Advanced Robotics

Lecture 3

Motion induces a mapping of the associated linear space into itself

$$\begin{array}{ccc} \overset{P}{\bullet} & \xrightarrow{m} & \overset{P'}{\bullet} \\ \varphi\downarrow & & \downarrow\varphi \\ \vec{x} & \xrightarrow{m} & \vec{x}\,' \end{array}$$

With a fixed origin O and mapping φ,

$m : \mathcal{P} \rightarrow \mathcal{P}$ induces a mapping $m : \mathcal{V} \rightarrow \mathcal{V}$

$$\begin{aligned} P' &= m(P) \\ \vec{x}\,' &= m(\vec{x}) \end{aligned}$$

$$\begin{aligned} \vec{x} &= \varphi(O, P) \\ \vec{x}\,' &= \varphi(O, m(P)) \end{aligned}$$

Motion characterisation in $\mathcal{V}$

$$
\begin{aligned}
\vec{x} &= \varphi(O, P) \\
\vec{x}\,' &= \varphi(O, m(P)) \\
\vec{y} &= \varphi(O, Q) \\
\vec{y}\,' &= \varphi(O, m(Q))
\end{aligned}
$$

$m : \mathcal{E} \rightarrow \mathcal{E}$ *motion* $\Rightarrow \forall P, Q \in \mathcal{P}$:

$$d(m(P), m(Q)) = d(P, Q)$$

$$
\begin{aligned}
d(P, Q) &= \sqrt{\varphi(P, Q) \cdot \varphi(P, Q)} \\
&= \sqrt{(\varphi(O, Q) - \varphi(O, P)) \cdot (\varphi(O, Q) - \varphi(O, P))} \\
&= \sqrt{(\vec{y} - \vec{x}) \cdot (\vec{y} - \vec{x})} \\
d(m(P), m(Q)) &= \sqrt{(\vec{y}\,' - \vec{x}\,') \cdot (\vec{y}\,' - \vec{x}\,')}
\end{aligned}
$$

$$\forall \vec{x}, \vec{y} \in L: \qquad \sqrt{(\vec{y} - \vec{x}) \cdot (\vec{y} - \vec{x})} = \sqrt{(\vec{y}\,' - \vec{x}\,') \cdot (\vec{y}\,' - \vec{x}\,')}$$

Motion characterisation in $\mathcal{V}$

$$d(m(P), m(Q)) \;=\; d(P, Q) \quad \text{for every } P, Q$$

$$\sqrt{(\vec{y} - \vec{x}) \cdot (\vec{y} - \vec{x})} \;=\; \sqrt{(\vec{y}\,' - \vec{x}\,') \cdot (\vec{y}\,' - \vec{x}\,')}$$

$$\vec{x} \cdot \vec{x} \geq 0 \Rightarrow \quad \Updownarrow$$

$$(\vec{y} - \vec{x}) \cdot (\vec{y} - \vec{x}) \;=\; (\vec{y}\,' - \vec{x}\,') \cdot (\vec{y}\,' - \vec{x}\,')$$

$$(\vec{y} - \vec{x}) \cdot (\vec{y} - \vec{x}) \;=\; (m(\vec{y}) - m(\vec{x})) \cdot (m(\vec{y}) - m(\vec{x})) \quad \text{for every } \vec{x}, \vec{y}$$

Motion is not linear in general

$f : \mathcal{V} \rightarrow \mathcal{V}$ is linear iff $\forall \alpha, \beta \in \mathbb{R}$ and $\forall \vec{x}, \vec{y} \in \mathcal{V}$ holds

$$f(\alpha\,\vec{x} + \beta\,\vec{y}) = \alpha\,f(\vec{x}) + \beta\,f(\vec{y})$$

A general motion m is not linear since, e.g.,

Translation $m_{\vec{o}}(\vec{x}) = \vec{x} + \vec{o}$ is a motion since

$$\begin{aligned}(m_{\vec{o}}(\vec{y}) - m_{\vec{o}}(\vec{x})) \cdot (m_{\vec{o}}(\vec{y}) - m_{\vec{o}}(\vec{x})) &= (\vec{y} + \vec{o} - \vec{x} - \vec{o}) \cdot (\vec{y} + \vec{o} - \vec{x} - \vec{o}) \\ &= (\vec{y} - \vec{x}) \cdot (\vec{y} - \vec{x})\end{aligned}$$

but

$$m_{\vec{o}}(\alpha\,\vec{x}) = \alpha\,\vec{x} + \vec{o} \neq \alpha\,\vec{x} + \alpha\,\vec{o} = \alpha\,m_{\vec{o}}(\vec{x})$$

for $\vec{o} \neq \vec{0}$ and $\alpha \neq 1$ . . . translations are not linear

But there are motions that are linear as we will show ...

Motion that fixes origin preserves the scalar product

For every motion holds

$$(\vec{y}-\vec{x})\cdot(\vec{y}-\vec{x}) = (m(\vec{y})-m(\vec{x}))\cdot(m(\vec{y})-m(\vec{x})) \quad \text{for every } \vec{x},\, \vec{y}$$

$$(\vec{y}-\vec{x})\cdot(\vec{y}-\vec{x}) = (\vec{y}\,'-\vec{x}\,')\cdot(\vec{y}\,'-\vec{x}\,')$$

moreover, if $m(O)=O$, i.e. $m(\vec{0})=\vec{0}$, then for $\vec{x}=\vec{0}$, we get

$$\vec{y}\,'\cdot\vec{y}\,' = \vec{y}\cdot\vec{y} \quad \text{for every } \vec{y} \text{ and } \vec{y}\,'=m(\vec{y})$$

and thus

$$\begin{aligned}
(\vec{y}-\vec{x})\cdot(\vec{y}-\vec{x}) &= (\vec{y}\,'-\vec{x}\,')\cdot(\vec{y}\,'-\vec{x}\,')\\
\vec{y}\cdot\vec{y}-2\,\vec{y}\cdot\vec{x}+\vec{x}\cdot\vec{x} &= \vec{y}\,'\cdot\vec{y}\,'-2\,\vec{y}\,'\cdot\vec{x}\,'+\vec{x}\,'\cdot\vec{x}\,'\\
\vec{y}\cdot\vec{y}-2\,\vec{y}\cdot\vec{x}+\vec{x}\cdot\vec{x} &= \vec{y}\cdot\vec{y}-2\,\vec{y}\,'\cdot\vec{x}\,'+\vec{x}\cdot\vec{x}\\
-2\,\vec{y}\cdot\vec{x} &= -2\,\vec{y}\,'\cdot\vec{x}\,'\\
\vec{y}\cdot\vec{x} &= \vec{y}\,'\cdot\vec{x}\,'
\end{aligned}$$

$$\vec{y}\cdot\vec{x} = m(\vec{y})\cdot m(\vec{x}) \quad \text{for every } \vec{x},\, \vec{y}$$

Motion that fixes origin is linear

There exists an orthonormal basis $\beta = (\vec{e}_1, \ldots, \vec{e}_n)$ in $\mathcal{V}$

It is mapped by m to vectors $\beta' = (\vec{e}_1{}', \ldots, \vec{e}_n{}')$ as

$$\vec{e}_i{}' = m(\vec{e}_i), \quad i = 1, \ldots, n$$

which are also an orthonormal basis

$$\vec{e}_i{}' \cdot \vec{e}_j{}' = m(\vec{e}_i) \cdot m(\vec{e}_j) = \vec{e}_i \cdot \vec{e}_j$$

Take a general vector $\vec{x}$ and its image $m(\vec{x})$

$$\vec{x} \;=\; x_1\vec{e}_1 + \ldots + x_n\vec{e}_n \qquad m(\vec{x}) \;=\; x_1'\vec{e}_1{}' + \ldots + x_n'\vec{e}_n{}'$$

Use the scalar product to compute the coordinates

$$\vec{e}_i \cdot \vec{x} = \sum_{j=1}^{n} x_j(\vec{e}_i \cdot \vec{e}_j) = x_i(\vec{e}_i \cdot \vec{e}_i) + \sum_{j \neq i}^{n} x_j(\vec{e}_i \cdot \vec{e}_j) = x_i\, 1 + \sum_{j \neq i}^{n} x_j\, 0 = x_i$$

$$x_{i\beta} = \vec{e}_i \cdot \vec{x} = m(\vec{e}_i) \cdot m(\vec{x}) = \vec{e}_i{}' \cdot \vec{x}' = x'_{i\beta'}$$

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Motion that fixes origin is linear

For every $\vec{x}$, $m(\vec{x})$ can be obtained in the following way

1. choose an orthonormal basis $\beta = (\vec{e}_1, \ldots, \vec{e}_n)$ in $\mathcal{V}$
2. find coordinates of $\vec{x}$ w.r.t. β: $x_{i\beta} = \vec{e}_i \cdot \vec{x}$
3. construct $m(\vec{x}) = x_{1\beta}\, m(\vec{e}_1) + \ldots + x_{n\beta}\, m(\vec{e}_n)$

Linearity:

$$(a\,\vec{x} + b\,\vec{y})_{i\beta} \;=\; \vec{e}_i \cdot (a\,\vec{x} + b\,\vec{y}) = a(\vec{e}_i \cdot \vec{x}) + b(\vec{e}_i \cdot \vec{y}) = a\,x_{i\beta} + b\,y_{i\beta}$$

$$\begin{aligned}
m(a\,\vec{x} + b\,\vec{y}) &= (a\,x_{1\beta} + b\,y_{1\beta})\,m(\vec{e}_1) + \ldots + (a\,x_{n\beta} + b\,y_{n\beta})\,m(\vec{e}_n) \\
&= (a\,x_{1\beta}\,m(\vec{e}_1) + \ldots + a\,x_{n\beta}\,m(\vec{e}_n)) \\
&+ (b\,y_{1\beta}\,m(\vec{e}_1) + \ldots + b\,y_{n\beta}\,m(\vec{e}_n)) \\
&= a(x_{1\beta}\,m(\vec{e}_1) + \ldots + x_{n\beta}\,m(\vec{e}_n)) \\
&+ b(y_{1\beta}\,m(\vec{e}_1) + \ldots + y_{n\beta}\,m(\vec{e}_n)) \\
&= a\,m(\vec{x}) + b\,m(\vec{y})
\end{aligned}$$

Motion that fixes origin is linear

Express vectors of the the basis $\beta' = m(\beta)$ in the basis β:

$$\vec{e}_i{}' = \sum_{j=1}^{n} a_{ji}\vec{e}_j, \quad \vec{x} = \sum_{i=1}^{n} x_i\vec{e}_i$$

Find the coordinates y_j of $m(\vec{x})$ w.r.t. the basis β:

$$\begin{aligned} m(\vec{x}) = \sum_{j=1}^{n} y_j\vec{e}_j = \sum_{i=1}^{n} x_i m(\vec{e}_i) &= \sum_{i=1}^{n} x_i'\vec{e}_i{}' \\ &= \sum_{i=1}^{n} x_i(\sum_{j=1}^{n} a_{ji}\vec{e}_j) \\ &= \sum_{j=1}^{n}(\sum_{i=1}^{n} a_{ji}x_i)\vec{e}_j \end{aligned}$$

$$\vec{e}_1{}'_\beta = \sum_{j=1}^{n} a_{j1}\vec{e}_j,$$

$$y_j = \sum_{i=1}^{n} a_{ji}x_i \qquad \Rightarrow \qquad \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

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Motion that fixes origin is linear

$$\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$m(\vec{x}) = \mathtt{A}\vec{x}$$

It holds $\forall \vec{x}, \vec{y} \in \mathcal{V}$: $m(\vec{x}) \cdot m(\vec{y}) = \vec{x} \cdot \vec{y}$

Writing $\vec{x} \cdot \vec{y} = \vec{x}^\top \vec{y}$ we get

$$\vec{x}^\top \mathtt{A}^\top \mathtt{A}\, \vec{y} = \vec{x}^\top \vec{y} \qquad \text{for all } \vec{x}, \vec{y} \in \mathcal{V}$$

In particular, for the standard basis $\vec{x} = \vec{e}_i, \vec{y} = \vec{e}_j$ we get

$$\vec{e}_i^\top \mathtt{A}^\top \mathtt{A}\, \vec{e}_j = \begin{pmatrix} a_{i1} & \ldots & a_{in} \end{pmatrix} \begin{pmatrix} a_{1j} \\ \vdots \\ a_{nj} \end{pmatrix} = \vec{e}_i \cdot \vec{e}_j = \left\langle \begin{array}{l} 0 \text{ for } i = j \\ \\ 1 \text{ for } i \neq j \end{array} \right.$$

A is an orthogonal matrix

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Motion that fixes origin is represented by an orhonormal matrix

Choose a Cartesian coordinate system $(O; A, B, C)$, $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$.
Motion m, which preserves the origin, maps it to the Cartesian coordinate system, $(O; m(A), m(B), m(C))$, $(\vec{e}_1{}', \vec{e}_2{}', \vec{e}_3{}')$

$$\vec{e}_i{}' = \mathtt{A}\,\vec{e}_i$$

for some orthogonal matrix A.
Every motion m is orientation preserving:

$$\forall O, A, B, C \in \mathcal{P}\colon\ o(O; A, B, C) = o(m(O); m(A), m(B), m(C))$$

i.e. it maps right-handed bases to the right-handed bases.

$$\vec{e}_3 = \vec{e}_1 \times \vec{e}_2 \ \xrightarrow{m}\ \vec{e}_3{}' = \vec{e}_1{}' \times \vec{e}_2{}'$$

$$\begin{aligned} 1 &= \vec{e}_3{}' \cdot \vec{e}_3{}' = \vec{e}_3{}' \cdot (\vec{e}_1{}' \times \vec{e}_2{}') = \det\begin{pmatrix}\vec{e}_1{}' & \vec{e}_2{}' & \vec{e}_3{}'\end{pmatrix} \\ &= \det(\mathtt{A}\begin{pmatrix}\vec{e}_1{}' & \vec{e}_2{}' & \vec{e}_3{}'\end{pmatrix}) = \det\mathtt{A}\det\begin{pmatrix}\vec{e}_1{}' & \vec{e}_2{}' & \vec{e}_3{}'\end{pmatrix} = \det\mathtt{A}\,1 \\ &= \det\mathtt{A} \end{aligned}$$

$\mathtt{A}$ is othonormal

Every motion is a linear transformation followed by a translation

Be m motion. Motion m maps $\vec{0}$ somewhere. Introduce $\vec{b} = m(\vec{0})$ and the new motion $m'(\vec{x}) = (t_{-\vec{b}} \circ m)(\vec{x})$, where $t_{-\vec{b}}(\vec{x}) = \vec{x} - \vec{b}$ is a translation. Motion m' maps origin to origin:

$$m'(\vec{0}) = (t_{-\vec{b}} \circ m)(\vec{0}) = t_{-\vec{b}}(m(\vec{0})) = t_{-\vec{b}}(\vec{b}) = \vec{b} - \vec{b} = \vec{0}$$

and therefore there is an orthonormal matrix $\mathtt{A}$ such that

$$\begin{aligned} m'(\vec{x}) &= \mathtt{A}\,\vec{x} \\ m(\vec{x}) - \vec{b} &= \mathtt{A}\,\vec{x} \\ m(\vec{x}) &= \mathtt{A}\,\vec{x} + \vec{b} \end{aligned}$$

Motion representation by a 4×4 matrix

For the motion in the three-dimensional Euclidean world

$$m(\vec{x}) \;=\; \mathtt{A}\,\vec{x} + \vec{b}$$

$\mathtt{A} \in \mathbb{R}^3$ and $\vec{b} \in \mathbb{R}^3$

can be written in a concise way as

$$m(\vec{x}) = \mathtt{A}\,\vec{x} + \vec{b} = \begin{bmatrix} & \mathtt{A} & & \vec{b} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \vec{x} \\ 1 \end{bmatrix}$$

non-linear mapping

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