Advanced Robotics

Lecture 9

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Polynomials in one variable

Leading term: a non-zero polynomial

$$f(x) = \underbrace{a_0 x^m}_{} + a_1 x^{m-1} + \dots + a_m \quad \in k[x]$$

$a_0 \neq 0$ $\qquad$ $LT(f) = a_0 x^m$ $\equiv$ the leading term

Example:

$$f = 2x^3 - 4x + 3 \quad \Rightarrow \quad LT(f) = 2x^3$$

Division of terms

$\alpha, \beta \in \mathbb{Z}^n_{\geq 0}$, $a_\alpha, b_\beta \in k$, $x^\alpha, x^\beta \in k[x_1, \dots, x_n]$ monomials

$a_\alpha x^\alpha$ divides $b_\beta x^\beta \overset{\text{def}}{\equiv} \beta_i - \alpha_i \geq 0$, $i = 1, \dots, n$

If $a_\alpha x^\alpha$ divides $b_\beta x^\beta$, then there is exactly one monomial

$$c_\gamma x^\gamma = \frac{b_\beta}{a_\alpha} \cdot x^{\beta - \alpha}$$

such that $b_\beta x^\beta = a_\alpha x^\alpha \cdot c_\gamma \cdot x^\gamma$

"Division" of polynomials in one variable

polynomials cannot be divided but can be "divided"

$$f : g \stackrel{def}{\equiv} f = qg + r \, , \quad r = 0 \vee \deg(r) < \deg(g)$$

Example $f = 2x^3 - 4x + 3$, $g(x) = x - 1$

$$f : g \equiv 2x^3 - 4x + 3 = 2x^2(x-1) + 2x^2 - 4x + 3 =$$
$$= (2x^2 + 2x)(x-1) - 2x + 3 = \underbrace{(2x^2 + 2x - 2)}_{q}(x-1) + \underbrace{1}_{r}$$

notice that: $\deg(f) = \deg(LT(f))$

$LT(g)$ divides $LT(f) \Leftrightarrow \deg(LT(g)) \leq \deg(LT(f)) \Leftrightarrow \deg(g) \leq \deg(f)$

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"Division theorem"

Let k be a field and g be a non-zero polynomial in $k[x]$.

(i) Then every $f \in k[x]$ can be written as

$$f = q\,g + r$$

where $q, r \in k[x]$, and either

$$r = 0 \quad \text{or} \quad \deg(r) < \deg(g)$$

(ii) Furthermore, q and r are unique.

Proof: "Division algorithm"

Input: g, f

Output: q, r

$q := 0$

$r := f$

WHILE $r \neq 0$ AND $LT(g)$ divides $LT(r)$ DO

{

$$q := q + \frac{LT(r)}{LT(g)}$$

$$r := r - \frac{LT(r)}{LT(g)} \cdot g$$

}

Observe that $f = q g + r$ holds true

(a) $q = 0$ & $r = f \Rightarrow 0 \cdot g + f = f$

(b) let q_i, r_i be such that $f = q_i g + r_i$, then

$$q_{i+1} g + r_{i+1} = \underbrace{\left(q_i + \frac{LT(r_i)}{LT(g)} \right)}_{q_{i+1}} g + \underbrace{\left(r_i - \frac{LT(r_i)}{LT(g)} \cdot g \right)}_{r_{i+1}} =$$

$$= q_i g + r_i = f$$

If the algorithm terminates, then either

$r = 0$ or

$LT(g)$ does not divide $LT(r) \Leftrightarrow \deg(r) < \deg(g)$

Let us show that the algorithm terminates

Assume that the algorithm does not terminate. Then,
LT(g) divides LT(r) and $r \neq 0$.

Observe that for $r_{i+1} = r_i - \frac{LT(r_i)}{LT(g)} \cdot g$ holds

$$r_{i+1} \begin{cases} \text{either} = 0 \\ \text{or} \quad \deg(r_{i+1}) < \deg(r_i) \end{cases}$$

write $r_i = a_0 x^m + a_1 x^{m-1} + \cdots + a_m$ with $m \geq \ell$

$g = b_0 x^\ell + b_2 x^{\ell-1} + \cdots + b_\ell$ (LT(g) divides LT(r_i))

$$r_{i+1} = r_i - \frac{LT(r_i)}{LT(g)} \cdot g = (\underbrace{a_0 x^m}_{} + a_1 x^{m-1} + \cdots) - \underbrace{\frac{a_0}{b_0} x^{m-\ell} (b_0 x^\ell}_{} + b_1 x^{\ell-1} + \cdots)$$

cancel

$$= (a_1 x^{m-1} + \cdots) - (\frac{a_0}{b_0} b_1 x^{m-1} + \cdots)$$

$$= (a_1 - \frac{a_0}{b_0} b_1) x^{m-1} + (a_2 - \frac{a_0}{b_0} b_2) x^{m-2} + \cdots$$

and therefore we see that

either $r_{i+1} = 0$ if all coefficients vanish

or $\deg(r_i+1) \leq m-1 < m = deg(r_i)$