S V D

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Linear mapping

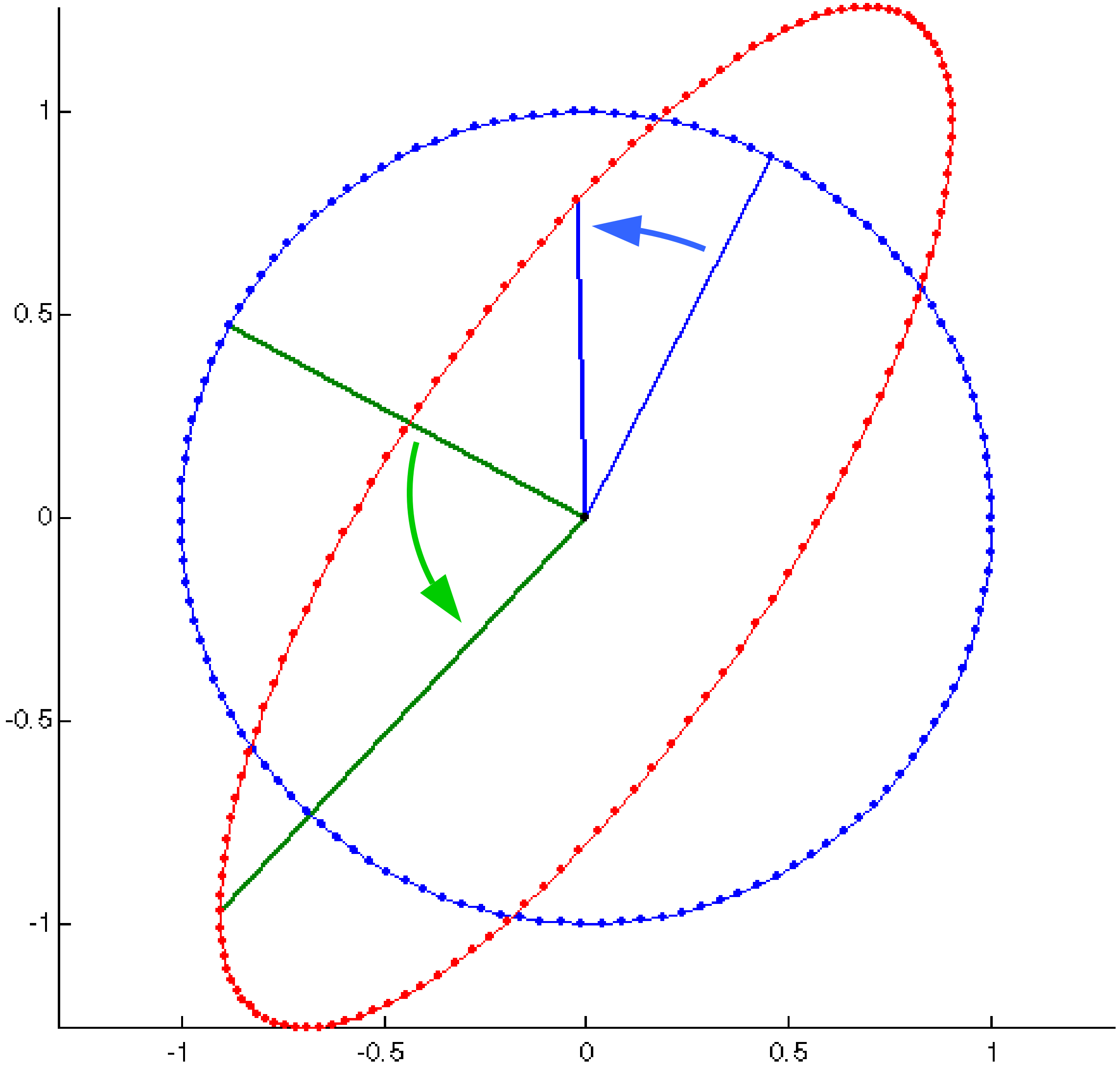
$A : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ ... a linear mapping

```
fi = 0:0.01:2*pi;
x = [cos(fi); sin(fi)];
A = randn(2,2);
y = A*x;

>> A

A =

    0.7942   -0.4284
    1.2336    0.2478
```



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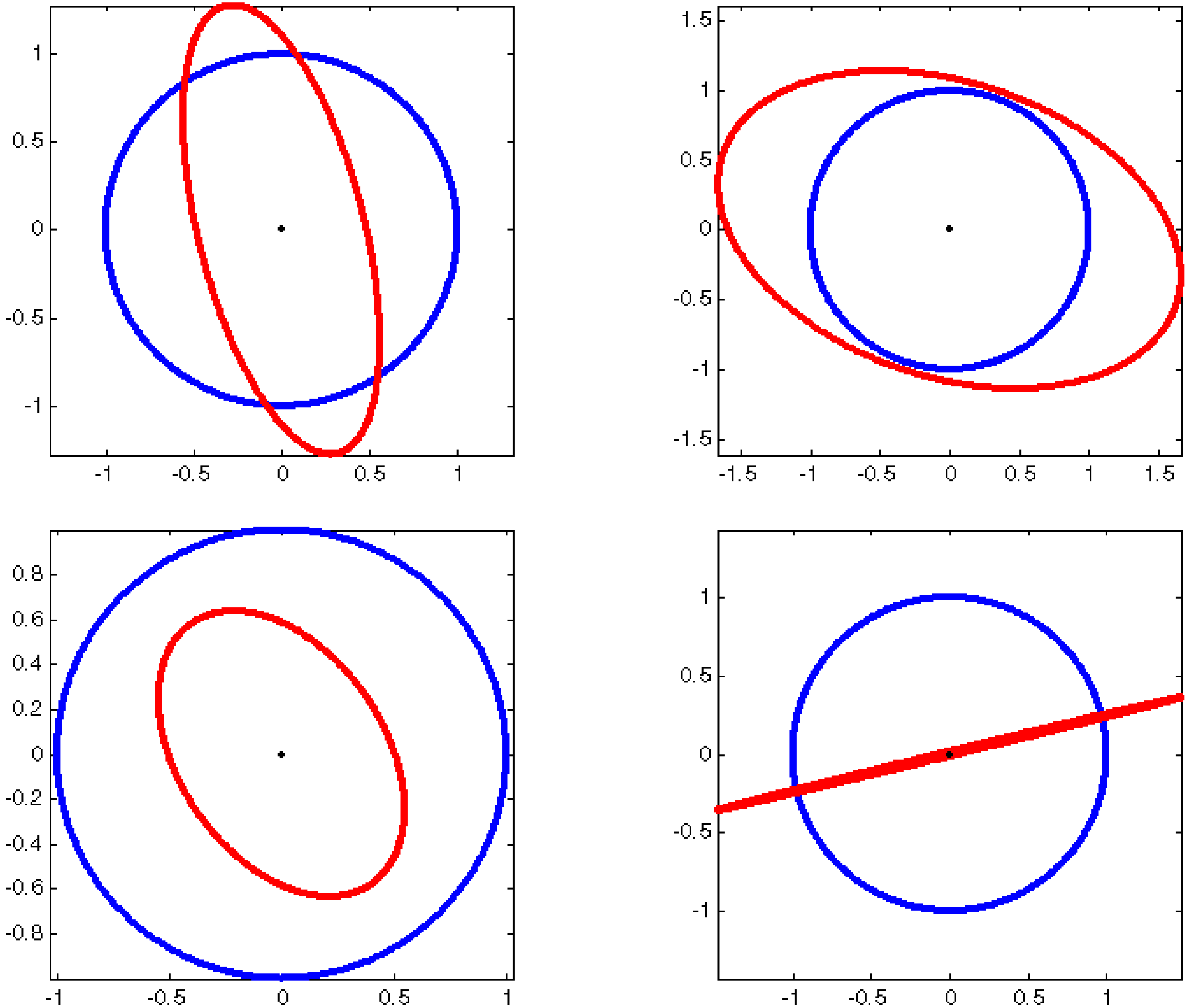
Observation: a linear mapping maps circles to ellipses or to line segments



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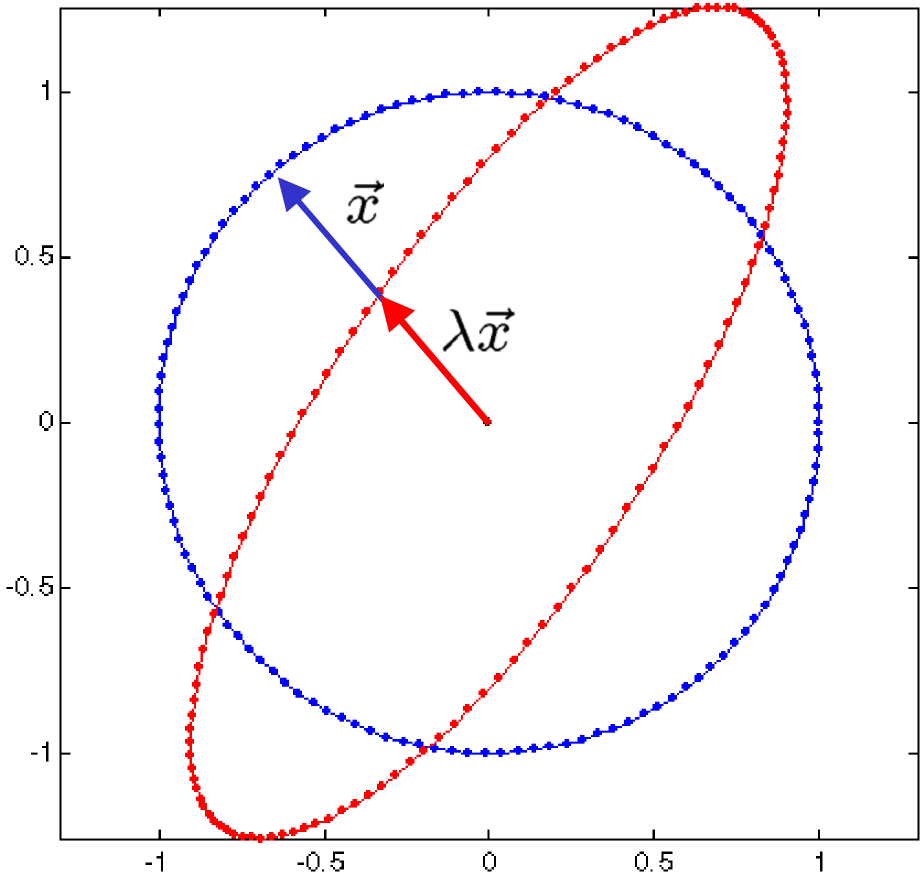
Ellipse is a squashed circle



A set Y is an ellipse $\Leftrightarrow$ Y is a conic and $\forall \vec{x}$ on an unit circle $\exists \lambda \geq 0$ such that $\lambda\vec{x}$ is on Y

S V D – Singular Value Decomposition

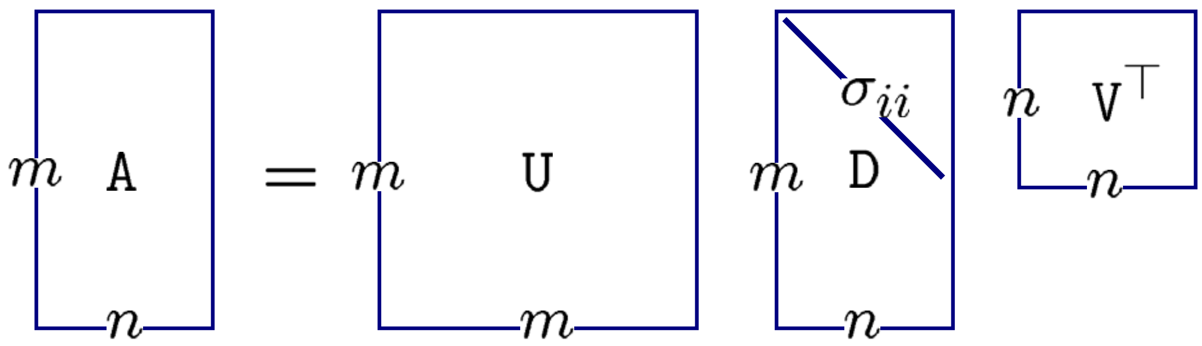
For every matrix $\mathtt{A} \in \mathbb{R}^{m \times n}$ exist matrices

$\mathtt{U} \in \mathbb{R}^{m \times m}, \mathtt{D} \in \mathbb{R}^{m \times n}, \mathtt{V} \in \mathbb{R}^{n \times n}$ such that

$\mathtt{U}^\top \mathtt{U} = \mathtt{I}$ and $\mathtt{V}^\top \mathtt{V} = \mathtt{I}$

$\mathtt{D} = \mathrm{diag}([\sigma_{11}, \ldots, \sigma_{nn}]), \sigma_{11} \geq \ldots \geq \sigma_{nn} \geq 0$

$\mathtt{A} = \mathtt{U}\,\mathtt{D}\,\mathtt{V}^\top$



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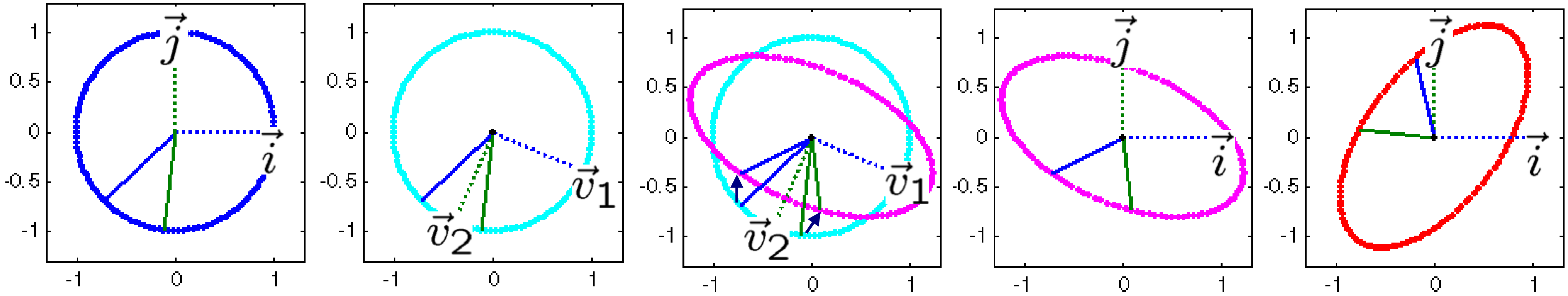
S V D – interpretation for regular 2×2 matrices



$\beta = (\vec{i}, \vec{j})$ $\beta' = (\vec{v}_1, \vec{v}_2)$

$$\mathbf{x}_\beta \xrightarrow{\mathtt{V}^{-1}(=\mathtt{V}^\top)} \mathbf{x}_{\beta'} \xrightarrow{\mathtt{D}} \mathbf{z}_{\beta'} \xrightarrow{\mathtt{V}} \mathbf{z}_\beta \xrightarrow{\mathtt{UV}^{-1}} \mathbf{y}_\beta$$

change of basis "squashing" along coordinate axes change of basis rotation

$$\mathtt{A} = \mathtt{U\,D\,V}^\top = (\mathtt{U\,V}^{-1}) \quad \mathtt{V} \quad \mathtt{D} \quad \mathtt{V}^{-1}$$

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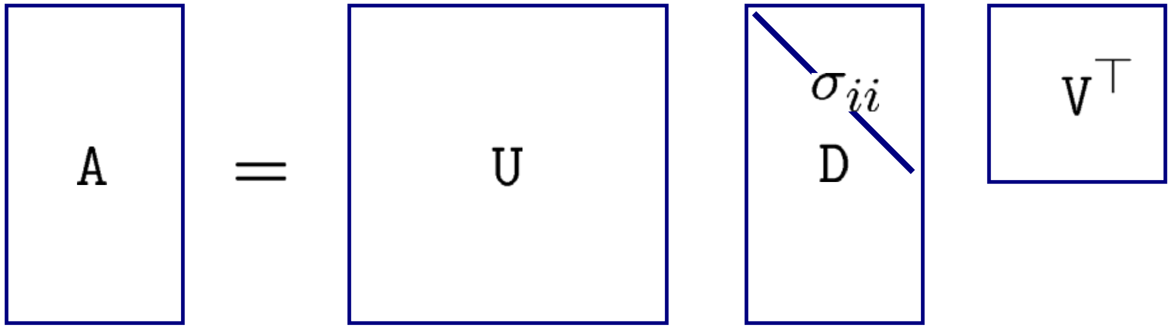
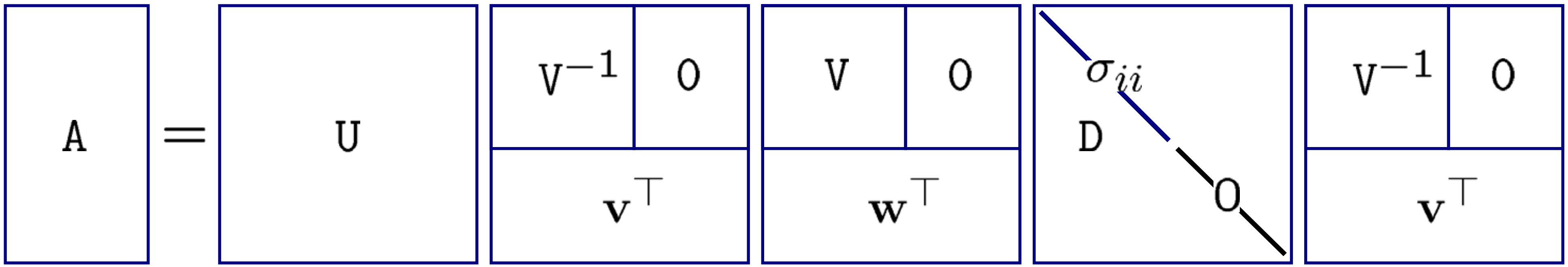
S V D – interpretation in general

$$\mathtt{A} = \mathtt{U}\,\mathtt{D}\,\mathtt{V}^\top$$



$$\mathtt{A} = \mathtt{U}\,\mathtt{D}\,\mathtt{V}^\top \quad = \quad (\mathtt{U}\,\mathtt{V}^{-1}) \quad \bar{\mathtt{V}} \quad \bar{\mathtt{D}} \quad \bar{\mathtt{V}}^{-1}$$



$\bar{\mathtt{V}}^{-1}$ $\quad$ $\bar{\mathtt{V}}$ $\quad$ $\bar{\mathtt{D}}$ $\quad$ $\bar{\mathtt{V}}^{-1}$

S V D – Low rank approximation

Let $\mathtt{A}^{m\times n}$ be a real matrix of rank r.

We are looking for a real matrix $\bar{\mathtt{A}}^{m\times n}$ of rank $k \le r$ that best approximates $\mathtt{A}$ in the sense that the largest difference between the matrices understood as linear mappings is minimized, i.e.

$$\bar{\mathtt{A}} = \arg \min_{\substack{\mathtt{B}\,\in\,\mathbb{R}^{m\times n}\\ rank\,\mathtt{B}\,=\,k}} \; \max_{\substack{\mathbf{y}\,\in\,\mathbb{R}^{n}\\ \|\mathbf{y}\|\,=\,1}} \underbrace{\|\mathtt{A}\,\mathbf{y}-\mathtt{B}\,\mathbf{y}\|}_{\text{vector norm}} = \arg \min_{\substack{\mathtt{B}\,\in\,\mathbb{R}^{m\times n}\\ rank\,\mathtt{B}\,=\,k}} \underbrace{\|\mathtt{A}-\mathtt{B}\|}_{\text{matrix norm}}$$

Interestingly, it is easy to find matrix $\bar{\mathtt{A}}$ using SVD of $\mathtt{A}$.

S V D – Low rank approximation

Theorem:

Let $\mathtt{A} = \mathtt{U}\,\mathtt{D}\,\mathtt{V}^\top$ be the singular value decomposition or a real matrix $\mathtt{A}^{m\times n}$. Then,

$$\mathtt{A}_k = \arg \min_{\substack{\mathtt{B} \in \mathbb{R}^{m\times n} \\ rank\,\mathtt{B} = k}} \|\mathtt{A} - \mathtt{B}\|$$

is obtained as

$$\mathtt{A}_k = \mathtt{U}\,\mathtt{D}_k\,\mathtt{V}^\top$$

where

$$\mathtt{A} = \mathtt{U}\,\mathtt{D}\,\mathtt{V}^\top,\ \mathtt{D} = \operatorname{diag}([\sigma_{11}, \ldots, \sigma_{nn}])$$

$$\mathtt{D}_k = \operatorname{diag}([\sigma_{11}, \ldots, \sigma_{kk}, 0, 0, \ldots])$$

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S V D – Proof of the low rank approximation

Lemma: $\mathtt{R}^{m \times m}$ and $\mathtt{R}^\top \mathtt{R} = \mathtt{I}$, then $\|\mathtt{R}\,\mathtt{A}\| = \|\mathtt{A}\|$

Proof:

$$\|\mathtt{R}\,\mathtt{A}\| = \max_{\substack{\mathbf{x} \in \mathbb{R}^n \\ \|\mathbf{x}\| = 1}} \|\mathtt{R}\,\mathtt{A}\,\mathbf{x}\| = \max_{\substack{\mathbf{x} \in \mathbb{R}^n \\ \|\mathbf{x}\| = 1}} (\mathbf{x}^\top \mathtt{A}^\top \mathtt{R}^\top \mathtt{R}\,\mathtt{A}\,\mathbf{x})^{\frac{1}{2}}$$

$$= \max_{\substack{\mathbf{x} \in \mathbb{R}^n \\ \|\mathbf{x}\| = 1}} (\mathbf{x}^\top \mathtt{A}^\top \mathtt{A}\,\mathbf{x})^{\frac{1}{2}} = \|\mathtt{A}\|$$

Lemma: $\mathtt{R}^{n \times n}$ and $\mathtt{R}^\top \mathtt{R} = \mathtt{I}$, then $\|\mathtt{A}\,\mathtt{R}\| = \|\mathtt{A}\|$

Proof:

$$\|\mathtt{A}\,\mathtt{R}\| = \max_{\substack{\mathbf{x} \in \mathbb{R}^n \\ \|\mathbf{x}\| = 1}} \|\mathtt{A}\,\mathtt{R}\,\mathbf{x}\| = \max_{\substack{\mathbf{y} \in \mathbb{R}^n \\ \|\mathbf{y}\| = 1}} \|\mathtt{A}\,\mathbf{y}\| = \|\mathtt{A}\|$$

since $\{\mathbf{y} \,|\, \mathbf{y} = \mathtt{R}\,\mathbf{x},\ \mathbf{x} \in \mathbb{R}^n,\ \|\mathbf{x}\| = 1\} = \{\mathbf{x} \,|\, \mathbf{x} \in \mathbb{R}^n,\ \|\mathbf{x}\| = 1\}$

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S V D – Proof of the low rank approximation

Lemma: $\|\mathtt{A} - \mathtt{A}_k\| = \sigma_{k+1,k+1}$

Proof:

$$
\begin{aligned}
\|\mathtt{A} - \mathtt{A}_k\| &= \|\mathtt{U}(\mathtt{D} - \mathtt{D}_k)\mathtt{V}^\top\| = \|\mathtt{D} - \mathtt{D}_k\| \\
&= \max_{\substack{\mathbf{x} \in \mathbb{R}^n \\ \|\mathbf{x}\| = 1}} \left((\sigma_{11} - \sigma_{11})^2 x_1^2 + \ldots + (\sigma_{kk} - \sigma_{kk})^2 x_k^2 + \sigma_{k+1,k+1}^2 x_{k+1}^2 + \ldots\right)^{\frac{1}{2}} \\
&= \max_{\substack{\mathbf{x} \in \mathbb{R}^n \\ \|\mathbf{x}\| = 1}} \left(0\, x_1^2 + \ldots + 0\, x_k^2 + \sigma_{k+1,k+1}^2 x_{k+1}^2 + \ldots + \sigma_{nn}^2 x_n^2\right)^{\frac{1}{2}} \\
&\leq \max_{\substack{\mathbf{x} \in \mathbb{R}^n \\ \|\mathbf{x}\| = 1}} \sigma_{k+1,k+1} \left(x_1^2 + \ldots + x_k^2 + x_{k+1}^2 + \ldots + x_n^2\right)^{\frac{1}{2}} = \sigma_{k+1,k+1}
\end{aligned}
$$

Since $\|(\mathtt{D} - \mathtt{D}_k)\mathtt{V}^\top \mathbf{v}_{k+1,k+1}\| = \sigma_{k+1,k+1}$ we conclude that $\|\mathtt{A} - \mathtt{A}_k\| = \sigma_{k+1,k+1}$

S V D – Proof of the low rank approximation

Proof of the theorem: By contradiction. If $k = n$, then $\mathtt{A}_k = \mathtt{A}$. Assume that there is a matrix $\mathtt{B}$ with $\operatorname{rank}\mathtt{B} = k < \operatorname{rank}\mathtt{A}$ such that $\|\mathtt{A}-\mathtt{B}\| < \|\mathtt{A}-\mathtt{A}_k\| = \sigma_{k+1,k+1}$.

The null space N of $\mathtt{B}$ has dimension $n-k > 0$, and thus there is $\mathbf{x} \in N$ such that $\|\mathbf{x}\| = 1$. For every $\mathbf{x} \in N$, $\mathtt{B}\,\mathbf{x} = \mathbf{0}$. Take $\mathbf{x} \in N$ such that $\|\mathbf{x}\| = 1$.

Then $$\|\mathtt{A}\,\mathbf{x}\| = \|(\mathtt{A}-\mathtt{B})\,\mathbf{x}\| \leq \|(\mathtt{A}-\mathtt{B})\| \overset{\text{assumption}}{<} \sigma_{k+1,k+1}$$

$$\forall \mathbf{x} \in \mathbb{R}^n : \|\mathtt{A}-\mathtt{B}\| = \max_{\mathbf{y}\in\mathbb{R}^n,\, \|\mathbf{y}\|=1} \|(\mathtt{A}-\mathtt{B})\mathbf{y}\| \geq \|(\mathtt{A}-\mathtt{B})\mathbf{x}\|$$

For every $\mathbf{x} \in M = span(\mathbf{v}_1, \ldots, \mathbf{v}_{k+1})$, such that $\|\mathbf{x}\| = 1$

$$\|\mathtt{A}\,\mathbf{x}\| = \|\mathtt{D}\begin{pmatrix}\mathbf{v}_1^\top\\ \vdots \\ \mathbf{v}_n^\top\end{pmatrix}\mathbf{x}\| = \begin{array}{c}\mathbf{x}\in M\\ \Downarrow \\ \mathbf{x} = \sum_{i=1}^{k+1} a_i \mathbf{v}_i\end{array} = \|\mathtt{D}\begin{pmatrix}\mathbf{v}_1^\top\\ \vdots \\ \mathbf{v}_n^\top\end{pmatrix}\sum_{i=1}^{k+1} a_i \mathbf{v}_i\| =$$

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S V D – Proof of the low rank approximation

$$
\begin{aligned}
&= \|\mathtt{D}\begin{pmatrix} a_1 \\ \vdots \\ a_{k+1} \\ 0 \\ \vdots \end{pmatrix}\| \\
&= (\sigma_{11}^2 a_1^2 + \ldots + \sigma_{k+1,k+1}^2 a_{k+1}^2)^{\frac{1}{2}} \\
&\geq (\sigma_{k+1,k+1}^2 a_1^2 + \ldots + \sigma_{k+1,k+1}^2 a_{k+1}^2)^{\frac{1}{2}} \\
&= \sigma_{k+1,k+1}(a_1^2 + \ldots + a_{k+1}^2)^{\frac{1}{2}} = \sigma_{k+1,k+1}
\end{aligned}
$$

since $1 = \|\mathbf{x}\| = (a_1^2 + \ldots + a_{k+1}^2)^{\frac{1}{2}}$.

$M \cap N \neq \{\mathbf{0}\}$, since $dim\ M = k+1$, $dim\ N = n-k$ and $k+1+n-k = n+1 > n$, and therefore there is a unit vector $\mathbf{x} \in M \cap N$ such that $\|\mathtt{A}\,\mathbf{x}\| < \sigma_{k+1,k+1}$ and $\|\mathtt{A}\,\mathbf{x}\| \geq \sigma_{k+1,k+1}$, which is absurd. Therefore, there is no such B.