Advanced Robotics

Lecture 12

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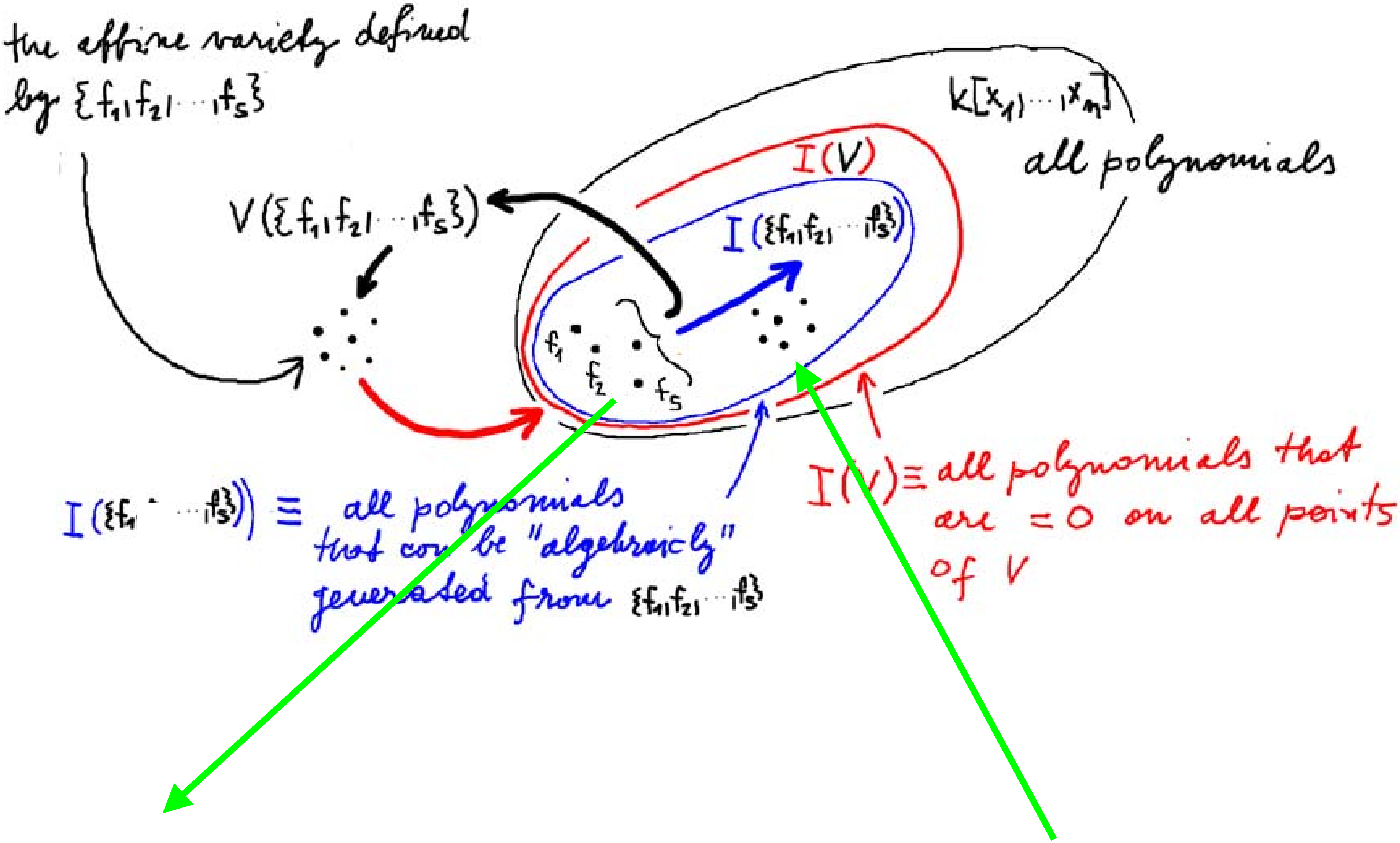
Groebner basis is a special basis of the Ideal



Basis: $B = \{f_1, f_2, \ldots, f_s\}$ $\xrightarrow[\text{manipulation}]{\text{Algebraic}}$ Groebner basis w.r.t. $<_{lex}$: $G = \{g_1, g_2, \ldots, g_n\}$

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Reading the solution out from a Groebner basis

Theorem 3: Let G be a Groebner basis constructed by the Buchberger algorithm w.r.t. $x_1 >_{lex} \cdots >_{lex} x_n$ from polynomials $\{f_1, \dots, f_s\} \in \mathbb{C}[x_1, \dots, x_n]$ for which equations $\{f_i = 0\}_{i=1,\dots,s}$ have a finite number of solutions. Then G contains a polynomial $g \in \mathbb{C}[x_n]$.

There is often even more:

G often consists of a set of polynomials

$g_n(x_n)$

$g_{n-1}(x_n, x_{n-1})$

$g_{n-2}(x_n, x_{n-1}, x_{n-2})$

$\vdots$

$g_1(x_n, x_{n-1}, x_{n-2}, \dots, x_1)$

A working definition of a

Groebner basis (of an ideal)

(A basis) $G = (g_1, \ldots, g_t)$ (of an ideal I) is a Groebner basis

if the remainder on division of $f \in k[x_1, \ldots, x_n]$

by G does not depend on the ordering of g_i in G.

Beware! only r is unique – a_i's need not be unique

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Least common multiple of monomials

Let $x^\alpha, x^\beta \in k[x_1, \dots, x_n]$ be monomials, then x^γ with

$\gamma_i = \max(\alpha_i, \beta_i)$, $i = 1, \dots, n$ is

the **least common multiple** – $LCM(x^\alpha, x^\beta)$ – of x^α, x^β

Example:

$$x^\alpha = x y^3 z^2 \quad , \quad x^\beta = y z^6$$

$$\downarrow \qquad\qquad \downarrow$$

$$\alpha = (1, 3, 2) \qquad \beta = (0, 1, 6)$$

$$\gamma = \max((1,3,2),(0,1,6)) = (1,3,6)$$

$$x^\gamma = x y^3 z^6$$

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The S-polynomial (designed to cancel the leading terms)

The S-polynomial of $f, g \in k[x_1, \dots, x_n]$ is the (algebraic) combination

$$S(f,g) = \frac{LCM(LM(f), LM(g))}{LT(f)} \cdot f - \frac{LCM(LM(f), LM(g))}{LT(g)} \cdot g$$

Example: $f = x^3y^2 - x^2y^3 + x \ , \quad g = 3x^4y + y^2 \ \in \mathbb{R}[x,y]$

with $x >_{lex} y$

$$S(f,g) = \frac{LCM(x^3y^2, x^4y)}{x^3y^2} \cdot f - \frac{LCM(x^3y^2, x^4y)}{3x^4y} \cdot g = \frac{x^4y^2}{x^3y^2} \cdot f - \frac{x^4y^2}{3x^4y} g$$

$$= x \cdot f - \frac{1}{3} y g = x^4y^2 - x^3y^3 + x^2 - x^4y^2 - \frac{1}{3}y^3 = -x^3y^3 + x^2 - \frac{1}{3}y^3$$

cancel

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Characterization of Groebner bases in terms of S-polynomials

A set $G = \{g_1, \dots, g_t\}$ of polynomials in $k[x_1, \dots, x_n]$ is a Groebner basis if for all $i, j \in 1, \dots, t$, $i \neq j$, the remainder on division of $S(g_i, g_j)$ by G (with arbitrary but fixed order of g_k) is zero

Algorithm:

$\{f_1, \dots, f_s\}$ polynomials in $k[x_1, \dots, x_n]$

Input: $F = (f_1, \dots, f_s)$ Output: a Groebner basis $G = (g_1, \dots, g_t)$

```
G := F
REPEAT
  { G' := G
    FOR each pair (p,q) ∈ {1,...,s}^2, p ≠ q DO
      { S = \overline{S(p,q)}^{G'}
        IF S ≠ 0 THEN { G := G ∪ {S} }
      }
  }
UNTIL G = G'
```

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Example: $k[x,y]$, $x >_{lex} y$ $\quad F = (f_1, f_2) = (x^3 - 2xy,\ x^2y - 2y^2 + x)$

F is not GB: $S(f_1, f_2) = \frac{x^3y}{x^3} f_1 - \frac{x^3y}{x^2y} f_2 = y f_1 - x f_2 = -2xy^2 + 2xy^2 - x^2 = -x^2$

and $\overline{S(f_1, f_2)}^F = -x^2 \neq 0$

$G_1 = F \cup \{-x^2\} = (f_1, f_2, -x^2)$

$$S(f_1, x^2) = \frac{x^3}{x^3} f_1 - \frac{x^3}{x^2} x^2 = -2xy \ ; \ \overline{S(f_1, x^2)}^{G_1} = -2xy$$

$$S(f_2, x^2) = \frac{x^2y}{x^2y} f_2 - \frac{x^2y}{x^2} x^2 = -2y^2 + x \ ; \ \overline{S(f_2, x^2)}^{G_1} = -2y^2 + x$$

$G_2 = (f_1, f_2, -x^2, -2xy, x - 2y^2) = (f_1, f_2, f_3, f_4, f_5)$

$$S(f_1, f_4) = \frac{x^3y}{x^3} f_1 - \frac{x^3y}{-2xy} f_4 = -2xy^2 \ ; \ \overline{S(f_1, f_4)}^{G_2} = 0$$

$$S(f_2, f_4) = \frac{x^2y}{x^2y} f_2 - \frac{x^2y}{-2xy} f_4 = x - 2y^2 \ ; \ \overline{S(f_2, f_4)}^{G_2} = 0$$

$\vdots$

$$S(f_4, f_5) = \frac{xy}{-2xy} f_4 - \frac{xy}{x} f_5 = 3y^3 \ ; \ \overline{3y^3}^{G_2} = 3y^3$$

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$$G_3 = (\underset{f_1}{x^3 - 2xy}, \underset{f_2}{x^2y - 2y^2 + x}, \underset{f_3}{-x^2}, \underset{f_4}{-2xy}, \underset{f_5}{x - 2y^2}, \underset{f_6}{3y^3})$$

$$\left.\begin{aligned} f_1 &= -x f_3 + f_4 \\ f_2 &= -y f_3 + f_5 \\ f_3 &= -x f_5 + y f_4 \\ f_4 &= -2y f_5 - \tfrac{4}{3} f_6 \end{aligned}\right\} \Rightarrow G_4 = (\underset{f_5}{x - 2y^2}, \underset{f_6}{3y^3})$$

$$S(f_5, f_6) = \frac{xy^3}{x} f_5 - \frac{xy^3}{3y^3} f_6 = -2y^5$$

$$\overline{-2y^5}^{G_4} = 0$$

Therefore G_3 is a Groebner basis. It containes f_1, f_2

G_4 is also a Groebner basis. It generates the same ideal as G_3.

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