Advanced Robotics

Lecture 3

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Space, coordinates, motion

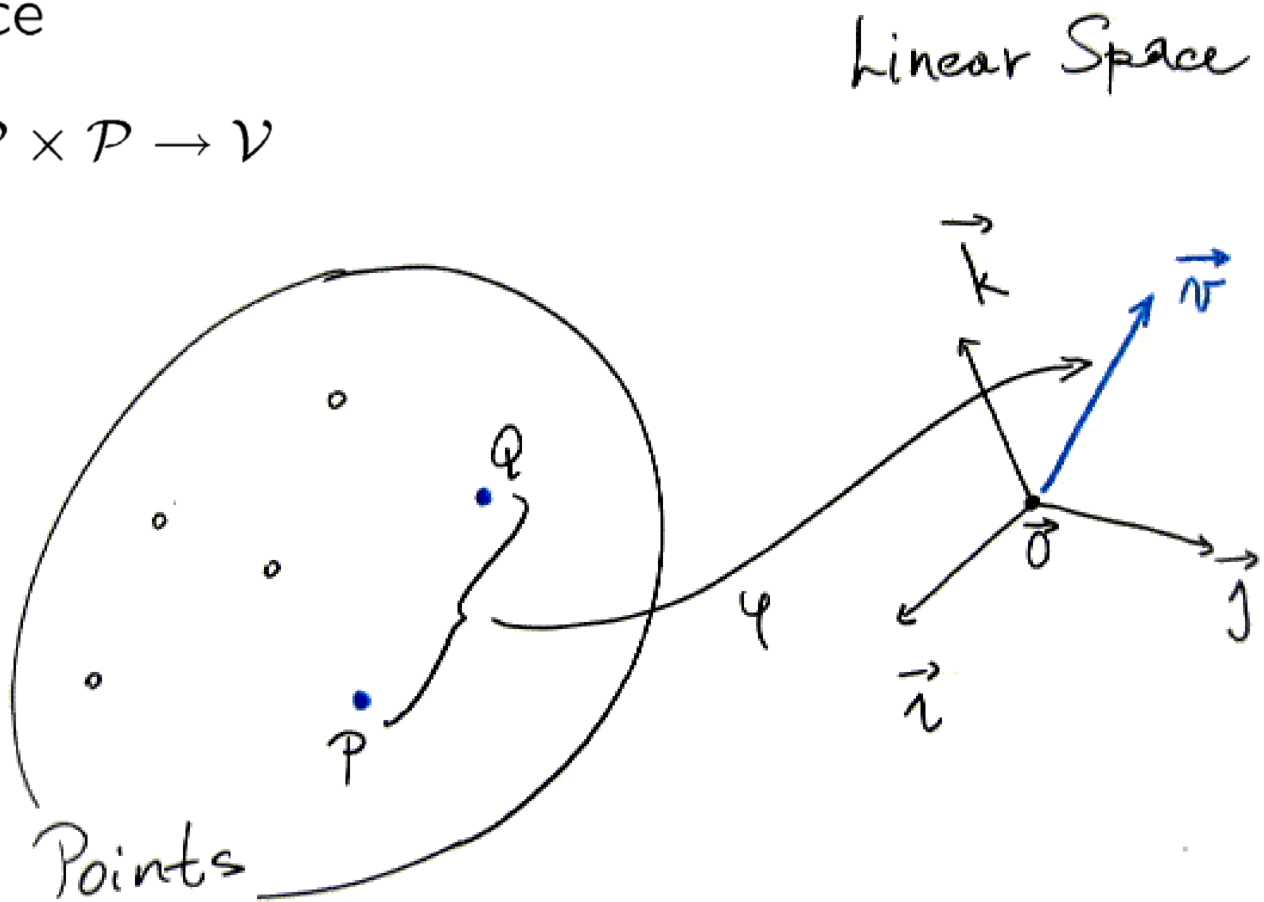
Affine Space

triple $\mathcal{A} = (\mathcal{P}, \mathcal{V}, \varphi)$

$\mathcal{P}$ . . . set of points

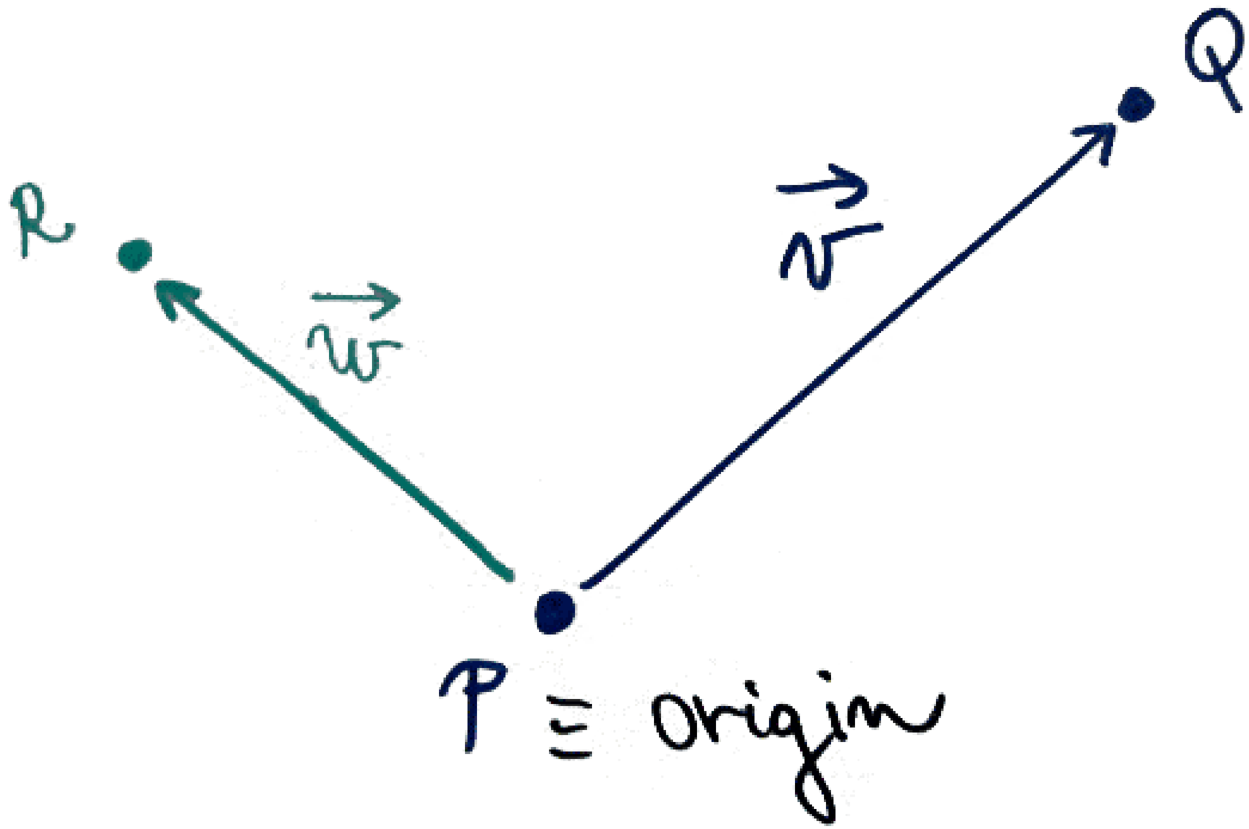
$\mathcal{V}$ . . . linear space

φ . . . function $\mathcal{P} \times \mathcal{P} \rightarrow \mathcal{V}$



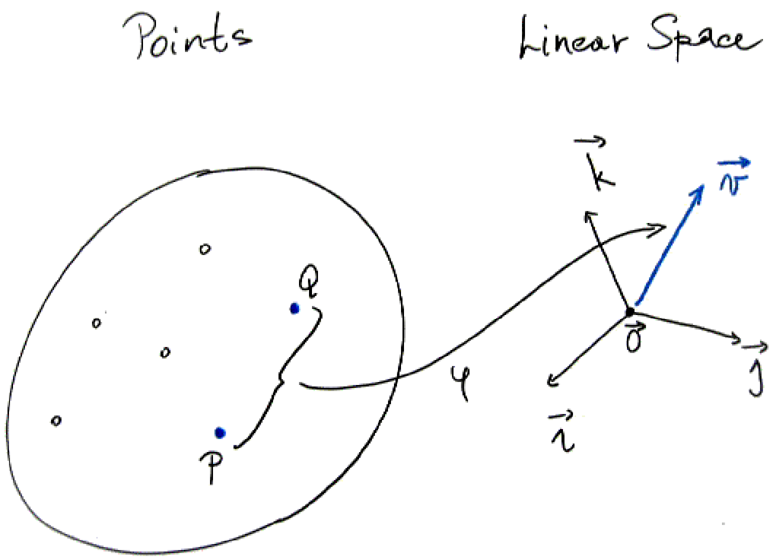
We always define φ when drawing vectors

$\varphi : \mathcal{P} \times \mathcal{P} \rightarrow \mathcal{V}$



Affine Space

triple $\mathcal{A} = (\mathcal{P}, \mathcal{V}, \varphi)$



$\mathcal{A}$ ... set of points

$\mathcal{V}$ ... linear space

φ ... function $\mathcal{P} \times \mathcal{P} \rightarrow \mathcal{V}$

1. $\forall P, Q \in \mathcal{P}\ \exists! \vec{v} \in \mathcal{V} : \varphi(P, Q) = \vec{v}$ (i.e. φ is a function)

2. $\forall P \in \mathcal{P}\ \forall \vec{v} \in \mathcal{V}\ \exists! Q \in \mathcal{P} : \varphi(P, Q) = \vec{v}$ ($\psi(P, \vec{v}) \rightarrow Q$ is a function)

3. $\forall P, Q, R \in \mathcal{P} : \varphi(P, Q) + \varphi(Q, R) - \varphi(P, R) = 0$ ($\triangle$-equality)

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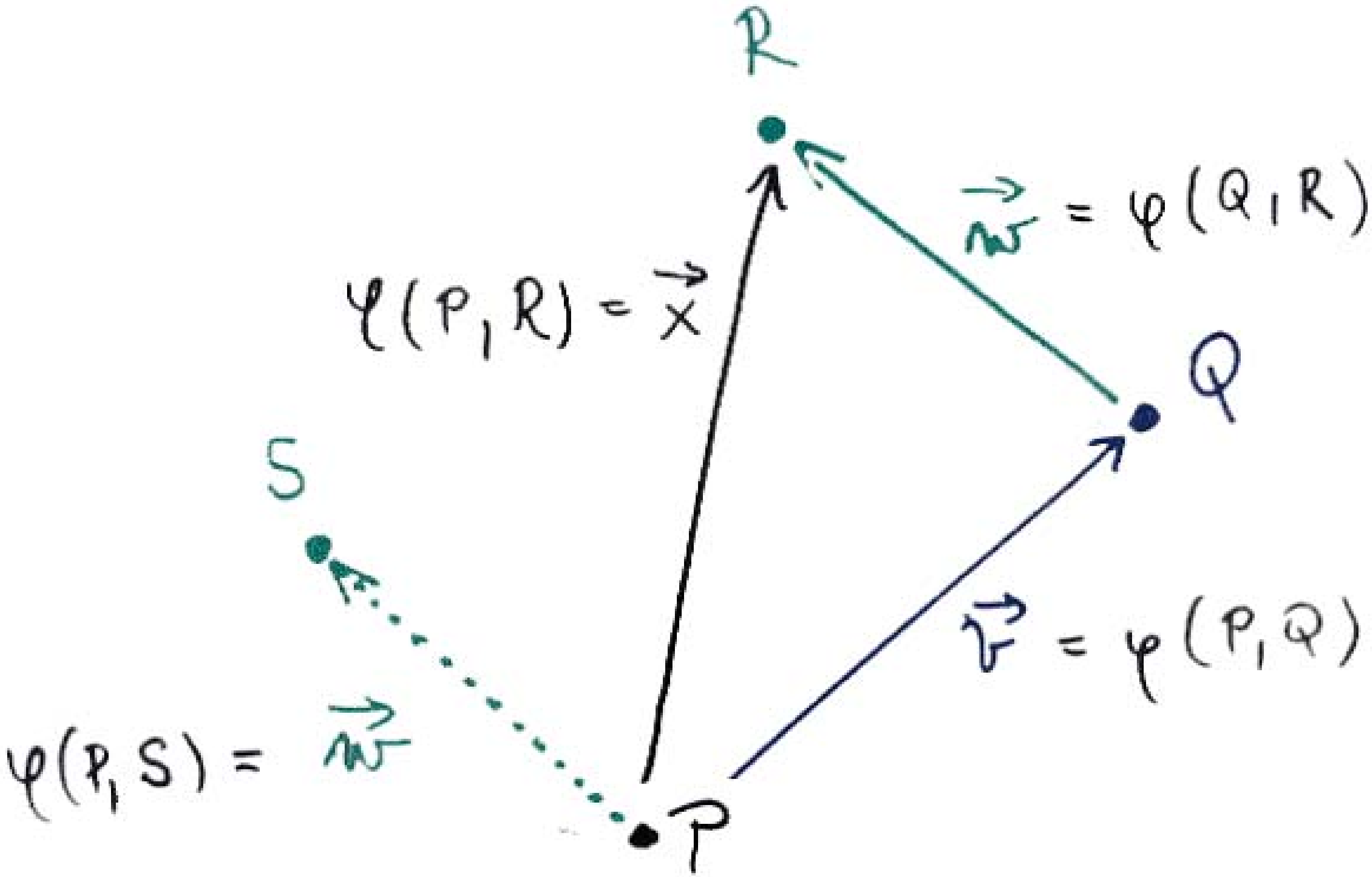
△-equality in detail

$$\forall P, Q, R \in \mathcal{P} : \varphi(P,Q) + \varphi(Q,R) - \varphi(P,R) = 0$$

$$\vec{v} = \varphi(P,Q)$$

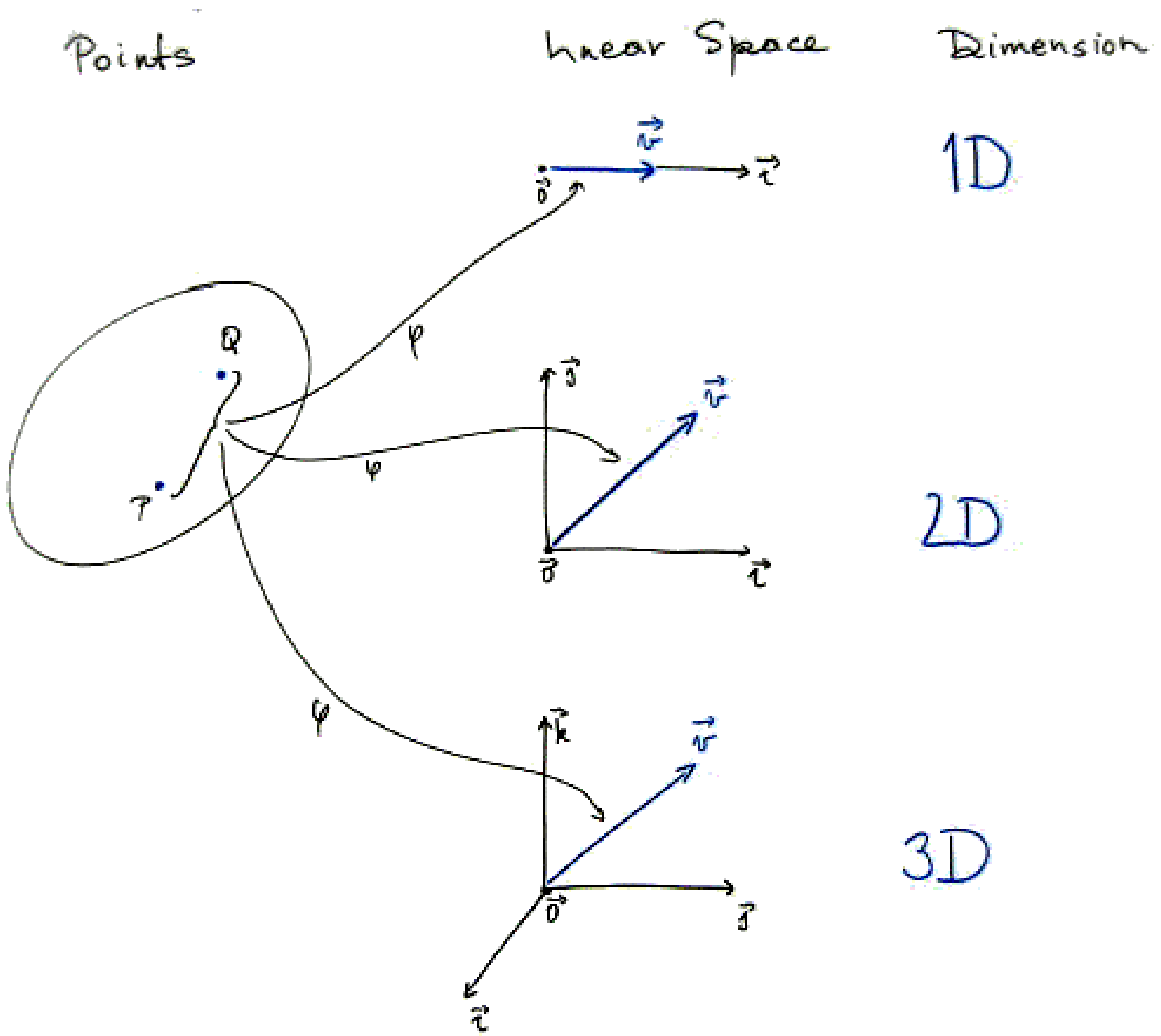
$$\vec{w} = \varphi(Q,R)$$

$$\vec{x} = \varphi(P,R)$$



Affine Space

Dimension of an affine space: $\dim \mathcal{A} \overset{def}{\equiv} \dim \mathcal{V}$



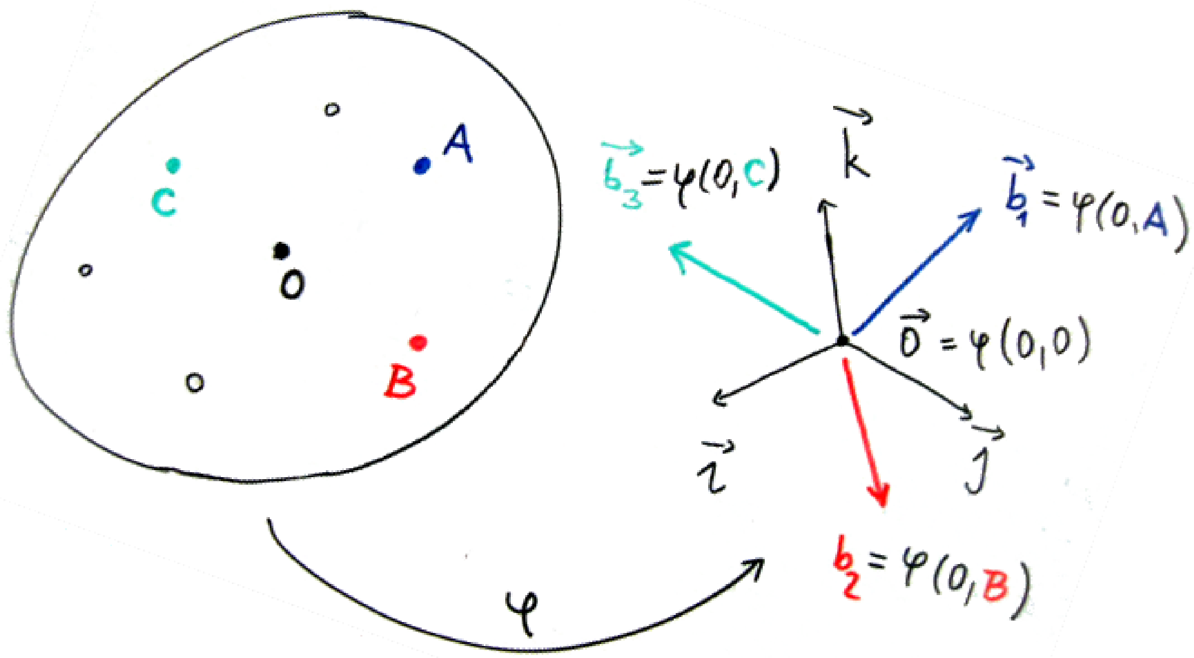
Affine Space

Affine coordinate system

1. ordered 4-tuple of points (O, A, B, C) and an
2. ordered basis $(\vec{b}_1, \vec{b}_2, \vec{b}_3)$

such that there is φ

1. satisfying conditions of the affine space
2. $\varphi(O, A) = \vec{b}_1$
   $\varphi(O, B) = \vec{b}_2$
   $\varphi(O, C) = \vec{b}_3$

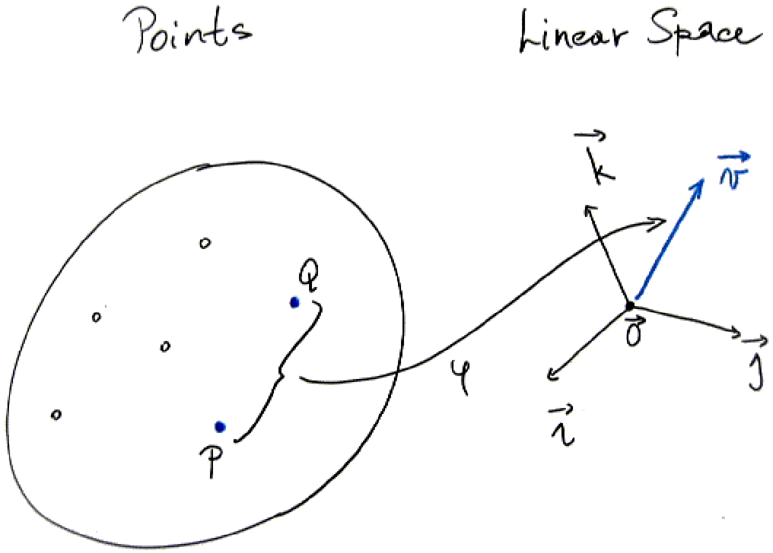


1. $\vec{x} = \varphi(O, X)_{(\vec{b}_1, \vec{b}_2, \vec{b}_3)}$ is a 1:1 corresponcence betwen $\mathcal{P}$ and $\mathcal{V}$
2. $\varphi(P, Q) = \varphi(O, Q) - \varphi(O, P)$

Euclidean Space

$\stackrel{def}{\equiv}$ an affine space + the Euclidean distance d

quadruple $\mathcal{E} = (\mathcal{P}, \mathcal{V}, \varphi, d)$



$\mathcal{P}$ ... set of points

$\mathcal{V}$ ... linear space

φ ... function $\mathcal{P} \times \mathcal{P} \rightarrow \mathcal{V}$

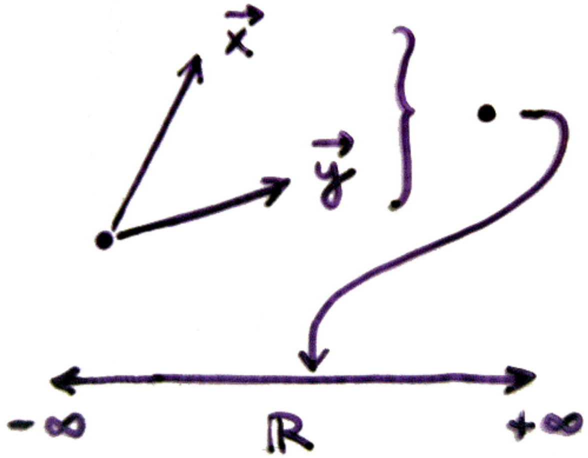
d ... distance $\mathcal{P} \times \mathcal{P} \rightarrow \mathbb{R}$ (the real numbers)

$$d(P, Q) = \sqrt{\vec{v} \cdot \vec{v}} = \sqrt{\varphi(P, Q) \cdot \varphi(P, Q))}$$

where $\cdot : \mathcal{V} \times \mathcal{V} \rightarrow \mathbb{R}$ is the "Euclidean" scalar product (spheres are "spherical")

Euclidean Space

Scalar product $\cdot : \mathcal{V} \times \mathcal{V} \rightarrow \mathbb{R}$



For every $\vec{x}, \vec{y}, \vec{z} \in \mathcal{V}$

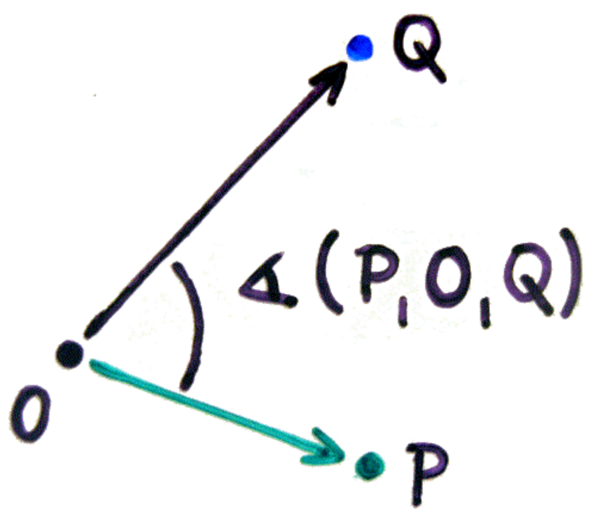
1. $\vec{x} \cdot \vec{y} = \vec{y} \cdot \vec{x}$
2. $(\vec{x} + \vec{y}) \cdot \vec{z} = \vec{x} \cdot \vec{z} + \vec{y} \cdot \vec{z}$
3. $\vec{x} \cdot \vec{x} \geq 0$
4. $\vec{x} \cdot \vec{x} = 0 \Leftrightarrow \vec{x} = \vec{0}$
5. $(\lambda \vec{x}) \cdot \vec{y} = \lambda (\vec{x} \cdot \vec{y})$ for all $\lambda \in \mathbb{R}$.

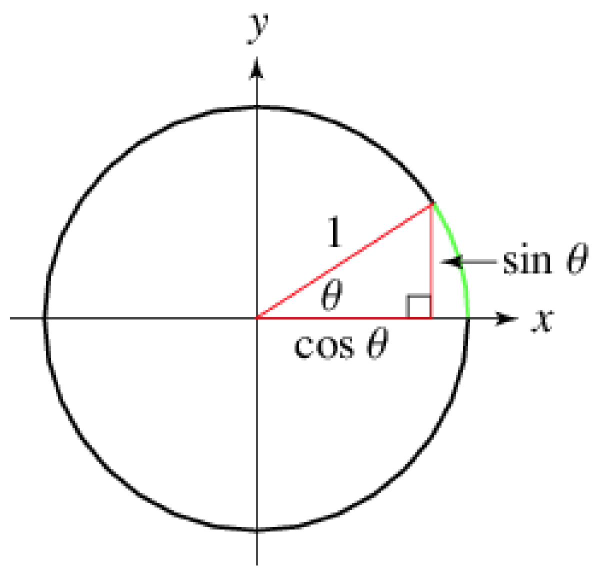
Euclidean Space

The "Euclidean" scalar product ($\exists$ non-Euclidean s. p's!)

$$\vec{x} \cdot \vec{y} = \sum_i^n x_i \, y_i$$

$\vec{x} = [x_1, x_2, \ldots, x_n]^\top$, $\vec{y} = [y_1, y_2, \ldots, y_n]^\top$ in the *standard basis*





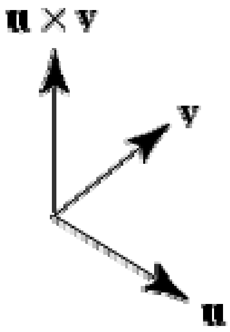
Angle

$$\cos \measuredangle(P,O,Q) = \frac{\varphi(O,P) \cdot \varphi(O,Q)}{\sqrt{\varphi(O,P) \cdot \varphi(O,P)}\sqrt{\varphi(O,Q) \cdot \varphi(O,Q)}}$$

Right-handed bases in a 3-dimensional linear space

An orthonormal basis $(\vec{b}_1, \vec{b}_2, \vec{b}_3)$ in a 3-dimensional linear space is right-handed if

$$\vec{b}_3 = \vec{b}_1 \times \vec{b}_2$$



$\times$ ... vector product of $\vec{a} = [a_1, a_2, a_3]^\top$, $\vec{b} = [b_1, b_2, b_3]^\top$

$$\vec{a} \times \vec{b} = \left[\det \begin{bmatrix} a_2 & a_3 \\ b_2 & b_3 \end{bmatrix}, -\det \begin{bmatrix} a_1 & a_3 \\ b_1 & b_3 \end{bmatrix}, \det \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \right]$$

It holds

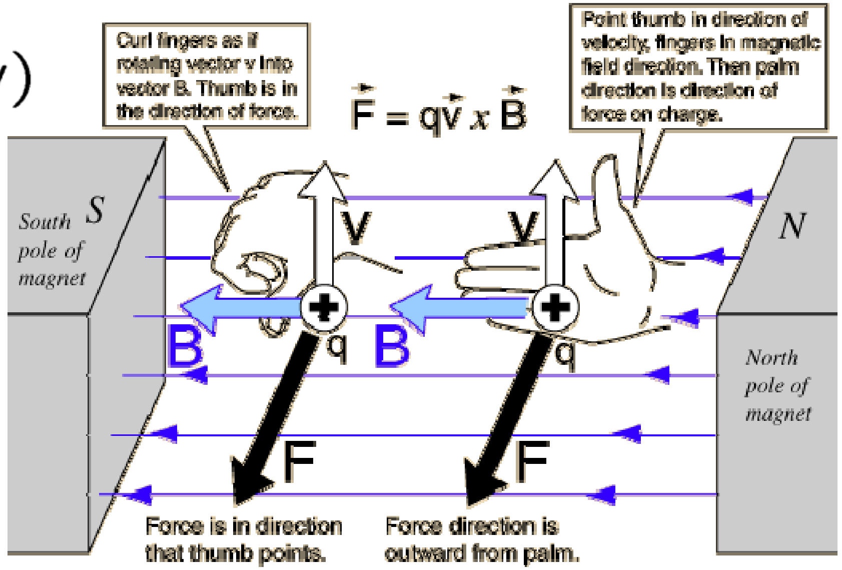
$$\begin{aligned} \vec{c} \cdot (\vec{a} \times \vec{b}) &= c_1 \det \begin{bmatrix} a_2 & a_3 \\ b_2 & b_3 \end{bmatrix} - c_2 \det \begin{bmatrix} a_1 & a_3 \\ b_1 & b_3 \end{bmatrix} + c_3 \det \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \\ &= \det \begin{bmatrix} \vec{a} & \vec{b} & \vec{c} \end{bmatrix} \end{aligned}$$

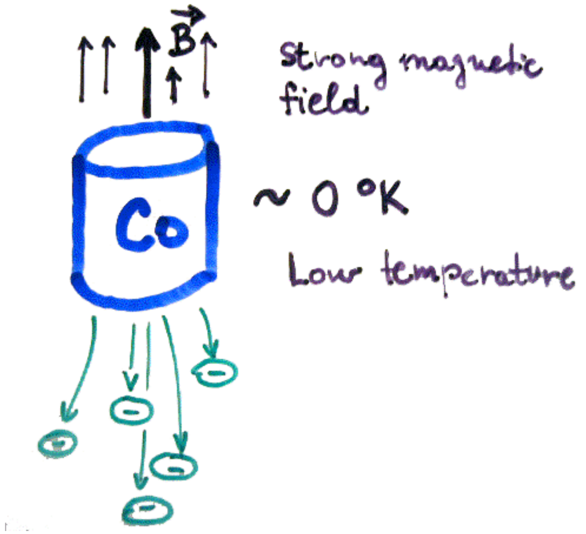
Orientation in 3-dimensional Euclidean space

Orientation $o(O; A, B, C) = \{-1, 0, 1\}$

a 4-tuple of points $(O; A, B, C)$ is positively oriented iff

1. $\varphi(O, A) \equiv \vec{B}$ (magnetic intenzity)
2. $\varphi(O, B) \equiv \vec{F}$ (force)
3. $\varphi(O, C) \equiv \vec{v}$ (velocity)





There are two equal choices of orientations in the macro-world but only one in the micro-world of quantum mechanics:

"When we put cobalt atoms in an extremely strong magnetic field, more disintegration electrons go down than up"

R. Feynman. Six Not-So-Easy Pieces. Addison-Wesley 1997.

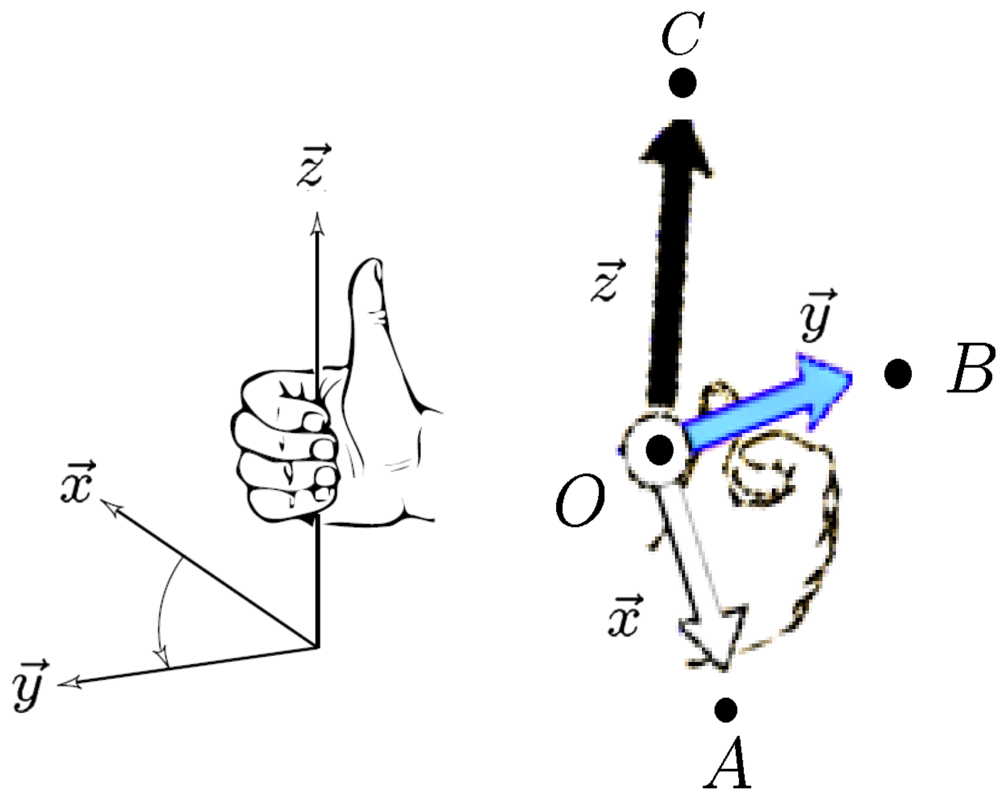
Orientation in 3-dimensional Euclidean space

Orientation $o(O; A, B, C) = \{-1, 0, 1\}$

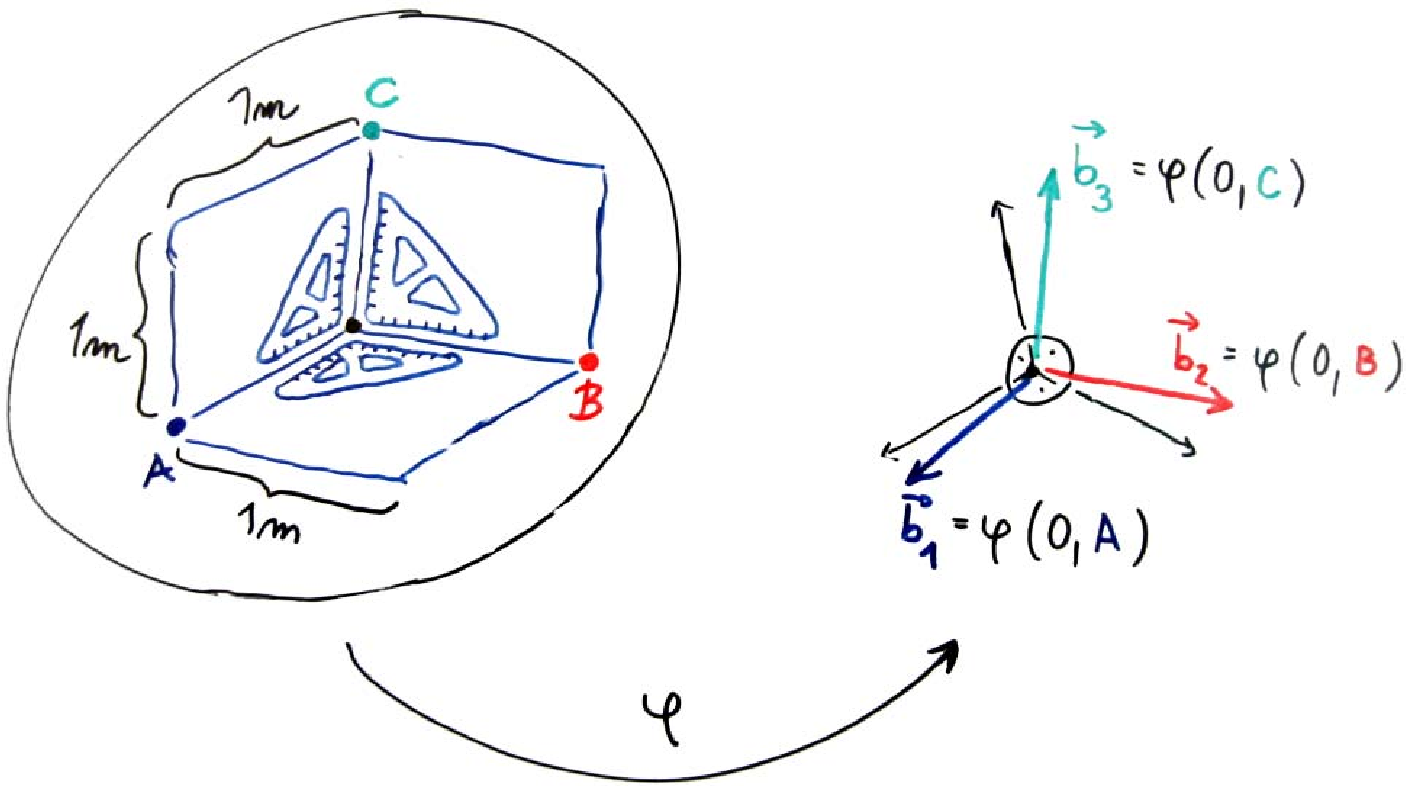
a 4-tuple of points $(O; A, B, C)$ is positively oriented iff the right hand corresponds to

1. $\varphi(O, A) \equiv \vec{x}$ (magnetic intenzity)
2. $\varphi(O, B) \equiv \vec{y}$ (force)
3. $\varphi(O, C) \equiv \vec{z}$ (velocity)

as in the following figure



Euclidean Space



Cartesian coordinate system (O, A, B, C) & $(\vec{b}_1, \vec{b}_2, \vec{b}_3)$

1. $(O; A, B, C)$ . . . positively oriented vertices of a unit cube
2. $(\vec{b}_1, \vec{b}_2, \vec{b}_3) = (\vec{i}, \vec{j}, \vec{k})$ . . . the standard basis (is right-handed and orthonormal)

Euclidean Space

Robot kinematics is done in a

subspace of the three-dimensional Euclidean space

line . . . 1D

plane . . . 2D

space . . . 3D

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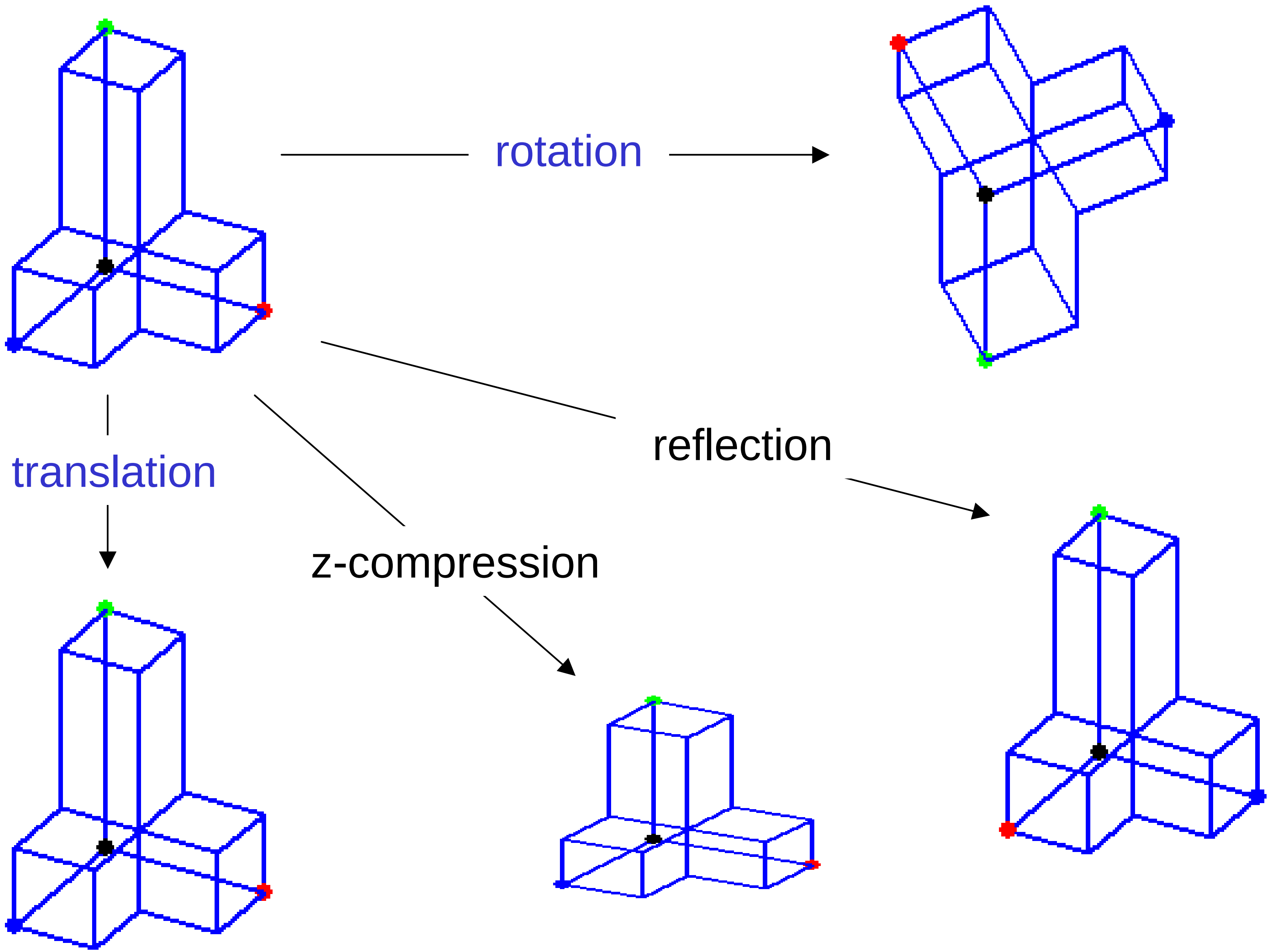
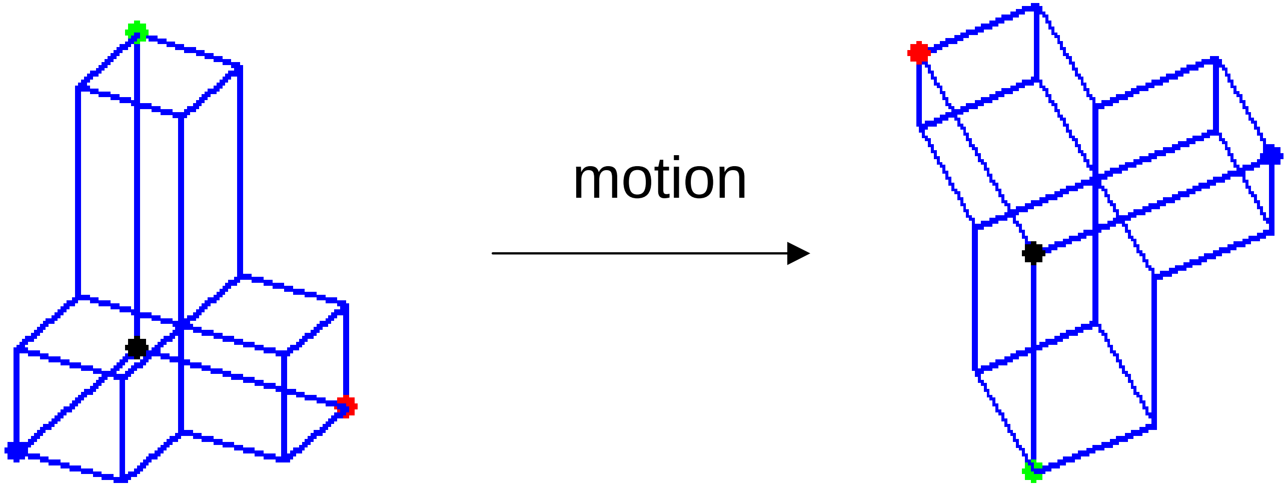
Motion in the 3-dimensional Euclidean space



Motion in the 3-dimensional Euclidean space



We say that $m : \mathcal{P} \rightarrow \mathcal{P}$ is *motion* iff for every $O, A, B, C \in \mathcal{P}$

1. distances preserved: $d(m(A), m(B)) = d(A, B)$

2. orientation preserved: $o(O; A, B, C) = o(m(O); m(A), m(B), m(C))$

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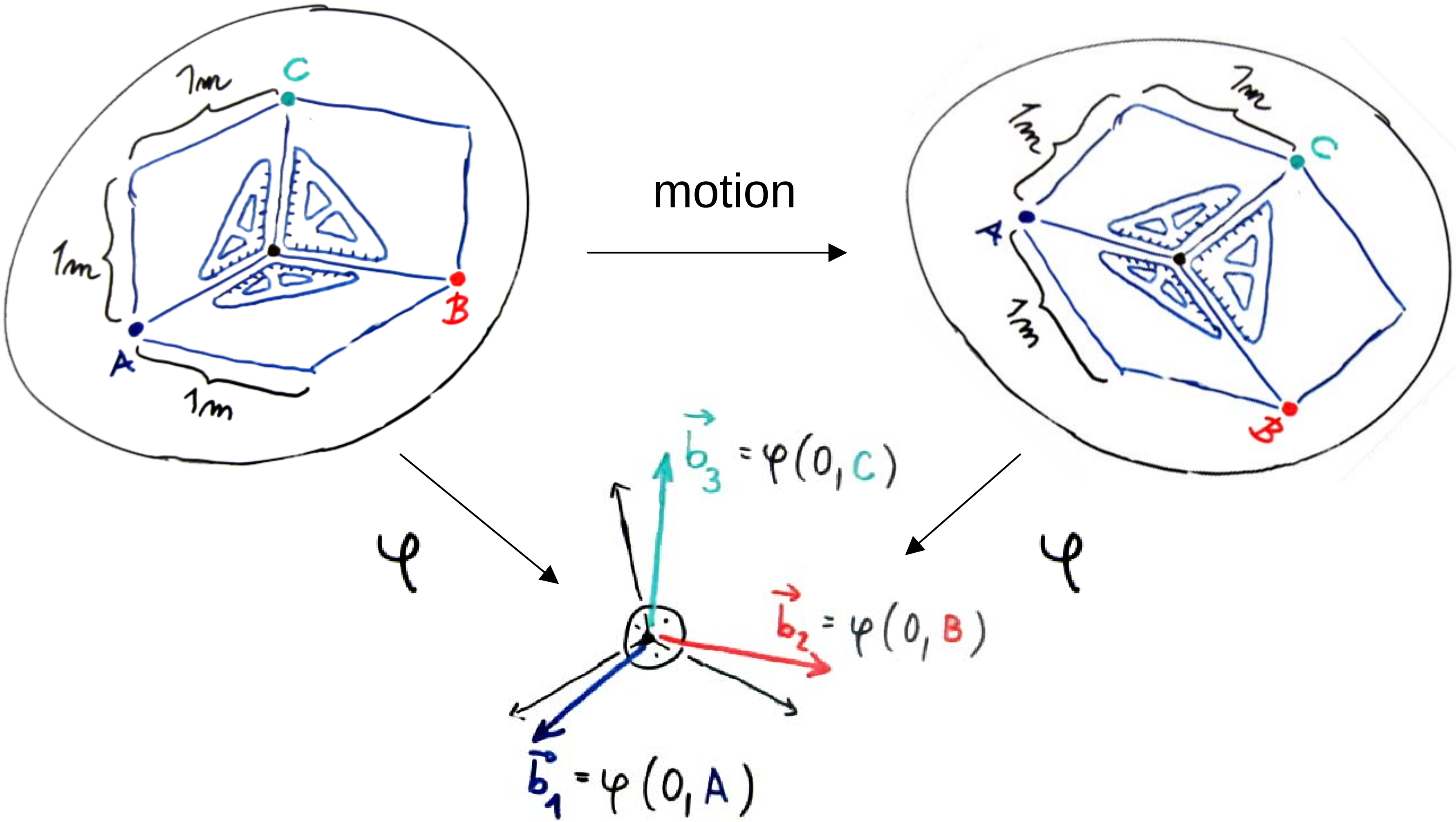
Motion in the 3-dimensional Euclidean space

Motion $m : \mathcal{P} \rightarrow \mathcal{P}$ maps Cartesian coordinate systems to Cartesian coordinate systems

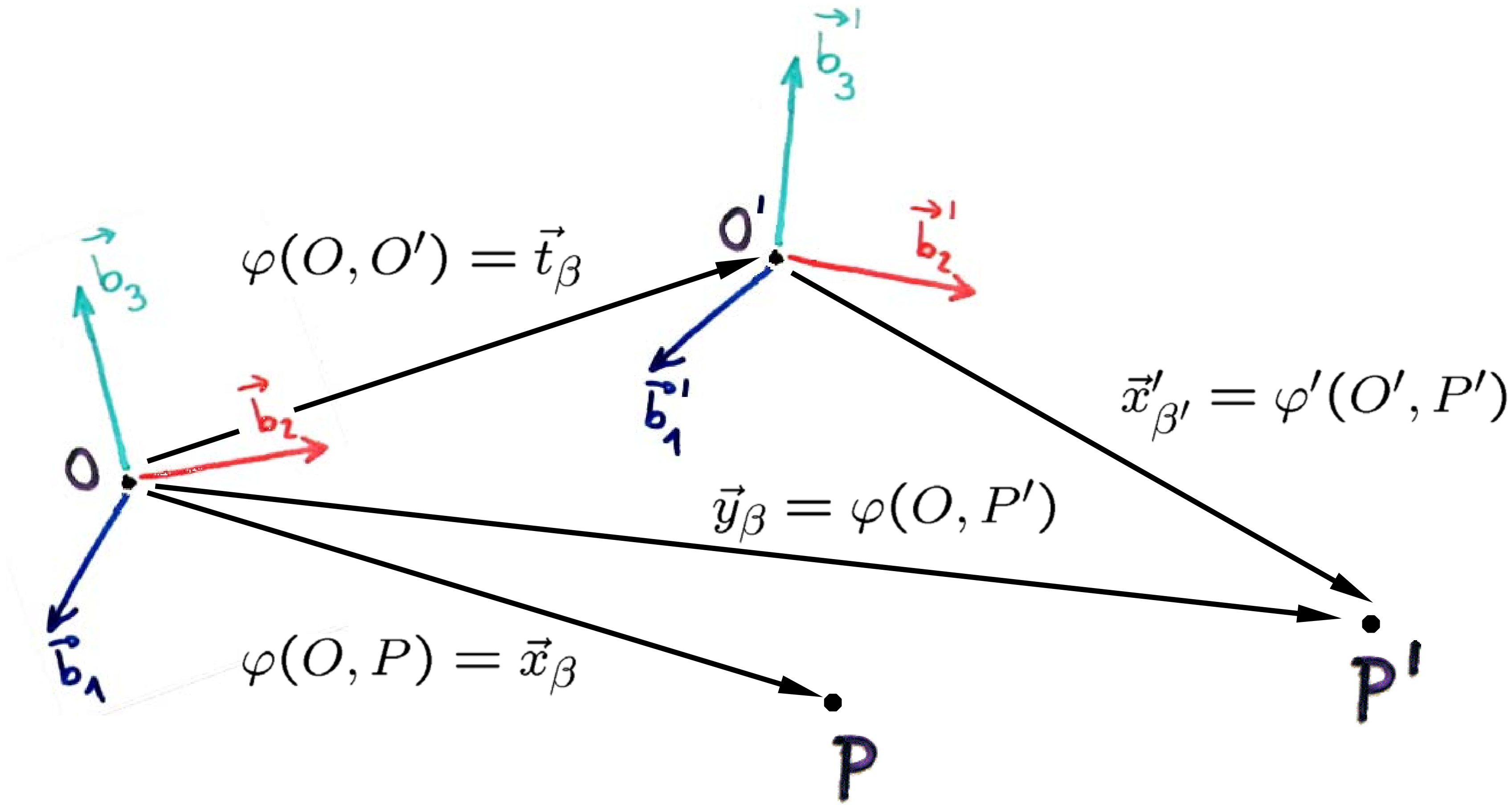


Motion in the 3-dimensional Euclidean space

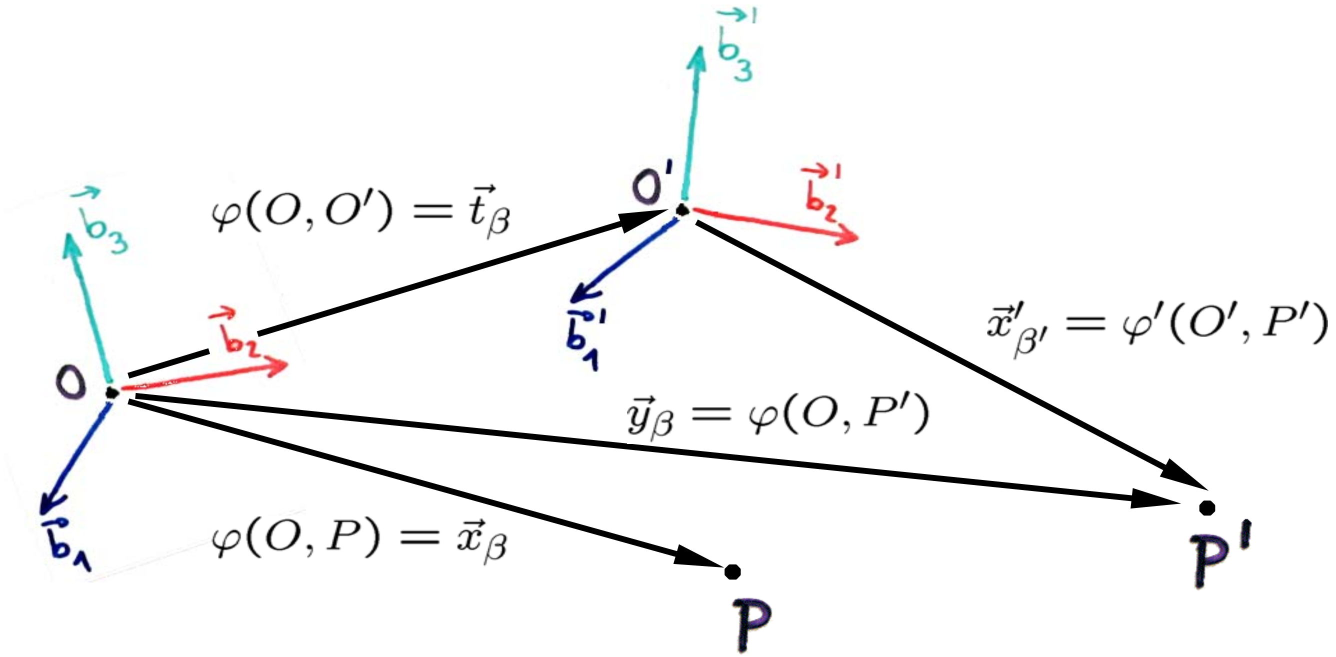
$$P' = m(P) : \vec{x}_\beta \to \vec{y}_\beta$$

$$\beta = (\vec{b}_1, \vec{b}_2, \vec{b}_3)$$

$$\beta' = (\vec{b}'_1, \vec{b}'_2, \vec{b}'_3)$$



Motion in the 3-dimensional Euclidean space



Motion: $\vec{x}'_{\beta'} := \vec{x}_\beta$

$\triangle$-equality: $\vec{y}_\beta = \vec{x}'_\beta + \vec{t}_\beta$

$$\vec{y}_\beta = \vec{x}'_\beta + \vec{t}_\beta = \mathtt{R}\,\vec{x}'_{\beta'} + \vec{t}_\beta = \mathtt{R}\,\vec{x}_\beta + \vec{t}_\beta = \begin{pmatrix} \vec{b}'_{1\beta} & \vec{b}'_{2\beta} & \vec{b}'_{3\beta} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_\beta + \vec{t}_\beta$$

$$\vec{x}_\beta = \mathtt{R}^{-1}(\vec{y}_\beta - \vec{t}_\beta) = \begin{pmatrix} \vec{b}_{1\beta'} & \vec{b}_{2\beta'} & \vec{b}_{3\beta'} \end{pmatrix} (\vec{y}_\beta - \vec{t}_\beta)$$

$\mathtt{R} = \begin{pmatrix} \vec{b}'_{1\beta} & \vec{b}'_{2\beta} & \vec{b}'_{3\beta} \end{pmatrix}$ . . . a rotation matrix

Rigid motion as a coordinate transformation

Motion induces a mapping of the associated linear space into itself

$$
\begin{array}{ccc}
\overset{P}{\bullet} & \xrightarrow{m} & \overset{P'}{\bullet} \\
\varphi \downarrow & & \downarrow \varphi \\
\vec{x} & \xrightarrow{m} & \vec{x}\,'
\end{array}
$$

With a fixed origin O and mapping φ,

$m : \mathcal{P} \rightarrow \mathcal{P}$ induces a mapping $m : \mathcal{V} \rightarrow \mathcal{V}$

$$
\begin{aligned}
P' &= m(P) \\
\vec{x}\,' &= m(\vec{x})
\end{aligned}
\qquad\qquad
\begin{aligned}
\vec{x} &= \varphi(O, P) \\
\vec{x}\,' &= \varphi(O, m(P))
\end{aligned}
$$

Motion characterisation in $\mathcal{V}$

$$\begin{aligned}\vec{x} &= \varphi(O,P)\\ \vec{x}' &= \varphi(O,m(P))\\ \vec{y} &= \varphi(O,Q)\\ \vec{y}' &= \varphi(O,m(Q))\end{aligned}$$

$m:\mathcal{E}\rightarrow\mathcal{E}$ *motion* $\Rightarrow \forall P,Q\in\mathcal{P}:$

$$d(m(P),m(Q)) = d(P,Q)$$

$$\begin{aligned}d(P,Q) &= \sqrt{\varphi(P,Q)\cdot\varphi(P,Q)}\\ &= \sqrt{(\varphi(O,Q)-\varphi(O,P))\cdot(\varphi(O,Q)-\varphi(O,P))}\\ &= \sqrt{(\vec{y}-\vec{x})\cdot(\vec{y}-\vec{x})}\\ d(m(P),m(Q)) &= \sqrt{(\vec{y}'-\vec{x}')\cdot(\vec{y}'-\vec{x}')}\end{aligned}$$

$\forall\vec{x},\vec{y}\in L$: $\quad\sqrt{(\vec{y}-\vec{x})\cdot(\vec{y}-\vec{x})} = \sqrt{(\vec{y}'-\vec{x}')\cdot(\vec{y}'-\vec{x}')}$

Motion characterisation in $\mathcal{V}$

$$d(m(P), m(Q)) \;=\; d(P, Q) \quad \text{for every } P,\, Q$$

$$\sqrt{(\vec{y} - \vec{x}) \cdot (\vec{y} - \vec{x})} \;=\; \sqrt{(\vec{y}\,' - \vec{x}\,') \cdot (\vec{y}\,' - \vec{x}\,')}$$

$$\vec{x} \cdot \vec{x} \geq 0 \Rightarrow \quad \Updownarrow$$

$$(\vec{y} - \vec{x}) \cdot (\vec{y} - \vec{x}) \;=\; (\vec{y}\,' - \vec{x}\,') \cdot (\vec{y}\,' - \vec{x}\,')$$

$$(\vec{y} - \vec{x}) \cdot (\vec{y} - \vec{x}) \;=\; (m(\vec{y}) - m(\vec{x})) \cdot (m(\vec{y}) - m(\vec{x})) \quad \text{for every } \vec{x},\, \vec{y}$$

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Motion is not linear in general

$f : \mathcal{V} \rightarrow \mathcal{V}$ is linear iff $\forall \alpha, \beta \in \mathbb{R}$ and $\forall \vec{x}, \vec{y} \in \mathcal{V}$ holds

$$f(\alpha\,\vec{x} + \beta\,\vec{y}) = \alpha\,f(\vec{x}) + \beta\,f(\vec{y})$$

A general motion m is not linear since, e.g.,

Translation $m_{\vec{o}}(\vec{x}) = \vec{x} + \vec{o}$ is a motion since

$$\begin{aligned}(m_{\vec{o}}(\vec{y}) - m_{\vec{o}}(\vec{x})) \cdot (m_{\vec{o}}(\vec{y}) - m_{\vec{o}}(\vec{x})) &= (\vec{y} + \vec{o} - \vec{x} - \vec{o}) \cdot (\vec{y} + \vec{o} - \vec{x} - \vec{o}) \\ &= (\vec{y} - \vec{x}) \cdot (\vec{y} - \vec{x})\end{aligned}$$

but

$$m_{\vec{o}}(\alpha\,\vec{x}) = \alpha\,\vec{x} + \vec{o} \neq \alpha\,\vec{x} + \alpha\,\vec{o} = \alpha\,m_{\vec{o}}(\vec{x})$$

for $\vec{o} \neq \vec{0}$ . . . translations are not linear

But there are motions that are linear as we will show ...

Motion that fixes origin preserves the scalar product

For every motion holds

$$(\vec{y}-\vec{x})\cdot(\vec{y}-\vec{x}) \;=\; (m(\vec{y})-m(\vec{x}))\cdot(m(\vec{y})-m(\vec{x})) \quad \text{for every } \vec{x},\, \vec{y}$$

$$(\vec{y}-\vec{x})\cdot(\vec{y}-\vec{x}) \;=\; (\vec{y}\,'-\vec{x}\,')\cdot(\vec{y}\,'-\vec{x}\,')$$

moreover, if $m(O)=O$, i.e. $m(\vec{0})=\vec{0}$, then for $\vec{x}=\vec{0}$, we get

$$\vec{y}\,'\cdot\vec{y}\,' \;=\; \vec{y}\cdot\vec{y} \quad \text{for every } \vec{y} \text{ and } \vec{y}\,'=m(\vec{y})$$

and thus

$$\begin{aligned}
(\vec{y}-\vec{x})\cdot(\vec{y}-\vec{x}) &= (\vec{y}\,'-\vec{x}\,')\cdot(\vec{y}\,'-\vec{x}\,')\\
\vec{y}\cdot\vec{y}-2\,\vec{y}\cdot\vec{x}+\vec{x}\cdot\vec{x} &= \vec{y}\,'\cdot\vec{y}\,'-2\,\vec{y}\,'\cdot\vec{x}\,'+\vec{x}\,'\cdot\vec{x}\,'\\
\vec{y}\cdot\vec{y}-2\,\vec{y}\cdot\vec{x}+\vec{x}\cdot\vec{x} &= \vec{y}\cdot\vec{y}-2\,\vec{y}\,'\cdot\vec{x}\,'+\vec{x}\cdot\vec{x}\\
-2\,\vec{y}\cdot\vec{x} &= -2\,\vec{y}\,'\cdot\vec{x}\,'\\
\vec{y}\cdot\vec{x} &= \vec{y}\,'\cdot\vec{x}\,'
\end{aligned}$$

$$\vec{y}\cdot\vec{x} \;=\; m(\vec{y})\cdot m(\vec{x}) \quad \text{for every } \vec{x},\, \vec{y}$$

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Motion that fixes origin is linear

There exists an orthonormal basis $\beta = (\vec{e}_1, \ldots, \vec{e}_n)$ in $\mathcal{V}$

It is mapped by m to vectors $\beta' = (\vec{e}_1{}', \ldots, \vec{e}_n{}')$ as

$$\vec{e}_i{}' = m(\vec{e}_i), \quad i = 1, \ldots, n$$

which are also an orthonormal basis

$$\vec{e}_i{}' \cdot \vec{e}_j{}' = m(\vec{e}_i) \cdot m(\vec{e}_j) = \vec{e}_i \cdot \vec{e}_j$$

Take a general vector $\vec{x}$ and its image $m(\vec{x})$

$$\vec{x} \;=\; x_1 \vec{e}_1 + \ldots + x_n \vec{e}_n \qquad m(\vec{x}) \;=\; x_1' \vec{e}_1{}' + \ldots + x_n' \vec{e}_n{}'$$

Use the scalar product to compute the coordinates

$$\vec{e}_i \cdot \vec{x} = \sum_{j=1}^{n} x_j (\vec{e}_i \cdot \vec{e}_j) = x_i (\vec{e}_i \cdot \vec{e}_i) + \sum_{j \neq i}^{n} x_j (\vec{e}_i \cdot \vec{e}_j) = x_i \, 1 + \sum_{j \neq i}^{n} x_j \, 0 = x_i$$

$$x_{i\beta} = \vec{e}_i \cdot \vec{x} = m(\vec{e}_i) \cdot m(\vec{x}) = \vec{e}_i{}' \cdot \vec{x}' = x'_{i\beta'}$$

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Motion that fixes origin is linear

For every $\vec{x}$, $m(\vec{x})$ can be obtained in the following way

1. choose an orthonormal basis $\beta = (\vec{e}_1, \ldots, \vec{e}_n)$ in $\mathcal{V}$
2. find coordinates of $\vec{x}$ w.r.t. β: $x_{i\beta} = \vec{e}_i \cdot \vec{x}$
3. construct $m(\vec{x}) = x_{1\beta}\, m(\vec{e}_1) + \ldots + x_{n\beta}\, m(\vec{e}_n)$

Linearity:

$$(a\,\vec{x} + b\,\vec{y})_{i\beta} = \vec{e}_i \cdot (a\,\vec{x} + b\,\vec{y}) = a(\vec{e}_i \cdot \vec{x}) + b(\vec{e}_i \cdot \vec{y}) = a\,x_{i\beta} + b\,y_{i\beta}$$

$$\begin{aligned}
m(a\,\vec{x} + b\,\vec{y}) &= (a\,x_{1\beta} + b\,y_{1\beta})\, m(\vec{e}_1) + \ldots + (a\,x_{n\beta} + b\,y_{n\beta})\, m(\vec{e}_n) \\
&= (a\,x_{1\beta}\, m(\vec{e}_1) + \ldots + a\,x_{n\beta}\, m(\vec{e}_n)) \\
&+ (b\,y_{1\beta}\, m(\vec{e}_1) + \ldots + b\,y_{n\beta}\, m(\vec{e}_n)) \\
&= a(x_{1\beta}\, m(\vec{e}_1) + \ldots + x_{n\beta}\, m(\vec{e}_n)) \\
&+ b(y_{1\beta}\, m(\vec{e}_1) + \ldots + y_{n\beta}\, m(\vec{e}_n)) \\
&= a\, m(\vec{x}) + b\, m(\vec{y})
\end{aligned}$$

Motion that fixes origin is linear

Express vectors of the the basis $\beta' = m(\beta)$ in the basis β:

$$m(\vec{e}_j) = \sum_{i=1}^{n} a_{ij}\,\vec{e}_i, \quad \vec{x} = \sum_{i=1}^{n} x_i\,\vec{e}_i, \quad m(\vec{x}) = \sum_{j=1}^{n} x'_j\, m(\vec{e}_j)$$

Find the coordinates y_k of $m(\vec{x})$ w.r.t. the basis β:

$$\begin{aligned} m(\vec{x}) = \sum_{k=1}^{n} y_k\,\vec{e}_k = \sum_{j=1}^{n} x'_j\, m(\vec{e}_j) &= \sum_{j=1}^{n} x_j\, m(\vec{e}_j) \\ &= \sum_{j=1}^{n} x_j\,(\sum_{i=1}^{n} a_{ij}\,\vec{e}_i) \\ &= \sum_{i=1}^{n}(\sum_{j=1}^{n} a_{ij}\,x_j)\vec{e}_i \end{aligned}$$

and since the coordinates of a vector w.r.t. a basis are unique, $k = i \Rightarrow y_k = \sum_{j=1}^{n} a_{ij}\,x_j$

$$\vec{e}_1{}'_{\beta} = \sum_{i=1}^{n} a_{i1}\vec{e}_i$$

$$y_i = \sum_{j=1}^{n} a_{ij}x_j \quad \equiv \quad \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

Motion that fixes origin is linear

$$\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$m(\vec{x}) = \mathtt{A}\vec{x}$$

It holds $\forall \vec{x}, \vec{y} \in \mathcal{V}$: $m(\vec{x}) \cdot m(\vec{y}) = \vec{x} \cdot \vec{y}$

Writing $\vec{x} \cdot \vec{y} = \vec{x}^\top \vec{y}$ we get

$$\vec{x}^\top \mathtt{A}^\top \mathtt{A}\, \vec{y} = \vec{x}^\top \vec{y} \qquad \text{for all } \vec{x}, \vec{y} \in \mathcal{V}$$

In particular, for the standard basis $\vec{x} = \vec{e}_i, \vec{y} = \vec{e}_j$ we get

$$\vec{e}_i^\top \mathtt{A}^\top \mathtt{A}\, \vec{e}_j = \begin{pmatrix} a_{i1} & \ldots & a_{in} \end{pmatrix} \begin{pmatrix} a_{1j} \\ \vdots \\ a_{nj} \end{pmatrix} = \vec{e}_i \cdot \vec{e}_j = \left\langle \begin{array}{l} 0 \text{ for } i = j \\ \\ 1 \text{ for } i \neq j \end{array} \right.$$

A is an orthogonal matrix

Motion that fixes origin is represented by an orhonormal matrix

Choose a Cartesian coordinate system $(O; A, B, C)$, $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$.
Motion m, which preserves the origin, maps it to the Cartesian coordinate system, $(O; m(A), m(B), m(C))$, $(\vec{e}_1{}', \vec{e}_2{}', \vec{e}_3{}')$

$$\vec{e}_i{}' = \mathtt{A}\, \vec{e}_i$$

for some orthogonal matrix A.
Every motion m is orientation preserving:

$$\forall O, A, B, C \in \mathcal{P}\colon\ o(O; A, B, C) = o(m(O); m(A), m(B), m(C))$$

i.e. it maps right-handed bases to the right-handed bases.

$$\vec{e}_3 = \vec{e}_1 \times \vec{e}_2 \ \xrightarrow{m}\ \vec{e}_3{}' = \vec{e}_1{}' \times \vec{e}_2{}'$$

$$\begin{aligned} 1 &= \vec{e}_3{}' \cdot \vec{e}_3{}' = \vec{e}_3{}' \cdot (\vec{e}_1{}' \times \vec{e}_2{}') = \det \begin{pmatrix} \vec{e}_1{}' & \vec{e}_2{}' & \vec{e}_3{}' \end{pmatrix} \\ &= \det (\mathtt{A} \begin{pmatrix} \vec{e}_1{}' & \vec{e}_2{}' & \vec{e}_3{}' \end{pmatrix}) = \det \mathtt{A} \det \begin{pmatrix} \vec{e}_1{}' & \vec{e}_2{}' & \vec{e}_3{}' \end{pmatrix} = \det \mathtt{A}\, 1 \\ &= \det \mathtt{A} \end{aligned}$$

$\mathtt{A}$ is othonormal

Every motion is a linear transformation followed by a translation

Be m motion. Motion m maps $\vec{0}$ somewhere. Introduce $\vec{b} = m(\vec{0})$ and the new motion $m'(\vec{x}) = (t_{-\vec{b}} \circ m)(\vec{x})$, where $t_{-\vec{b}}(\vec{x}) = \vec{x} - \vec{b}$ is a translation. Motion m' maps origin to origin:

$$m'(\vec{0}) = (t_{-\vec{b}} \circ m)(\vec{0}) = t_{-\vec{b}}(m(\vec{0})) = t_{-\vec{b}}(\vec{b}) = \vec{b} - \vec{b} = \vec{0}$$

and therefore there is an orthonormal matrix $\mathtt{A}$ such that

$$\begin{aligned} m'(\vec{x}) &= \mathtt{A}\,\vec{x} \\ m(\vec{x}) - \vec{b} &= \mathtt{A}\,\vec{x} \\ m(\vec{x}) &= \mathtt{A}\,\vec{x} + \vec{b} \end{aligned}$$

Motion representation by a 4×4 matrix

For the motion in the three-dimensional Euclidean world

$$m(\vec{x}) \;=\; \mathtt{A}\,\vec{x} + \vec{b}$$

$\mathtt{A} \in \mathbb{R}^3$ and $\vec{b} \in \mathbb{R}^3$

can be written in a concise way as

$$m(\vec{x}) = \mathtt{A}\,\vec{x} + \vec{b} = \begin{bmatrix} & \mathtt{A} & & \vec{b} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \vec{x} \\ 1 \end{bmatrix}$$

non-linear mapping