Advanced Robotics

Lecture 4

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Motion and Screws

Motion and Screws

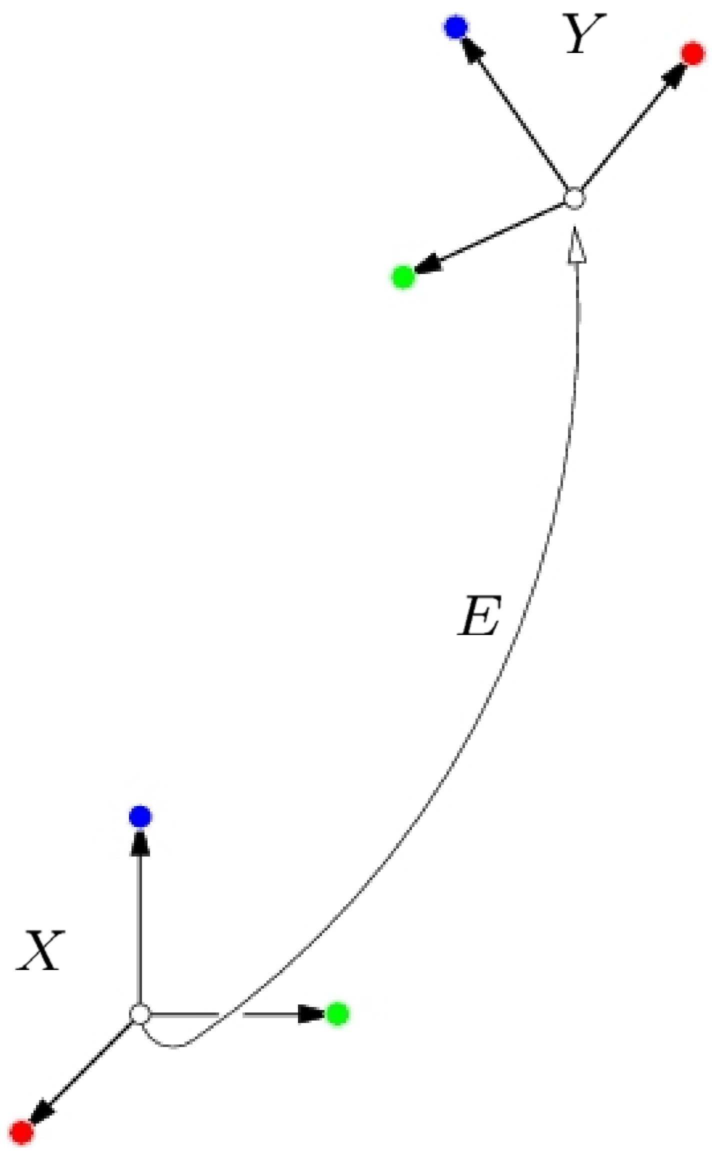
A rigid motion of a set of points $X = \begin{pmatrix} x & y & z & 1 \end{pmatrix}^\top$ into the set of points Y can be expressed by a Euclidean transform

$$\begin{aligned} E &= \begin{pmatrix} r & t \\ 0_3^\top & 1 \end{pmatrix}, \; r \in \mathbb{R}^{3\times 3}, \; t \in \mathbb{R}^3, \; r^\top r = r\,r^\top = I, \; \det r = 1 \\ Y &= E X \end{aligned}$$

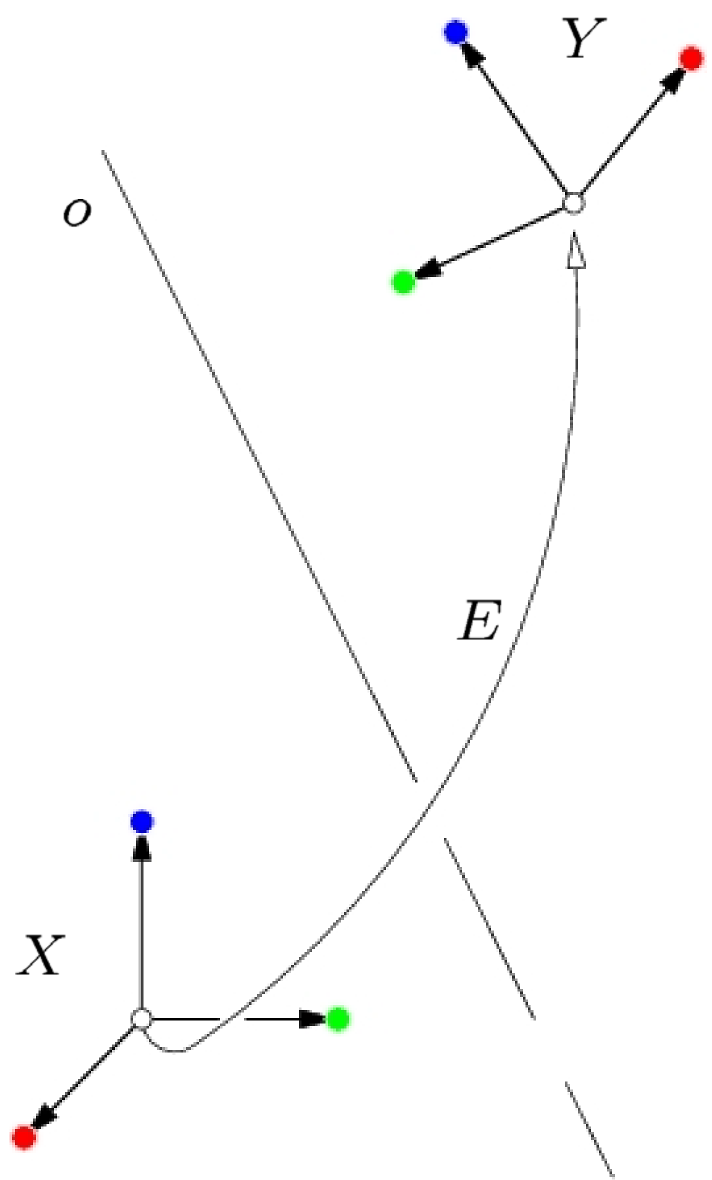
We will show that

Theorem 1 *For all motions E, there exists such a fixed line o, called the axis of motion, that E can be written as a composition of two one-parametric motions $E = E(s,\varphi) = E_2(s)\,E_1(\varphi)$, where $E_1(\varphi)$ is a rotation around o by angle φ and $E_2(s)$ is a translation along o by length s. Two-parametric motions $E(s,\varphi)$ are called* screws.

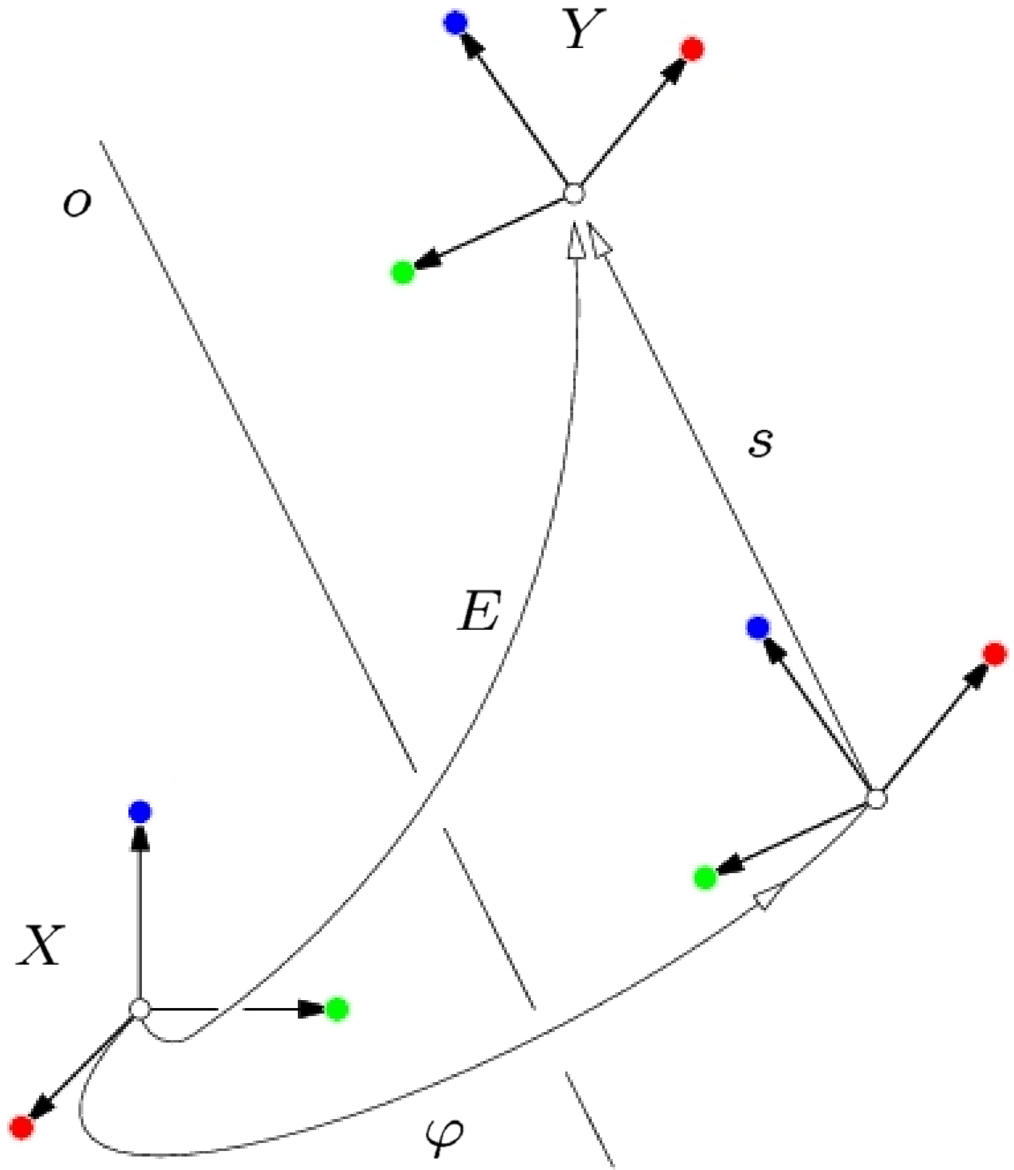
Motion and Screws



Motion and Screws



Motion and Screws



Motion and Screws

Assume motion

$$
\begin{aligned}
E &= \begin{pmatrix} r & t \\ 0_3^\top & 1 \end{pmatrix}, \; r \in \mathbb{R}^{3\times 3}, \; t \in \mathbb{R}^3, \; r^\top r = r\, r^\top = I, \; \det r = 1 \\
Y &= E X
\end{aligned}
$$

The existence of the axis does not depend on the choice of the coordinate system. Thus, we will choose a particular coordinate system with respect to which the E will take so simple form that the above statement will become evident. E will become a rotation around the z axis followed by a translation along the z axis.

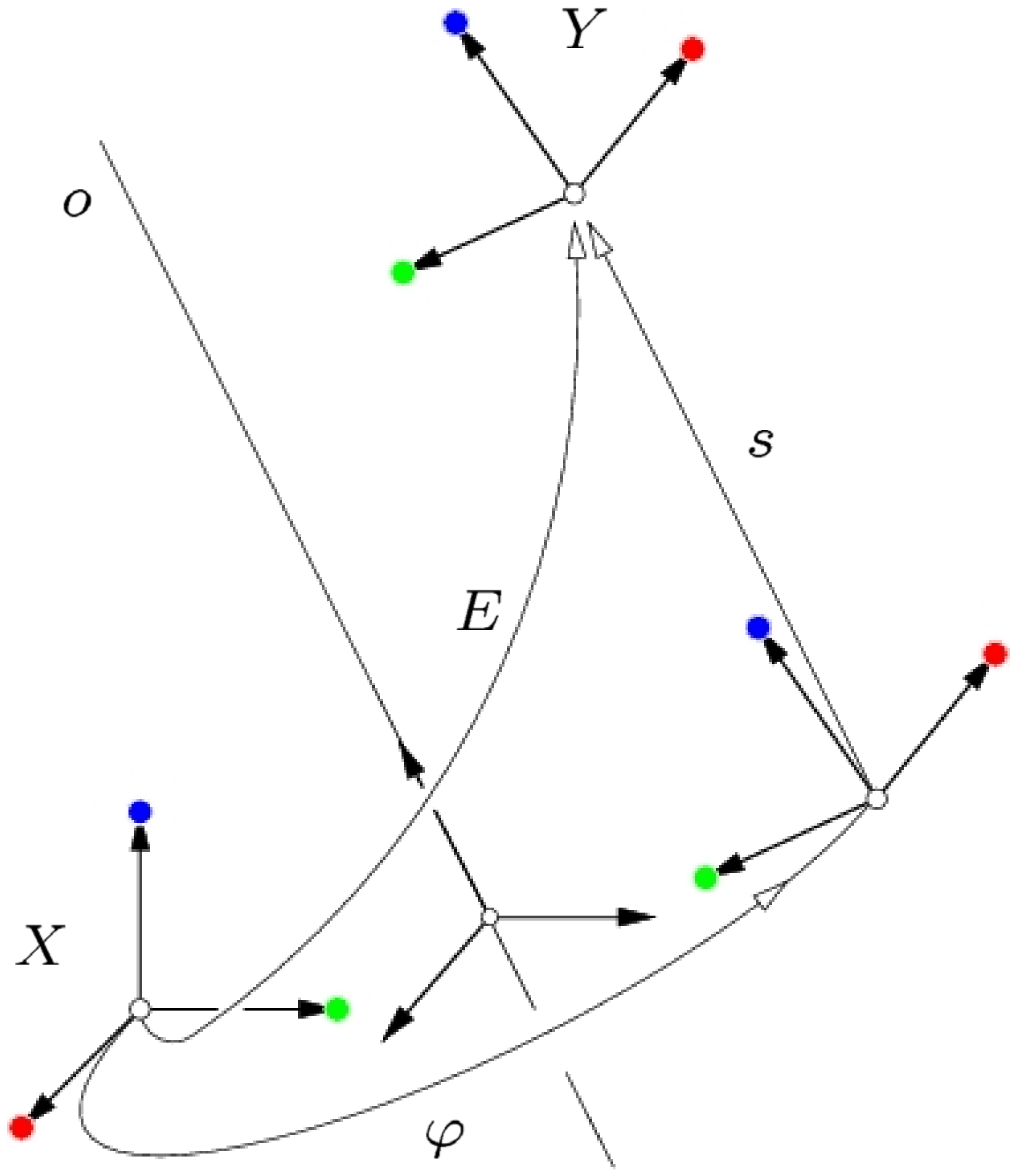
Let the change of the coordinate system be represented by

$$
\begin{aligned}
P &= \begin{pmatrix} R & T \\ 0_3^\top & 1 \end{pmatrix}^{-1}, \; R \in \mathbb{R}^{3\times 3}, \; T \in \mathbb{R}^3, \; R^\top R = R\, R^\top = I, \; \det R = 1 \\
X' &= P X \\
Y' &= P Y \\
Y' &= P E P^{-1} X' = P' X
\end{aligned}
$$

Thus

$$E' = P E P^{-1}$$

Motion and Screws



An elementary proof

Motion and Screws

Let us look at

$$E' = PEP^{-1}$$

Expand elements of E'

$$\begin{aligned}
P^{-1} &= \begin{bmatrix} R & T \\ 0_3^\top & 1 \end{bmatrix} \\
P &= \begin{bmatrix} R^{-1} & -R^{-1}T \\ 0_3^\top & 1 \end{bmatrix} \\
\\
E' &= PEP^{-1} \\
E' &= \begin{bmatrix} R^{-1} & -R^{-1}T \\ 0_3^\top & 1 \end{bmatrix} \begin{bmatrix} r & t \\ 0_3^\top & 1 \end{bmatrix} \begin{bmatrix} R & T \\ 0_3^\top & 1 \end{bmatrix} \\
E' &= \begin{bmatrix} R^{-1}r & R^{-1}t - R^{-1}T \\ 0_3^\top & 1 \end{bmatrix} \begin{bmatrix} R & T \\ 0_3^\top & 1 \end{bmatrix} \\
E' &= \begin{bmatrix} R^{-1}rR & R^{-1}rT + R^{-1}t - R^{-1}T \\ 0_3^\top & 1 \end{bmatrix} \\
E' &= \begin{bmatrix} R^{-1}rR & R^{-1}((r-I)T + t) \\ 0_3^\top & 1 \end{bmatrix}
\end{aligned}$$

and choose P to make E' as simple as possible.

It is not possible to make E' exacly diagonal but it is possible to make it almost diagonal.

Observation 1 1 is an eigenvalue of r.

To proof the statement, consider that

$$\|rx\|^2 = (rx)^\top (rx) = x^\top r^\top r x = x^\top (r^\top r) x = x^\top I x = x^\top x = \|x\|^2$$

If there is a $\lambda \in \mathbb{C}$ such that

$$r\,x = \lambda\,x$$

then $|\lambda| = 1$ since

$$|\lambda|^2 \|x\|^2 = \|\lambda x\|^2 = \|rx\|^2 = \|x\|^2$$

There is a real unit eigenvalue since r is a real matrix with the characteristic polynomial

$$\begin{aligned}
p(\lambda) &= \det(\lambda I - r) = \det\left(\begin{bmatrix} \lambda - r_{11} & -r_{12} & -r_{13} \\ -r_{21} & \lambda - r_{22} & -r_{23} \\ -r_{31} & -r_{32} & \lambda - r_{33} \end{bmatrix}\right) \\
&= \lambda^3 + (-r_{11} - r_{22} - r_{33})\lambda^2 \\
&+ (r_{11}r_{22} - r_{21}r_{12} + r_{11}r_{33} - r_{31}r_{13} + r_{22}r_{33} - r_{23}r_{32})\lambda \\
&+ r_{11}(r_{23}r_{32} - r_{22}r_{33}) - r_{21}(r_{32}r_{13} - r_{12}r_{33}) + r_{31}(r_{13}r_{22} - r_{12}r_{23}) \\
&= \lambda^3 - \mathrm{trace}(r)\lambda^2 + b\lambda - \det(r) = \lambda^3 - \mathrm{trace}(r)\lambda^2 + b\lambda - 1
\end{aligned}$$

which has always a real solution. Since $p(0) = -1$, $\lim_{\lambda\to\infty} p(\lambda) = +\infty$, and $p(\lambda)$ is a continuous function, it must cross the zero value at a positive real number. We conclude that 1 is an eigenvalue.

Alternatively, we can replace the analytic part of the previous proof by the following, purely algebraic, argument to further characterize the eigenvalues.

Algebraic equation

$$0 = p(\lambda) \;=\; \lambda^3 + a\lambda^2 + b\lambda - 1$$

has three roots in $\mathbb{C}$, which can be written as $\lambda_1, \lambda_2 = x + iy, \lambda_3 = x - iy$ with all $\lambda_1, x, y \in \mathbb{R}$, since the complex roots appear in complex conjugate pairs. Thus

$$\begin{aligned}
p(\lambda) &= (\lambda - \lambda_1)(\lambda - x - iy)(\lambda - x + iy) \\
&= (\lambda - \lambda_1)(\lambda^2 - 2x\lambda + x^2 + y^2) \\
&= \lambda^3 - (2x + \lambda_1)\lambda^2 + \lambda(x^2 + y^2 + 2x\lambda_1) - \lambda_1(x^2 + y^2) \\
&= \lambda^3 - (2x + \lambda_1)\lambda^2 + \lambda(\|\lambda_2\|^2 + 2x\lambda_1) - \lambda_1\|\lambda_2\|^2 \\
&= \lambda^3 - (2x + \lambda_1)\lambda^2 + \lambda(1 + 2x\lambda_1) + \lambda_1 1
\end{aligned}$$

We see that there holds for the absolute term

$$1 \;=\; \lambda_1 1$$

and therefore

$$1 \;=\; \lambda_1$$

Finally we get

$$p(\lambda) \;=\; \lambda^3 - (2x + 1)\lambda^2 + (2x + 1)\lambda - 1$$

Observation 2 There is a rotation matrix R reducing r to a rotation around the z axis.

There is a real unit eigenvector X such that $r\,X = X$. Construct a rotation matrix S with columns $R = \begin{bmatrix} R_1 & R_2 & X \end{bmatrix}$ by the Gram-Schmidt othogonalization. Observe that

$$R^{-1} r R \;=\; R^\top r R = \begin{bmatrix} R_1^\top r \\ R_2^\top r \\ X^\top r \end{bmatrix} \begin{bmatrix} R_1 & R_2 & X \end{bmatrix} = \begin{bmatrix} R_1^\top r\, R_1 & R_1^\top r\, R_2 & R_1^\top r\, X \\ R_2^\top r\, R_1 & R_2^\top r\, R_2 & R_2^\top r\, X \\ X^\top r\, R_1 & X^\top r\, R_2 & X^\top r\, X \end{bmatrix}$$

$$= \begin{bmatrix} R_1^\top r\, R_1 & R_1^\top r\, R_2 & R_1^\top X \\ R_2^\top r\, R_1 & R_2^\top r\, R_2 & R_2^\top X \\ X^\top R_1 & X^\top R_2 & X^\top X \end{bmatrix} = \begin{bmatrix} R_1^\top r\, R_1 & R_1^\top r\, R_2 & 0 \\ R_2^\top r\, R_1 & R_2^\top r\, R_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} e & f & 0 \\ g & h & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which is a rotation matrix since r and R are both rotations. Clearly, it is a rotation around the z axis. We used the fact that $r\,X = X$ and thus $X^\top = X^\top r^\top = X^\top r^{-1} \Rightarrow X^\top r = X^\top$.

Proof of the **Theorem 1**

In general

$$E' = \begin{pmatrix} R^{-1}rR & R^{-1}((r-I)T+t) \\ 0_3^\top & 1 \end{pmatrix}$$

We saw that we can choose $R = \begin{pmatrix} R_1 & R_2 & R_3 \end{pmatrix}$ to make $rR = R\begin{pmatrix} e & f & 0 \\ g & h & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Let us choose T to reduce t to a translation along the z axis.

Either $t \in \text{span}\,(r - I)$ or $t \notin \text{span}\,(r - I)$.

If $t \in \text{span}\,(r - I)$, then we can choose T such that $(r - I)T + t = 0$ to get

$$E' = \begin{pmatrix} R^{-1}rR & 0_3 \\ 0_3^\top & 1 \end{pmatrix} = \begin{pmatrix} e & f & 0 & 0 \\ g & h & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

which coresponds to a rotation around the z axis.

If $t \notin \text{span}\,(r - I)$, then we can distinguish two cases according to possible rank of $(r - I)$, which can be either 0 or 2.

First of all, $\text{rank}\,(r - I) < 3$ because $r \in \mathbb{R}^{3\times 3}$ and 1 is an eigenvalue of r, and therefore there is $\vec{x} \neq 0$ such that $r\,\vec{x} = \vec{x}$. We see that $(r - I)$ has a non-trivial nullspace since $(r - I)\,\vec{x} = r\,\vec{x} - \vec{x} = \vec{x} - \vec{x} = 0$, and thus $r - I$ cannot be of full rank.

Secondly, assume $\text{rank}\,(r - I) = 1$. Consider unit eigenvectors $\vec{x}_1, \vec{x}_2, \vec{x}_3$ of r corresponding to $1, \lambda_2, \lambda_3$. Then, $0, (x-1)+iy, (x-1)-iy$ are eigenvalues of $r-I$ since $(r-I)\vec{x}_1 = \vec{x}_1 - \vec{x}_1 = 0\,\vec{x}_1$, $(r - I)\vec{x}_2 = \lambda_2\vec{x}_2 - \vec{x}_2 = ((x - 1) + iy)\vec{x}_2$, and $(r - I)\vec{x}_3 = \lambda_3\vec{x}_3 - \vec{x}_3 = ((x - 1) - iy)\vec{x}_3$.

Notice that linearly dependent $\vec{x}_2, \vec{x}_3$ imply $\lambda_2 = \lambda_3$ since $\lambda_2\vec{x}_3 = \lambda_2\alpha\,\vec{x}_2 = A(\alpha\,\vec{x}_2) = A\vec{x}_3 = \lambda_3\vec{x}_3$ with $\vec{x}_3 \neq \vec{0}$ and $\alpha \neq 0$. We have used $\lambda_2\vec{x}_2 = A\vec{x}_2$ implies $\lambda_2(\alpha\,\vec{x}_2) = A(\alpha\,\vec{x}_2)$ for all α.

Now, for $((x - 1) + iy)\vec{x}_2$, $((x - 1) - iy)\vec{x}_3$ to be dependent, either one of the eigenvalues must be zero, or the eigenvectors must be dependent and thus the eigenvalues must be same. If one is zero, then $y = 0$ and $x - 1 = 0$. If they are same, then again $y = 0$ and thus $x = 1$ since $\|x + iy\| = 1$. In all cases, all eigenvalues of r are equal to 1.

That implies $r = I$ since $p(\lambda) = (\lambda - 1)^3 = \lambda^3 - 3\lambda^2 + 3\lambda - 1$ and so $\text{trace}(r) = r_{11} + r_{22} + r_{33} = 3$ but that means that $r_{11} = r_{22} = r_{33} = 1$ since $0 \leq r_{11}, r_{22}, r_{33} \leq 1$. Thus we conclude that $\text{rank}\,(r - I) = 1 \Rightarrow \text{rank}\,(r - I) = \text{rank}\,(I - I) = \text{rank}\,0 = 0$.

Assuming $t \notin \text{span}\,(r - I)$ we can distinguish the two cases: $\text{rank}\,(r - I) = 0$ and $\text{rank}\,(r - I) = 2$.

1. Let $\text{rank}\,(r - I) = 0$.

Then $r = I$ and

$$E' = \begin{pmatrix} I & R^{-1}t \\ 0_3^\top & 1 \end{pmatrix}$$

If $t = 0$, then we are done since E' is the indentity and it can be written as the rotation by $\varphi = 0$ followed by the traslation by $s = 0$ around resp. along any line through the origin. If $t \neq 0$, then we can construct a rotation by the Gram-Schmidt orthogonalization as $R = \begin{pmatrix} R_1 & R_2 & t/\|t\| \end{pmatrix}$ to make

$$\begin{aligned} R^{-1}t &= \begin{pmatrix} R_1^\top t & R_2^\top t & t^\top t/\|t\| \end{pmatrix}^\top = \begin{pmatrix} 0 & 0 & \|t\| \end{pmatrix}^\top \\ E' &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \|t\| \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

which coresponds to the rotation by $\varphi = 0$ followed by the traslation by $s = \|t\|$ around resp. along the line corresponding to the z axis.

2. Let $\text{rank}\,(r-I)=2$.

Then, we can generate by $(r-I)T+t$ any one-dimensional subspace of $\mathbb{R}^3$, which is not in $\text{span}\,(r-I)$, just by choosing T. Recall that $rR_3=R_3$, i.e. $(r-I)R_3=0$. Therefore, $R_3 \notin \text{span}\,(r-I)$ and we can make $(r-I)T+t=sR_3$ and $R^{-1}((r-I)T+t)=R^{-1}sR_3=s\begin{pmatrix}R_1^\top R_3 & R_2^\top R_3 & R_3^\top R_3\end{pmatrix}^\top=\begin{pmatrix}0 & 0 & s\end{pmatrix}^\top$. Finally we obtain

$$E' = \begin{pmatrix} R^{-1}rR & R^{-1}((r-I)T+t) \\ 0_3^\top & 1 \end{pmatrix} = \begin{pmatrix} e & f & 0 & 0 \\ g & h & 0 & 0 \\ 0 & 0 & 1 & s \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The matrix $\begin{pmatrix} e & f & 0 \\ g & h & 0 \\ 0 & 0 & 1 \end{pmatrix} = R^{-1}rR$ is a rotation since it is a composition of tree rotations.

Therefore, E' coresponds to a rotation around and a traslation along the z axis.

The teorem is proven.