Rigid motion & screws

Intelligent Robotics

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A rigid motion of a set of points $X = \begin{pmatrix} x & y & z & 1 \end{pmatrix}^\top$ into the set of points Y can be expressed by a Euclidean transform

$$
\begin{aligned}
E &= \begin{pmatrix} r & t \\ 0_3^\top & 1 \end{pmatrix}, \; r \in \mathbb{R}^{3\times 3}, \; t \in \mathbb{R}^3, \; r^\top r = r\,r^\top = I, \; \det r = 1 \\
Y &= EX
\end{aligned}
$$

We will show that for all rigid motions there exists such a fixed line o, called the axis of motion, that E can be written as a composition of two one-parametric motions $E = E_2(s)\,E_1(\varphi)$, where $E_1(\varphi)$ is a rotation around o by angle φ and $E_2(s)$ is a translation along o by length s. Two-parametric motions $E(s, \varphi)$ are called *screws*.

The existence of the axis does not depend on the choice of the coordinate system. Thus, we will choose a particular coordinate system with respect to which the E will take so simple form that the above statement will become evident.

Let the change of the coordinate system be represented by

$$
\begin{aligned}
P &= \begin{pmatrix} R & T \\ 0_3^\top & 1 \end{pmatrix}^{-1}, \; R \in \mathbb{R}^{3\times 3}, \; T \in \mathbb{R}^3, \; R^\top R = R\,R^\top = I, \; \det R = 1 \\
X' &= PX \\
Y' &= PY \\
Y' &= PEP^{-1}X' = P'X
\end{aligned}
$$

Thus

$$E' = PEP^{-1}$$

An elementary proof

Expand elements of E'

$$
\begin{aligned}
P &= \begin{pmatrix} R^{-1} & -R^{-1}T \\ 0_3^\top & 1 \end{pmatrix} \\
P^{-1} &= \begin{pmatrix} R & T \\ 0_3^\top & 1 \end{pmatrix} \\
E' &= \begin{pmatrix} R^{-1} & -R^{-1}T \\ 0_3^\top & 1 \end{pmatrix} \begin{pmatrix} r & t \\ 0_3^\top & 1 \end{pmatrix} \begin{pmatrix} R & T \\ 0_3^\top & 1 \end{pmatrix} \\
E' &= \begin{pmatrix} R^{-1}r & R^{-1}t - R^{-1}T \\ 0_3^\top & 1 \end{pmatrix} \begin{pmatrix} R & T \\ 0_3^\top & 1 \end{pmatrix} \\
E' &= \begin{pmatrix} R^{-1}rR & R^{-1}r\,T + R^{-1}t - R^{-1}T \\ 0_3^\top & 1 \end{pmatrix} \\
E' &= \begin{pmatrix} R^{-1}rR & R^{-1}((r-I)T + t) \\ 0_3^\top & 1 \end{pmatrix}
\end{aligned}
$$

and choose P to make E' as simple as possible. We would like to make E' diagonal. However, it is not possible as we will see later. Now, let us simplify it as much as possible.

We shall start with an observation about matrices r.

There holds rank $(r - I)$ is either 0 or 2. To proof the statement, consider that

$$\|y\|^2 = y^\top y = (rx)^\top (rx) = x^\top r^\top rx = x^\top (r^\top r)x = x^\top Ix = x^\top x = \|x\|^2$$

If there is a $\lambda \in \mathbb{C}$ such that

$$r\,x = \lambda x$$

then $|\lambda| = 1$ since

$$|\lambda|^2 \|x\| = \|\lambda x\|^2 = \|x\|^2$$

There is a real unit eigenvalue since r is a real matrix with the chracteristic polynomial

$$p(\lambda) = \det(\lambda I - r) = \lambda^3 - \mathrm{trace}\,(r)\lambda^2 + b\lambda - \det(r) = \lambda^3 + a\lambda^2 + b\lambda - 1$$

which has always a real solution. Since $p(0) = -1$, $\lim_{\lambda\to\infty} p(\lambda) = +\infty$, and $p(\lambda)$ is a continuous function, it must cross the zero value at a positive real number. We conclude that 1 is an eigenvalue.

Furthermore

$$\lambda^3 + a\lambda^2 + b\lambda - 1 = (\lambda^2 + (a+1)\lambda + a + b + 1)(\lambda - 1) + (a + b)$$

implies that $a + b = 0$, i.e. $b = \mathrm{trace}\,(r)$. Thus

$$\frac{p(\lambda)}{\lambda - 1} = \lambda^2 + (a+1)\lambda + 1$$

Now, if all eigenvalues are real, then they must be $1, 1, 1$ or $1, -1, -1$ since

$$(\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2$$

Otherwise, the eigenvalues must be $1, a + bi, a - bi$.

r is a real matrix with the chracteristic polynomial

$$p(\lambda) = \det(\lambda I - r) = \lambda^3 - \mathrm{trace}\,(r)\lambda^2 + \mathrm{trace}\,(r)\lambda - \det(r)$$

and thus there holds

$$\begin{aligned} p(1) &= 1 - \mathrm{trace}\,(r) + \mathrm{trace}\,(r) - 1 = 0 \\ p(-1) &= -1 - \mathrm{trace}\,(r) - \mathrm{trace}\,(r) - 1 = -2\,(\mathrm{trace}\,(r) + 1) \end{aligned}$$

so $p(-1) = 0 \Leftrightarrow \mathrm{trace}\,(r) = -1$

Eigenvalues of r may have only one of the following values

1. $1, 1, 1$:
   $r = I$ since $(\lambda - 1)^3 = \lambda^3 - 3\lambda^2 + 3\lambda - 1$ implies that trace$(r) = r_{11} + r_{22} + r_{33} = 3$ but that means that $r_{11} = r_{22} = r_{33} = 1$ since $r_{11}, r_{22}, r_{33} \leq 1$.

2. $1, a + bi, a - bi$:
   For each λ_i there is an eigenvector $x_i \neq 0$. Suppose that there is i, j such that $x_i = x_j$. Then $\lambda_i x_i = r x_i = r x_j = \lambda_j x_j = \lambda_j x_i$ implies that $\lambda_i = \lambda_j$. That is not the case. Therefore, there is a rank 3 matrix R such that $rR = R \begin{pmatrix} a+bi & 0 & 0 \\ 0 & a-bi & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

3. $1, -1, -1$:
   Using a similar argument as above, we see that there is a rank 3 matrix R such that $rR = R \begin{pmatrix} e & f & 0 \\ g & h & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Matris R is a rotations. Matrix $\begin{pmatrix} e & f & 0 \\ g & h & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is diagonal.

By the Gram-Schmidt orthogonalization, there exists an othonormal matrix U such that $S = U^{-1}\, r\, U$ is upper triangular, i.e.

$$S = \begin{pmatrix} s_{11} & s_{12} & s_{13} \\ 0 & s_{22} & s_{23} \\ 0 & 0 & s_{33} \end{pmatrix}$$

It holds $S^\top S = (U^{-1}\, r\, U)^\top (U^{-1}\, r\, U) = (U^\top\, r\, U)^\top (U^\top\, r\, U) = I$ and also $S\, S^\top = I$. Therefore,

$$I = S\, S^\top = \begin{pmatrix} s_{11} & s_{12} & s_{13} \\ 0 & s_{22} & s_{23} \\ 0 & 0 & s_{33} \end{pmatrix} \begin{pmatrix} s_{11} & 0 & 0 \\ s_{12} & s_{22} & 0 \\ s_{13} & s_{23} & s_{33} \end{pmatrix} = \begin{pmatrix} s_{11}^2 + s_{12}^2 + s_{13}^2 & s_{12}s_{22} + s_{13}s_{23} & s_{13}s_{33} \\ s_{12}s_{22} + s_{13}s_{23} & s_{22}^2 + s_{23}^2 & s_{23}s_{33} \\ s_{13}s_{33} & s_{23}s_{33} & s_{33}^2 \end{pmatrix}$$

implies $s_{33}^2 = 1 \Rightarrow s_{13} = s_{23} = 0 \Rightarrow s_{22}^2 = 1 \Rightarrow s_{12} = 0 \Rightarrow s_{11}^2 = 1$. Thus

$$S = \begin{pmatrix} s_{11} & 0 & 0 \\ 0 & s_{22} & 0 \\ 0 & 0 & s_{33} \end{pmatrix}$$

Since $r\, U = U\, S$, the columns of U are eigenvectors and thus $U = R$ shows that R is a rotation.

We can choose S such that $\begin{pmatrix} e & f & 0 \\ g & h & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Let $\operatorname{rank}(r - I) = 0$. Then $r = I$ and

$$E' = \begin{pmatrix} I & R^{-1}t \\ 0_3^\top & 1 \end{pmatrix}$$

If $t = 0$, then we are done since E' is the indentity and it can be written as the rotation by $\varphi = 0$ followed by the traslation by $s = 0$ around resp. along any line through the origin. If $t \neq 0$, then by choosing $R = \begin{pmatrix} R_1 & R_2 & t/\|t\| \end{pmatrix}$ we make

$$\begin{aligned} R^{-1}t &= \begin{pmatrix} R_1^\top t & R_2^\top t & t^\top t/\|t\| \end{pmatrix}^\top = \begin{pmatrix} 0 & 0 & \|t\| \end{pmatrix}^\top \\ E' &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \|t\| \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

which coresponds to the rotation by $\varphi = 0$ followed by the traslation by $s = \|t\|$ around resp. along the line corresponding to the z axis.

Let $\text{rank}\,(r - I) = 2$. Then either $t \in \text{span}\,(r - I)$ or $t \notin \text{span}\,(r - I)$.

If $t \in \text{span}\,(r - I)$, then we can choose T such that $(r - I)T + t = 0$ to get

$$E' \;=\; \begin{pmatrix} R^{-1}rR & 0_3 \\ 0_3^\top & 1 \end{pmatrix} = \begin{pmatrix} e & f & 0 & 0 \\ g & h & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

which coresponds to a rotation around the z axis.

If $t \notin \text{span}\,(r - I)$, then we can choose T such that we can generate by $(r - I)T + t$ any one dimensional subspace that is not in $\text{span}\,(r - I)$. Choose $R = \begin{pmatrix} R_1 & R_2 & R_3 \end{pmatrix}$ to make $rR = R\begin{pmatrix} e & f & 0 \\ g & h & 0 \\ 0 & 0 & 1 \end{pmatrix}$, in particular $rR_3 = R_3$, i.e. $(r - I)R_3 = 0$. Therefore, $R_3 \notin \text{span}\,(r - I)$ and we can make $(r - I)T + t = sR_3$ and $R^{-1}((r - I)T + t) = R^{-1}sR_3 = s\begin{pmatrix} R_1^\top R_3 & R_2^\top R_3 & R_3^\top R_3 \end{pmatrix}^\top = \begin{pmatrix} 0 & 0 & s \end{pmatrix}^\top$. Finally we obtain

$$\begin{aligned} E' \;&=\; \begin{pmatrix} R^{-1}rR & R^{-1}((r - I)T + t) \\ 0_3^\top & 1 \end{pmatrix} \\ &=\; \begin{pmatrix} e & f & 0 & 0 \\ g & h & 0 & 0 \\ 0 & 0 & 1 & s \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

which coresponds to a rotation around and a traslation along the z axis.

A proof via the Jordan canonical form

Jordan canonical form:

Let A be a matrix of a linear transformation on N-dimensional linear space over $\mathbb{C}$ with characteristic polynomial

$$p(\lambda) = \prod_{i=1}^{K} (\lambda - \lambda_i)^{N_i} \qquad N_i > 0 \quad \& \quad N_1 + N_2 + \cdots + N_K = N,$$

then there exist transformation T such that

$$TAT^{-1} = \begin{pmatrix} J_1 & 0 & ... & 0 \\ 0 & J_2 & ... & 0 \\ & & ... & \\ 0 & 0 & ... & J_K \end{pmatrix}, \qquad J_i = \begin{pmatrix} J_{i1} & 0 & ... & 0 \\ 0 & J_{i2} & ... & 0 \\ & & ... & \\ 0 & 0 & ... & J_{iL_i} \end{pmatrix}, \qquad J_{il} = \begin{pmatrix} \lambda_i & 1 & 0 & ... & 0 \\ 0 & \lambda_i & 1 & ... & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & ... & \lambda_i \end{pmatrix}$$

with J_{il} of such size $M_{il} \times M_{il}$ that

$$M_{il} \geq M_{im} > 0 \ \text{ for } l < m, \qquad M_{i1} + M_{i2} + \cdots M_{iL_i} = N_i$$

Examples:

$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \qquad \begin{pmatrix} a+bi & 0 & 0 \\ 0 & a-bi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} a+bi & 0 & 0 & 0 \\ 0 & a-bi & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Screws:

$$E = \begin{pmatrix} r & t \\ 0_3^\top & 1 \end{pmatrix}, \; r \in \mathbb{R}^{3\times 3}, \; t \in \mathbb{R}^3, \; r^\top r = r\,r^\top = I, \; \det r = 1$$

The characteristic polynomial of E

$$p(\lambda) = (\lambda - 1)\det(\lambda I - r) = (\lambda - 1)(\lambda - 1)\left(\lambda^2 + (-\mathrm{trace}\,(r) + 1)\lambda + 1\right)$$

has solutions

1. $1, 1, 1, 1$ for $r = I$,

2. $-1, -1, 1, 1$, or

3. $a + bi, a - bi, 1, 1$.

Therefore $\exists Q : \; QEQ^{-1} =$

1. $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

2. $\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

3. $\begin{pmatrix} a+bi & 0 & 0 & 0 \\ 0 & a-bi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} a+bi & 0 & 0 & 0 \\ 0 & a-bi & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Thus, there is an ordered basis $\beta = (\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4)$ w.r.t. which E has the form

$$E_\beta = \begin{pmatrix} e & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & g \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ with } g \in \{0, 1\}$$

Let $\beta = (\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4)$ be an ordered basis w.r.t. which E has the form $E_\beta = \begin{pmatrix} e & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & g \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

We can write

$$\begin{aligned} x'\mathbf{b}_1 &= e\,x\,\mathbf{b}_1 \\ y'\mathbf{b}_2 &= f\,y\,\mathbf{b}_2 \\ z'\mathbf{b}_3 &= z\,\mathbf{b}_3 + g\,w\,\mathbf{b}_4 \\ w'\mathbf{b}_4 &= w\,\mathbf{b}_4 \end{aligned}$$

Taking $\beta' = (\mathbf{b}'_1, \mathbf{b}'_2, \mathbf{b}'_3, \mathbf{b}'_4) = (\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \frac{\mathbf{b}_4}{\|\mathbf{b}_4\|})$ instead of β we get

$$\begin{aligned} x'\mathbf{b}'_1 &= e\,x\,\mathbf{b}'_1 \\ y'\mathbf{b}'_2 &= f\,y\,\mathbf{b}'_2 \\ z'\mathbf{b}'_3 &= z\,\mathbf{b}'_3 + \|\mathbf{b}_4\|\,g\,w\,\mathbf{b}'_4 \\ w'\mathbf{b}'_4 &= w\,\mathbf{b}'_4 \end{aligned}$$

we get $E'_\beta = \begin{pmatrix} e & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & \|\mathbf{b}_4\|\,g \\ 0 & 0 & 0 & 1 \end{pmatrix}$