Advanced Robotics

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September 27, 2010

1 Change of coordinates induced by the change of basis

Let us discuss the relationship between the coordinates of a vector in a linear space, which is induced by passing from one basis to another. We shall derive the relationship between the coordinates in a three-dimensional linear space V^3 over real numbers, which is the most important when modeling the geometry around us. The formulas for all other n-dimensional spaces are obtained by passing from 3 to n.

Coordinates Let us consider an ordered basis $\beta = (\vec{b}_1, \vec{b}_2, \vec{b}_3)$ of V^3. A vector $\vec{x} \in V^3$ is uniquely expressed as a linear combination of the basic vectors by its *coordinates* x, y, z, i.e. $\vec{x} = x\,\vec{b}_1 + y\,\vec{b}_2 + z\,\vec{b}_3$ and can be represented as an ordered triple of coordinates, as a coordinate vector $\vec{x}_\beta = \begin{bmatrix} x & y & z \end{bmatrix}^\top$.

Two bases Having two ordered bases $\beta = (\vec{b}_1, \vec{b}_2, \vec{b}_3)$ and $\beta' = (\vec{b}'_1, \vec{b}'_2, \vec{b}'_3)$ leads to expressing one vector $\vec{x}$ in two ways as $\vec{x} = x\,\vec{b}_1 + y\,\vec{b}_2 + z\,\vec{b}_3$ and $\vec{x} = x'\,\vec{b}'_1 + y'\,\vec{b}'_2 + z'\,\vec{b}'_3$. The vectors of the basis β can also be expressed in the basis β' using their coordinates. Let us introduce

$$\begin{aligned} \vec{b}_1 &= a_{11}\,\vec{b}'_1 + a_{21}\,\vec{b}'_2 + a_{31}\,\vec{b}'_3 \\ \vec{b}_2 &= a_{12}\,\vec{b}'_1 + a_{22}\,\vec{b}'_2 + a_{32}\,\vec{b}'_3 \\ \vec{b}_3 &= a_{13}\,\vec{b}'_1 + a_{23}\,\vec{b}'_2 + a_{33}\,\vec{b}'_3 \end{aligned} \tag{1.1}$$

Change of coordinates We will next use the above equations to relate the coordinates of $\vec{x}$ w.r.t. the basis β to the coordinates of $\vec{x}$ w.r.t. the basis β'

$$\begin{aligned} \vec{x} &= x\,\vec{b}_1 + y\,\vec{b}_2 + z\,\vec{b}_3 \\ &= x\,(a_{11}\,\vec{b}'_1 + a_{21}\,\vec{b}'_2 + a_{31}\,\vec{b}'_3) + y\,(a_{12}\,\vec{b}'_1 + a_{22}\,\vec{b}'_2 + a_{32}\,\vec{b}'_3) + z\,(a_{13}\,\vec{b}'_1 + a_{23}\,\vec{b}'_2 + a_{33}\,\vec{b}'_3) \\ &= (a_{11}\,x + a_{12}\,y + a_{13}\,z)\,\vec{b}'_1 + (a_{21}\,x + a_{22}\,y + a_{23}\,z)\,\vec{b}'_2 + (a_{31}\,x + a_{32}\,y + a_{33}\,z)\,\vec{b}'_3 \\ &= x'\,\vec{b}'_1 + y'\,\vec{b}'_2 + z'\,\vec{b}'_3 \end{aligned}$$

Since coordinates are unique, we get

$$\begin{aligned} x' &= a_{11}\,x + a_{12}\,y + a_{13}\,z \\ y' &= a_{21}\,x + a_{22}\,y + a_{23}\,z \\ z' &= a_{31}\,x + a_{32}\,y + a_{33}\,z \end{aligned}$$

Coordinate vectors $\vec{x}_\beta$ and $\vec{x}'_\beta$ are thus related by a matrix multiplication

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

which we concisely write as

$$\vec{x}_{\beta'} = \mathtt{A}\,\vec{x}_{\beta} \tag{1.2}$$

The columns of matrix A can be viewed as coordinate vectors representing basic vectors, $\vec{b}_1, \vec{b}_2, \vec{b}_3$ of β in the basis β'

$$\mathtt{A} = \begin{bmatrix} | & | & | \\ \vec{b}_{1_{\beta'}} & \vec{b}_{2_{\beta'}} & \vec{b}_{3_{\beta'}} \\ | & | & | \end{bmatrix}$$

and the matrix multiplication can be interpreted as linear combination of columns of A by coordinates of $\vec{x}$ w.r.t. β

$$\vec{x}_{\beta'} = x\,\vec{b}_{1_{\beta'}} + y\,\vec{b}_{2_{\beta'}} + z\,\vec{b}_{3_{\beta'}}$$

Matrix A plays such an important role here that it deserves its own name. Matrix A is very often called the *change of basis matrix from basis β to β'* or the *transition matrix from basis β to basis β'* [1, 2] since it can be used to pass from coordinates w.r.t. β to coordinates w.r.t. β' by Equation 1.2.

However, literature [3] calls A the *change of basis matrix from basis β' to β*, i.e. it (seemingly illogically) swaps the bases. This choice is motivated by the fact that A can also be used to obtain the vectors of β from vectors of β' using Equation 1.1 as

$$\begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix} = \begin{bmatrix} a_{11}\,\vec{b}'_1 + a_{21}\,\vec{b}'_2 + a_{31}\,\vec{b}'_3 & a_{12}\,\vec{b}'_1 + a_{22}\,\vec{b}'_2 + a_{32}\,\vec{b}'_3 & a_{13}\,\vec{b}'_1 + a_{23}\,\vec{b}'_2 + a_{33}\,\vec{b}'_3 \end{bmatrix} \tag{1.3}$$

$$= \begin{bmatrix} \vec{b}'_1 & \vec{b}'_2 & \vec{b}'_3 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$= \begin{bmatrix} \vec{b}'_1 & \vec{b}'_2 & \vec{b}'_3 \end{bmatrix} \mathtt{A} \tag{1.4}$$

where the multiplication of a row of column vectors by a matrix from the right in Equation 1.4 has the meaning given by Equation 1.3 above. Yet another variation of the naming appeared in [4, 5] where $\mathtt{A}^{-1}$ was named the *change of basis matrix from basis β to β'*.

We have to conclude that the meaning associated with the *change of basis matrix* varies in the literature and hence we will avoid this confusing name and talk about A as about the *matrix transforming coordinates of a vector from basis β to basis β'*.

There is the following interesting variation of Equation 1.4

$$\begin{bmatrix} \vec{b}'_1 \\ \vec{b}'_2 \\ \vec{b}'_3 \end{bmatrix} = \mathtt{A}^{-\top} \begin{bmatrix} \vec{b}_1 \\ \vec{b}_2 \\ \vec{b}_3 \end{bmatrix} \tag{1.5}$$

where the basic vectors of β and β' are understood as elements of column vectors. The following questions arises: When are the coordinates of a vector $\vec{x}$ (Equation 1.2) and the basic vectors themselves (Equation 1.5) transformed in the same way? In other words, when $\mathtt{A} = \mathtt{A}^{-\top}$.

2 Motion

Let us introduce a mathematical model of rigid motion in three-dimensional Euclidean space. The important property of motion is that it only relocates objects without changing their shape. Distances between points on moving objects remain equal.

2.1 Change of position vector coordinates induced by motion

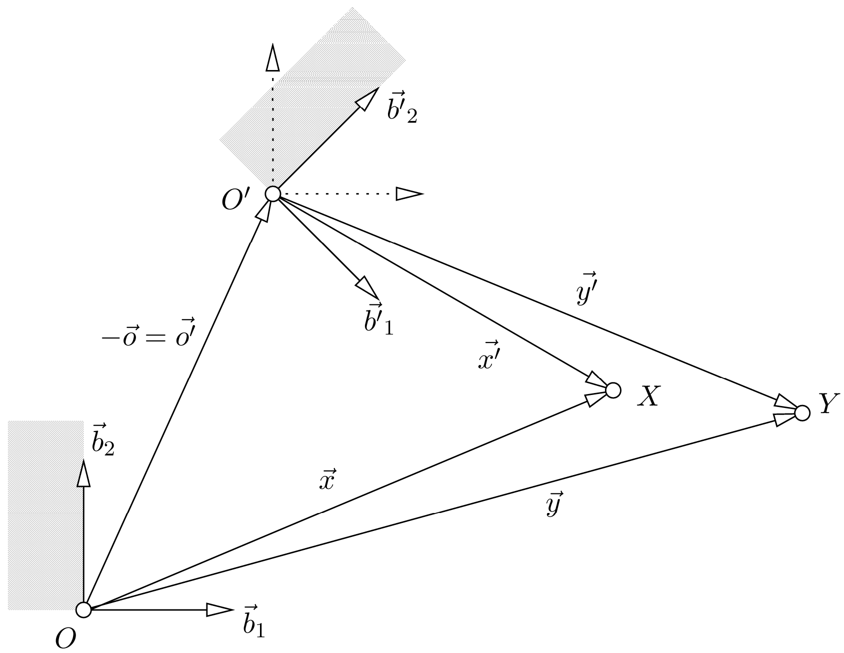


Figure 2.1: Representation of rigid motion.

Figure 2.1 illustrates the model of rigid motion using coordinate systems, points and their position vectors. A coordinate system (O, β) with origin O and basis β is attached to a moving rigid body. As the body moves to a new position, a new coordinate system (O', β') is constructed. Assume a point X in a general position w.r.t. the body, which is represented in the frame (O, β) by its position vector $\vec{x}$. The same point X is represented in the frame (O', β') by its position vector $\vec{x'}$. The motion induces a mapping $\vec{x}_\beta \mapsto \vec{x'}_{\beta'}$ of vectors. Such defined mapping also determines the motion itself and provides its convenient mathematical model.

Let us derive the formula for the mapping $\vec{x}_\beta \mapsto \vec{x'}_{\beta'}$ between the coordinates $\vec{x'}_{\beta'}$ of vector $\vec{x'}$ and coordinates $\vec{x}_\beta$ of vector $\vec{x}$. Consider the following equations:

$$\vec{x} \quad = \quad \vec{x'} + \vec{o'} \tag{2.1}$$

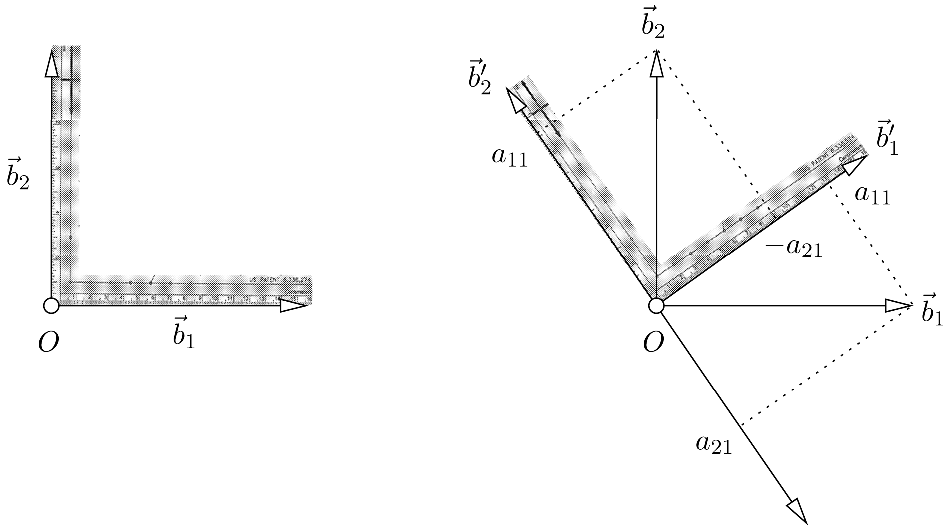
$$
\begin{aligned}
\vec{x'} &= \vec{x} - \vec{o'} && (2.2)\\
\vec{x'}_{\beta'} &= \vec{x}_{\beta'} - \vec{o'}_{\beta'} && (2.3)\\
\vec{x'}_{\beta'} &= \begin{bmatrix} \vec{b}_{1_{\beta'}} & \vec{b}_{2_{\beta'}} & \vec{b}_{3_{\beta'}} \end{bmatrix} \left(\vec{x}_{\beta} - \vec{o'}_{\beta}\right) && (2.4)\\
\vec{x'}_{\beta'} &= \mathtt{R}\left(\vec{x}_{\beta} - \vec{o'}_{\beta}\right) && (2.5)\\
\vec{x'}_{\beta'} &= \mathtt{R}\,\vec{x}_{\beta} - \mathtt{R}\,\vec{o'}_{\beta} && (2.6)\\
\vec{x'}_{\beta'} &= \mathtt{R}\,\vec{x}_{\beta} - \vec{o'}_{\beta'} && (2.7)\\
\vec{x'}_{\beta'} &= \mathtt{R}\,\vec{x}_{\beta} + \vec{o}_{\beta'} && (2.8)
\end{aligned}
$$

Vector $\vec{x}$ is the sum of vectors $\vec{x'}$ and $\vec{o}$, Eq. 2.1, 2.2. We can express all vectors in (the same) basis β', Eq. 2.3. To pass to the basis β of the frame (O, β) we introduce matrix $\mathtt{R} = \begin{bmatrix} \vec{b}_{1_{\beta'}} & \vec{b}_{2_{\beta'}} & \vec{b}_{3_{\beta'}} \end{bmatrix}$, which transforms the coordinates of vectors from β to β', Eq. 2.4. Columns of matrix $\mathtt{R}$ are coordinates $\vec{b}_{1_{\beta'}}, \vec{b}_{2_{\beta'}}, \vec{b}_{3_{\beta'}}$ of basic vectors $\vec{b}_1, \vec{b}_2, \vec{b}_3$ of basis β in basis β'. Equations 2.6, 2.7, 2.8 present frequent variations of the mapping. Notice that equation 2.8 introduced vector $\vec{o} = -\vec{o'}$.

Alternative model of the rigid motion can be developed from the relationship between the points X and Y and their position vectors in Figure 2.1. The point Y is obtained by moving point X altogether with the moving object. It means that the coordinates $\vec{y'}_{\beta'}$ of the position vector $\vec{y'}$ of Y in the coordinate system (O', β') equal the coordinates $\vec{x}_{\beta}$ of the position vector of X in the coordinate system (O, β), i.e.

$$
\begin{aligned}
\vec{y'}_{\beta'} &= \vec{x}_{\beta}\\
\vec{y}_{\beta'} + \vec{o}_{\beta'} &= \vec{x}_{\beta}\\
\mathtt{R}\left(\vec{y}_{\beta} + \vec{o}_{\beta}\right) &= \vec{x}_{\beta}\\
\vec{y}_{\beta} &= \mathtt{R}^{\top}\vec{x}_{\beta} - \vec{o}_{\beta} && (2.9)
\end{aligned}
$$

Equation 2.9 describes how is the point X moved to point Y w.r.t. the coordinate system (O, β).



(a) (b)

Figure 2.2: Rotation in two-dimensional space.

2.2 Rotation matrix

Motion that leaves at least one point fixed is called rotation. Choosing such a fixed point as the origin leads to $O = O'$ and hence to $\vec{o} = \vec{0}$. The motion is then fully described by matrix $\mathtt{R}$, which is called *rotation matrix*

To understand the matrix $\mathtt{R}$, we shall start with an experiment in two-dimensional plane. Imagine a right-angled triangle ruler as shown in Figure 2.2(a) with arms of equal length and let us define a coordinate system as in the figure. Next, rotate the triangle ruler around its tip, i.e. around the origin O of the coordinate system. We know, and we can verify it by direct physical measurement, that thanks to the symmetry of the situation, the parallelograms through the tips of $\vec{b}_1$ and $\vec{b}_2$ and along $\vec{b}_1'$ and $\vec{b}_2'$ will be rotated by 90 degrees. Hence we see that

$$\begin{aligned} \vec{b}_1 &= a_{11}\,\vec{b}_1' + a_{21}\,\vec{b}_2' \\ \vec{b}_2 &= -a_{21}\,\vec{b}_1' + a_{11}\,\vec{b}_2' \end{aligned}$$

for some real numbers a_{11} and a_{21}. By comparing Equations 1.1, 1.2, and 2.8 we conclude that coordinates $\vec{x}_\beta$ are transformed to coordinates $\vec{x}'_{\beta'}$ as

$$\vec{x}'_{\beta'} = \begin{bmatrix} a_{11} & -a_{21} \\ a_{21} & a_{11} \end{bmatrix} \vec{x}_\beta$$

and thus $\mathtt{R} = \begin{bmatrix} a_{11} & -a_{21} \\ a_{21} & a_{11} \end{bmatrix}$. We immediately see that

$$\mathtt{R}^\top\mathtt{R} = \begin{bmatrix} a_{11} & -a_{21} \\ a_{21} & a_{11} \end{bmatrix} \begin{bmatrix} a_{11} & a_{21} \\ -a_{21} & a_{11} \end{bmatrix} = \begin{bmatrix} a_{11}^2 + a_{21}^2 & 0 \\ 0 & a_{11}^2 + a_{21}^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

since $(a_{11}^2 + a_{21}^2)$ is the squared length of the basic vector of b_1, which is one. We hence arrived at an interesting result

$$\begin{aligned} \mathtt{R}^{-1} &= \mathtt{R}^\top \\ \mathtt{R} &= \mathtt{R}^{-\top} \end{aligned}$$

An important observation is that for coordinates $\vec{x}_\beta$ and $\vec{x'}_{\beta'}$ related by motion there holds

$$(x')^2 + (y')^2 = \vec{x'}^\top_{\beta'}\vec{x'}_{\beta'} = (\mathtt{R}\,\vec{x}_\beta)^\top\,\mathtt{R}\,\vec{x}_\beta = \vec{x}_\beta^\top\left(\mathtt{R}^\top\mathtt{R}\right)\vec{x}_\beta = \vec{x}_\beta^\top\vec{x}_\beta = x^2 + y^2$$

Now, if the basis β was constructed as in Figure 2.2, in which case it is called *orthonormal*, then the parallelogram used to measure coordinates x, y of $\vec{x}$ is a rectangle and hence $x^2 + y^2$ is the squared length of $\vec{x}$ by the Pythagoras theorem. If β' is related by rotation, then also $(x')^2 + (y')^2$ is the squared length of $\vec{x}$, again thanks to the Pythagoras theorem.

We see that $\vec{x}_\beta^\top\vec{x}_\beta$ is the squared length of $\vec{x}$ when β is orthonormal and that this length is preserved by computing it in the same way from the new coordinates of $\vec{x}$ in the new coordinate system after motion. The change of coordinates induced by motion is modeled by rotation matrix $\mathtt{R}$, which has the desired property $\mathtt{R}^\top\mathtt{R} = \mathtt{I}$, when the bases β, β' are both orthonormal.

Let us now think about three dimensions. It would be possible to generalize Figure 2.2 to three dimensions, construct orthonormal bases and use rectangular parallelograms to establish the relationship between elements of $\mathtt{R}$ in three dimensions. However, the figure and the derivations would become much more complicated.

We shall follow a more intuitive path instead. Consider that we have found that with two dimensional orthonormal bases, the length of vectors could be computed by the Pythagoras theorem since the parallelograms determining the coordinates were rectangular. To achieve this in three dimensions, we need (and can!) use bases consisting from three orthogonal vectors. Then again the parallelograms will be rectangular and hence the Pythagoras theorem for three dimensions can be used analogically to two dimensions. Now, considering orthonormal bases β, β' we require the following to hold for all vectors $\vec{x}$ with $\vec{x}_\beta = \begin{bmatrix} x & y & z \end{bmatrix}^\top$ and $\vec{x}'_{\beta'} = \begin{bmatrix} x' & y' & z' \end{bmatrix}^\top$

$$
\begin{aligned}
(x')^2 + (y')^2 + (z')^2 &= x^2 + y^2 + z^2 \\
\vec{x'}_{\beta'}^\top \vec{x'}_{\beta'} &= \vec{x}_\beta^\top \vec{x}_\beta \\
(\mathtt{R}\,\vec{x}_\beta)^\top \mathtt{R}\,\vec{x}_\beta &= \vec{x}_\beta^\top \vec{x}_\beta \\
\vec{x}_\beta^\top \left(\mathtt{R}^\top \mathtt{R}\right) \vec{x}_\beta &= \vec{x}_\beta^\top \vec{x}_\beta \\
\vec{x}_\beta^\top \mathtt{C}\,\vec{x}_\beta &= \vec{x}_\beta^\top \vec{x}_\beta
\end{aligned} \tag{2.10}
$$

Equation 2.10 must hold for all vectors $\vec{x}$ and hence also for special vectors such as those with coordinates

$$
\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}
$$

Let us see what that implies, e.g., for the first one

$$
\begin{aligned}
\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \mathtt{C} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} &= 1 \\
c_{11} &= 1
\end{aligned}
$$

The second and the third vector similarly lead to $c_{22} = c_{33} = 1$. Now, let's try the fourth vector

$$
\begin{aligned}
\begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \mathtt{C} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} &= 2 \\
1 + c_{12} + c_{21} + 1 &= 2 \\
c_{12} + c_{21} &= 0
\end{aligned}
$$

The fifth and the sixth vector similarly lead to $c_{13} + c_{31} = c_{23} + c_{32} = 0$. This brings us to the following form of $\mathtt{C}$

$$
\mathtt{C} = \begin{bmatrix} 1 & c_{12} & c_{13} \\ -c_{12} & 1 & c_{23} \\ -c_{13} & -c_{23} & 1 \end{bmatrix}
$$

Moreover, we see that $\mathtt{C}$ is symmetric since

$$
\mathtt{C}^\top = \left(\mathtt{R}^\top \mathtt{R}\right)^\top = \mathtt{R}^\top \mathtt{R} = \mathtt{C}
$$

which leads to $-c_{12} = c_{12}$, $-c_{13} = c_{13}$ and $-c_{23} = c_{23}$, i.e. $c_{12} = c_{13} = c_{23} = 0$ and allows us to conclude that

$$
\mathtt{R}^\top \mathtt{R} = \mathtt{C} = \mathtt{I} \tag{2.11}
$$

Interestingly, not all matrices $\mathtt{R}$ satisfying Equation 2.11 represent motions in three dimensional space.

Consider, e.g., matrix

$$\mathtt{S} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Matrix S does not correspond to any rotation of the space since it keeps the plane xy fixed and reflects all other points w.r.t. this xy plane. We see that there are matrices satisfying Equation 2.11, which are rotations but also are not rotations. Can we somehow distinguish them?

Notice that $\det \mathtt{S} = -1$ while $\det \mathtt{I} = 1$. It might be therefore interesting to study the determinant of $\mathtt{C}$. Consider that

$$1 = \det \mathtt{I} = \det (\mathtt{R}^\top \mathtt{R}) = \det \mathtt{R}^\top \det \mathtt{R} = \det \mathtt{R} \det \mathtt{R} = (\det \mathtt{R})^2$$

which gives that $\det \mathtt{R} = \pm 1$. We see that the sign of the determinant splits all matrices satisfying Equation 2.11 into two groups – rotations, which have positive determinant, and reflections, which have negative determinant. Product of any two rotations will again be rotation, product of a rotation and a reflection will be a reflection and product of two reflections will be a rotation.

To summarize, rotation in three dimensional space is represented by 3×3 matrices $\mathtt{R}$ with $\mathtt{R}^\top \mathtt{R} = \mathtt{I}$ and $\det \mathtt{R} = 1$.

2.3 Coordinate vectors

We see that the matrix $\mathtt{R}$ induced by motion has the property that coordinates and the basic vectors are transformed in the same way. This is particularly useful observation when β is formed by the standard basis, i.e.

$$\beta = \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

For motion Equation 1.5 becomes

$$\begin{bmatrix} \vec{b}'_1 \\ \vec{b}'_2 \\ \vec{b}'_3 \end{bmatrix} = \mathtt{R} \begin{bmatrix} \vec{b}_1 \\ \vec{b}_2 \\ \vec{b}_3 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \begin{bmatrix} \vec{b}_1 \\ \vec{b}_2 \\ \vec{b}_3 \end{bmatrix} = \begin{bmatrix} r_{11}\,\vec{b}_1 + r_{12}\,\vec{b}_2 + r_{13}\,\vec{b}_3 \\ r_{21}\,\vec{b}_1 + r_{22}\,\vec{b}_2 + r_{23}\,\vec{b}_3 \\ r_{31}\,\vec{b}_1 + r_{32}\,\vec{b}_2 + r_{33}\,\vec{b}_3 \end{bmatrix}$$

and hence

$$\vec{b}'_1 = r_{11}\,\vec{b}_1 + r_{12}\,\vec{b}_2 + r_{13}\,\vec{b}_3 = r_{11} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + r_{12} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + r_{13} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} r_{11} \\ r_{12} \\ r_{13} \end{bmatrix}$$

and similarly for $\vec{b}'_2$ and $\vec{b}'_3$. We conclude that

$$\begin{bmatrix} \vec{b}'_1 & \vec{b}'_2 & \vec{b}'_3 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{21} & r_{31} \\ r_{12} & r_{22} & r_{32} \\ r_{13} & r_{23} & r_{33} \end{bmatrix} = \mathtt{R}^\top$$

This also corresponds to solving for $\mathtt{R}$ in Equation 1.4 with $\mathtt{A} = \mathtt{R}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \vec{b}'_1 & \vec{b}'_2 & \vec{b}'_3 \end{bmatrix} \mathtt{R}$$

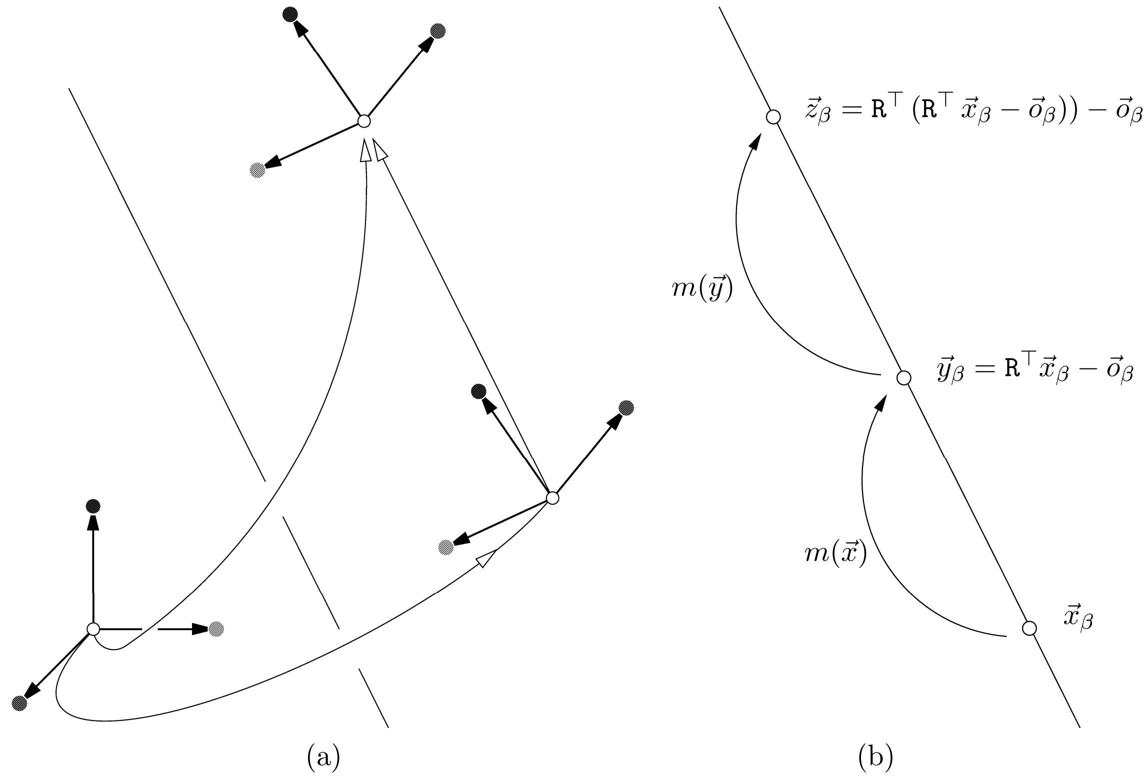


Figure 2.3: Axis of motion.

2.4 Axis of motion

We will study the motion and show that every motion in three dimensional space has an axis of motion. Axis of motion is a line of fixed points that remain fixed under the motion, Figure 2.3.

Let us consider Equation 2.9 and denote the motion so defined as $m(\vec{x}_\beta) = \mathtt{R}^\top\vec{x}_\beta - \vec{o}_\beta$. Apply the motion $m(\vec{x})$ twice to obtain $\vec{y_\beta} = m(\vec{x}_\beta)$ and $\vec{z_\beta} = m(\vec{y}_\beta)$. It then holds

$$\begin{aligned}
\vec{z}_\beta - \vec{y}_\beta &= \vec{y}_\beta - \vec{x}_\beta \\
\mathtt{R}^\top(\mathtt{R}^\top\vec{x}_\beta - \vec{o}_\beta)) - \vec{o}_\beta - \mathtt{R}^\top\vec{x}_\beta + \vec{o}_\beta &= \mathtt{R}^\top\vec{x}_\beta - \vec{o}_\beta - \vec{x}_\beta \\
(\mathtt{R}^\top)^2\vec{x}_\beta - \mathtt{R}^\top\vec{o}_\beta - \mathtt{R}^\top\vec{x}_\beta &= \mathtt{R}^\top\vec{x}_\beta - \vec{o}_\beta - \vec{x}_\beta \\
(\mathtt{R}^\top)^2\vec{x}_\beta - 2\,\mathtt{R}^\top\vec{x}_\beta + \vec{x}_\beta &= \mathtt{R}^\top\vec{o}_\beta - \vec{o}_\beta \\
\left((\mathtt{R}^\top)^2 - 2\,\mathtt{R}^\top + \mathtt{I}\right)\vec{x}_\beta &= (\mathtt{R}^\top - \mathtt{I})\vec{o}_\beta \\
(\mathtt{R}^\top - \mathtt{I})(\mathtt{R}^\top - \mathtt{I})\,\vec{x}_\beta &= (\mathtt{R}^\top - \mathtt{I})\,\vec{o}_\beta \\
(\mathtt{R}^\top - \mathtt{I})\left((\mathtt{R}^\top - \mathtt{I})\,\vec{x}_\beta - \vec{o}_\beta\right) &= 0 \qquad (2.12)
\end{aligned}$$

Equation 2.12 has always a solution since if $\vec{o}_\beta$ is in the span of $(\mathtt{R}^\top - \mathtt{I})$, then we can find $\vec{x}_\beta$ to make $(\mathtt{R}^\top - \mathtt{I})\,\vec{x}_\beta - \vec{o}_\beta = 0$. Otherwise, if $\vec{o}_\beta$ is not in the span of $(\mathtt{R}^\top - \mathtt{I})$, then $(\mathtt{R}^\top - \mathtt{I})$ must be of rank lower than 3 and $\vec{o}_\beta \neq \vec{0}$. It can be shown that either $rank(\mathtt{R}^\top - \mathtt{I}) = 2$ or $\mathtt{R}^\top - \mathtt{I} = \mathtt{0}$.

In the former case $(\mathtt{R}^\top - \mathtt{I})\,\vec{x}_\beta - \vec{o}_\beta$ spans $\mathbb{R}^3$ and hence can generate a nonzero vector from the null space of $\mathtt{R}^\top - \mathtt{I}$. In the latter case the equation clearly holds for all $\vec{x}_\beta$.

Bibliography

[1] Carl D. Meyer. *Matrix Analysis and Applied Linear Algebra.* SIAM: Society for Industrial and Applied Mathematics, Philadelphia, PA, USA, 2001.

[2] Jim Hefferon. *Linear Algebra.* Virginia Commonwealth University Mathematics, 2009.

[3] Petr Olšák. Úvod do algebry, zejména lineární. FEL ČVUT, Praha, 2007.

[4] Gilbert Strang. *Introduction to Linear Algebra.* Wellesley Cambridge, 3rd edition, 2003.

[5] Seymour Lipschutz. *3,000 Solved Problems in Linear Algebra.* McGraw-Hill, 1st edition, 1989.