Advanced Robotics

Lecture 10

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Monomial ordering

Monomials in one variable are easy to order by their degree, i.e.

$$x^0 <_{deg} x^1 <_{deg} x^2 <_{deg} \dots$$

also notice that $x^m <_{deg} x^n \iff x^m$ divides x^n

Not so simple with more variables

consider xy^2, x^2y ... neither one divides the the other but

$$\deg(xy^2) = 1+2 = 3 = 2+1 = \deg(x^2y)$$

A monomial ordering on $k[x_1, \ldots, x_n]$ is any ordering relation $<$ on $\mathbb{Z}^n_{\geq 0}$ satisfying :

(i) $\forall \alpha, \beta : \alpha > \beta$ or $\alpha < \beta$

(ii) $\alpha > \beta$ & $\gamma \in \mathbb{Z}^n_{\geq 0} \Rightarrow \alpha + \gamma > \beta + \gamma$

(iii) $\forall \alpha : \alpha > 0$

we write $x^\alpha > x^\beta \stackrel{\text{def}}{\equiv} \alpha > \beta$

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Lexicographic order

$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$, $\beta = (\beta_1, \beta_2, \dots, \beta_n) \in \mathbb{Z}^n_{\geq 0}$

$\alpha >_{lex} \beta$ if the left-most non-zero element of $\alpha - \beta$ is positive or $\alpha - \beta = 0$.

Examples $(1,2,0) >_{lex} (0,3,4) \Leftarrow (1,-1,-4)$

$(3,2,4) >_{lex} (3,2,1) \Leftarrow (0,0,3)$

Behold! $x,y,z \xrightarrow{\text{rename}} x_1, x_2, x_3 \Rightarrow$

$$\begin{array}{ccccc} x & & y & & z \\ | & & | & & | \\ (1,0,0) & >_{lex} & (0,1,0) & >_{lex} & (0,0,1) \\ | & & | & & | \\ z & & y & & x \end{array}$$

There is $n!$ lex orders

$x,y,z \xrightarrow{\text{rename}} x_3, x_2, x_1 \Rightarrow$

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The lex ordering on $\mathbb{Z}^n_{\geq}$ is a monomial ordering

$<_{lex}$ is an ordering ($\alpha > \alpha$; $\alpha > \beta$ & $\beta > \gamma \Rightarrow \alpha > \gamma$, $\alpha > \beta$ & $\beta > \alpha \Rightarrow \alpha = \beta$)

(a) $\alpha - \alpha = 0 \Rightarrow \alpha >_{lex} \beta$

$\exists\, i, j \in \mathbb{Z}^n_{\geq 0}$ such that $(\alpha - \beta)_k = 0$ and $(\beta - \gamma)_m = 0$ for $k < i$, $m < j$ &

(b) $\alpha >_{lex} \beta$, $\beta >_{lex} \gamma$ $\quad (\alpha - \beta)_i > 0$ & $(\beta - \gamma)_j > 0$

$(\alpha - \gamma)_k = 0 \quad k = 1, \dots, \min(i,j) - 1 \qquad \alpha_k = \beta_k = \gamma_k$

$$(\alpha - \gamma)_{\min(i,j)} > 0 \quad \begin{cases} \min(i,j) = i & \alpha_i \geq \beta_i = \gamma_i \\ \min(i,j) = j & \alpha_j = \beta_j \geq \gamma_j \end{cases}$$

$\Rightarrow \alpha >_{lex} \gamma$

(c) $\alpha >_{lex} \beta$ & $\beta >_{lex} \alpha \Rightarrow$ $\left.\begin{array}{l}\text{either } \alpha - \beta = 0 \text{ or} \\ \exists\, i \in \mathbb{Z}_{\geq 0}\ ((\alpha - \beta)_i > 0\ \&\ (\beta - \alpha)_i > 0)\end{array}\right\} \Rightarrow \alpha - \beta = 0$

The lex ordering is a monomial ordering

(i) $\forall \alpha, \beta : \quad \alpha \underset{lex}{>} \beta \quad$ or $\quad \beta \underset{lex}{>} \alpha$:

$c = \alpha - \beta = 0 \Rightarrow \alpha = \beta$ or there is the first non-zero element c_i. If $c_i > 0$, then $\alpha \underset{lex}{>} \beta$, $\beta \underset{lex}{>} \alpha$ otherwise.

(ii) $\alpha \underset{lex}{>} \beta \quad \& \quad \gamma \in \mathbb{Z}^m_{\geq 0} \quad \Rightarrow \quad \alpha + \gamma \underset{lex}{>} \beta + \gamma$

$\alpha + \gamma - (\beta + \gamma) = \alpha - \beta$

(iii) $\forall \alpha \quad : \quad \alpha \underset{lex}{>} 0$

$(\alpha - 0)_i \geq 0$

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a non-zero $f = \sum_{\alpha} a_\alpha x^\alpha \in k[x_1, \dots, x_n]$ & a monomial ordering $>$

multidegree of f $\quad \text{multideg}(f) = \max_{>} \left(\alpha \in \mathbb{Z}^n_{\geq 0} \mid a_\alpha \neq 0 \right)$

leading term $\longrightarrow$ $LT(f) = LC(f) \cdot LM(f)$

leading coefficient $\quad$ leading monomial

$LC(f) = a_{\text{multideg}(f)}$ $\quad$ $LM(f) = x^{\text{multideg}(f)}$

Example: $f = 4xy^2z + 4z^2 - 5x^3 + 7x^2z^2$, $>_{lex}$

$= 4x^{(1,2,1)} + 4x^{(0,0,2)} - 5x^{(3,0,0)} + 7x^{(2,0,2)}$

$\text{multideg}(f) = (3,0,0)$

$LC(f) = -5$

$LM(f) = x^3$

$LT(f) = -5x^3$