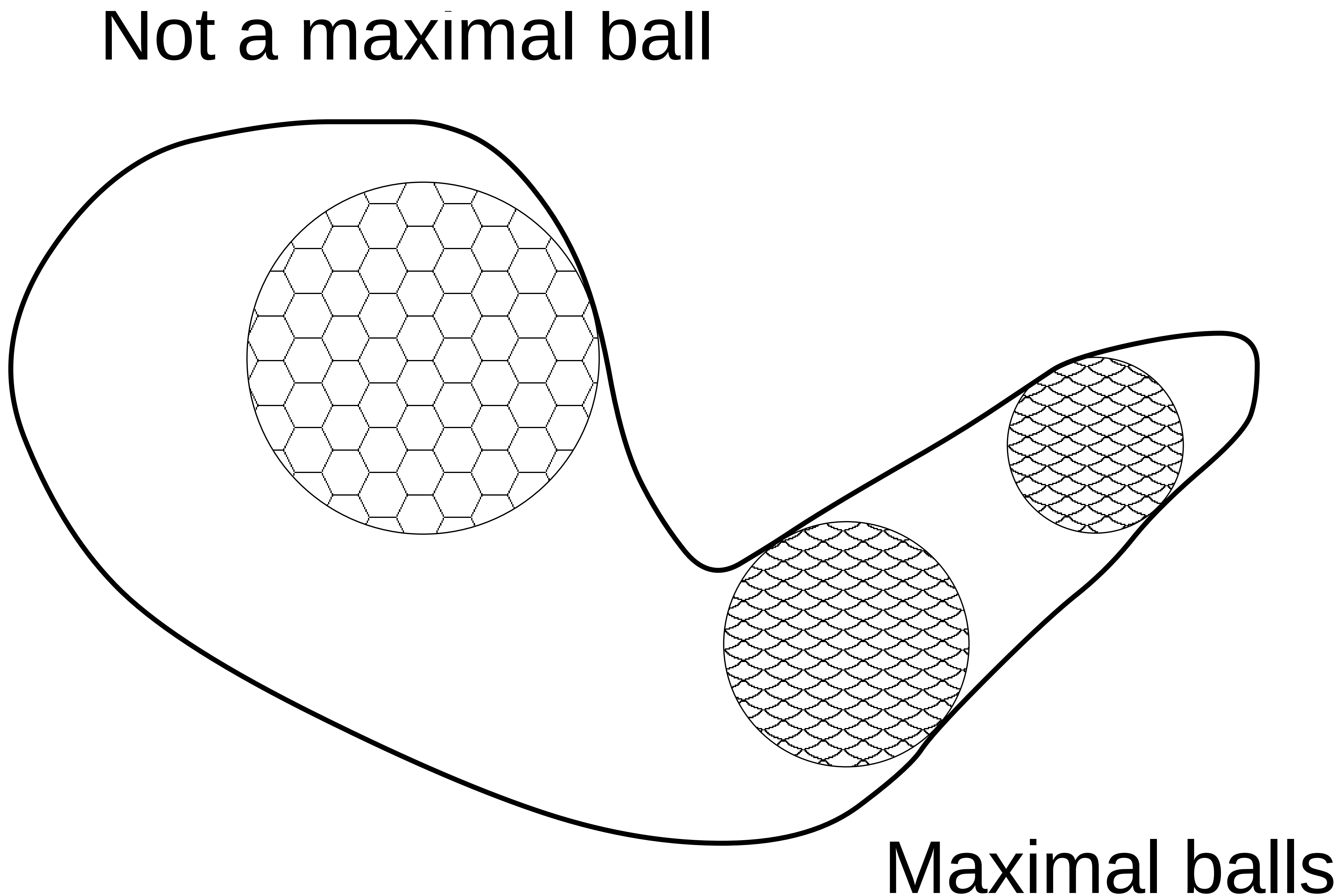







r_1
h_1
r_2
r_1
h_1
r_2
b
r_1
r_2
h_1




Not a maximal ball
Maximal balls