Advanced Robotics

Lecture 4

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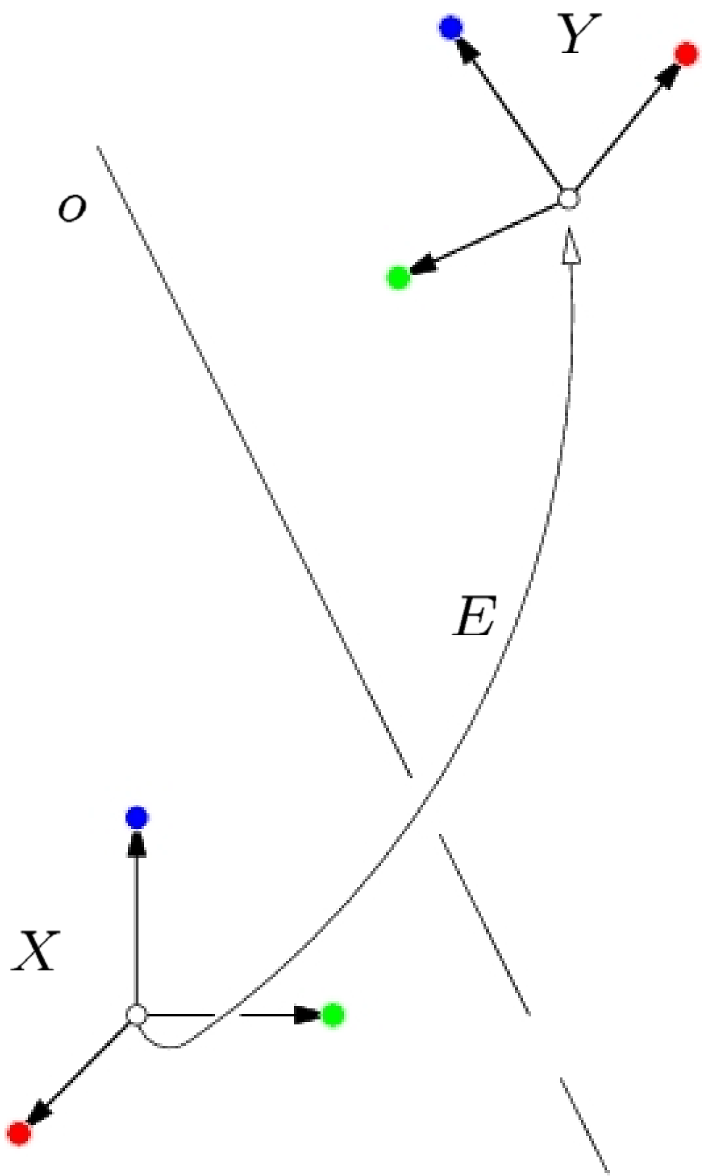
Motion and its axis



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Motion axis

Definitions:

Motion is a mapping $m: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ such that for all $\vec{x}, \vec{y}$ in $\mathbb{R}^3$ there holds $\|m(\vec{x}) - m(\vec{y})\| = \|\vec{x} - \vec{y}\|$.

Motion axis is a line a in $\mathcal{A}^3$ such that for all $\vec{x} \in a$ there holds $m(\vec{x}) \in a$.

Lemma: In a fixed coordinate system, every motion m can be described as $m(x) = r\,(x - t)$, where $r \in \mathbb{R}^{3\times 3}$, $r^\top r = I$, $det(r) = 1$ and $\vec{t} \in \mathbb{R}^3$.

Theorem: *Every motion has a motion axis.*

The proof follows next.

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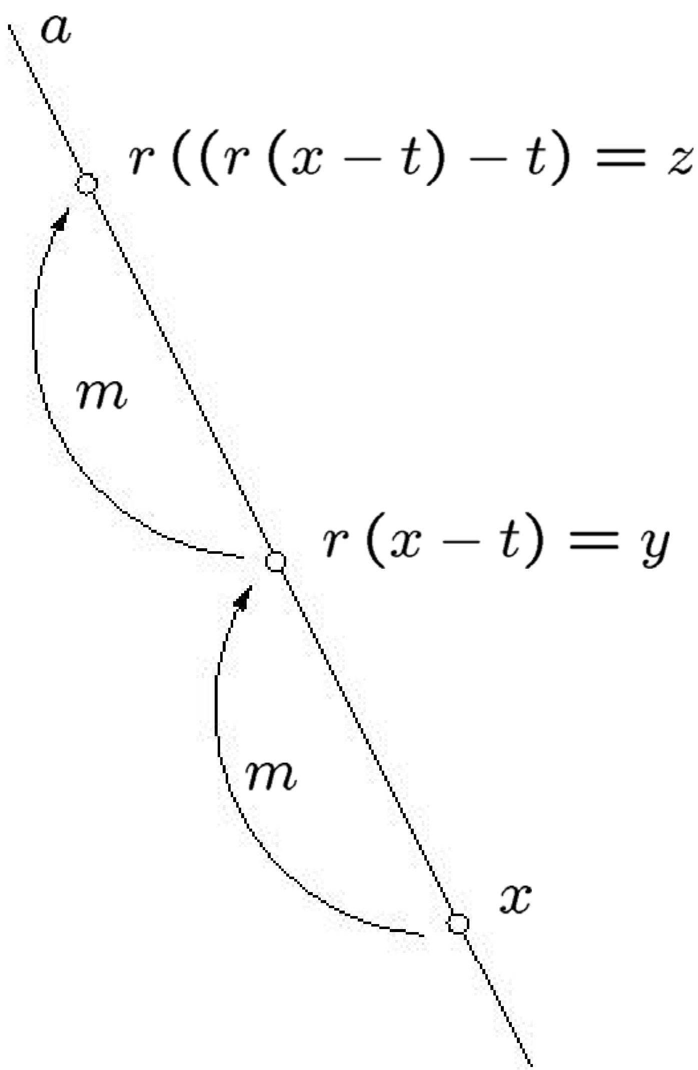
Motion axis



$$
\begin{aligned}
z-y &= y-x \\
r\,(r\,(x-t)-t)-r\,(x-t) &= r\,(x-t)-x \\
r^2\,x-r^2\,t &= 2\,r\,x-r\,t-x \\
r^2\,x-2\,r\,x+x &= r^2\,t-r\,t \\
(r^2-2\,r+I)\,x &= (r^2-r)\,t
\end{aligned}
$$

Thus we are getting the set of linear equations

$$A\,x = B\,t$$

with

$$A = (r^2-2\,r+I) \in \mathbb{R}^{3\times 3}$$
$$B = (r^2-r) \in \mathbb{R}^{3\times 3}$$

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Let us see that the set of linear equations

$$A\,x = B\,t$$

has always at least one dimensional space of solutions.

We will show that the vector $B\,t$ is in the span of columns of A. It follows from the fact that the columns of A and B span the same subspace of $\mathbb{R}^3$.

First, the sets of vectors perpendicular to columns of A and B are equal since $x^\top B = 0 \iff x^\top (r - I)\,r = 0 \iff x^\top (r - I) = 0$ (r is invertible) and $x^\top A = 0 \iff x^\top (r - I)(r - I) = 0 \iff x^\top (r - I) = 0$.

$x^\top A = 0$ either if $x^\top (r - I) = 0$ or if $y^\top = x^\top (r - I) \neq 0$ and $y^\top (r - I) = 0$. But the second case implies $(r - I)\,y = 0$ which means that y is at the same time in the span of the columns of $(r - I)$ and othogonal to the columns of $(r - I)$, thus it is 0, hence a contradicts $y \neq 0$.

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Secondly, we see that the columns of A and B generate the same subspaces since two linear subspaces in $\mathbb{R}^n$ are identical if and only if the sets of all vectors perpendicular to them are identical.

Finally, if $x \in \mathbb{R}^3$ is a solution to $A\,x = B\,t$, then also $x + \alpha\,v$, for all $\alpha \in \mathbb{R}$ and $r\,v = v$ is also a solution since

$$A\,(x + \alpha\,v) = \alpha\,A\,v = \alpha\,(r^2 - 2\,r + I)\,v = \alpha\,(v - 2\,v + v) = 0$$

This shows that one can choose a coordinate system with the origin on the motion axis o and with the z axis along o. In this coordinate system, the motion corresponds to a rotation around the axis and translation along the axis.

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