Advanced Robotics

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Contents

1 Notation

$\mathbb{R}$	...	real numbers
$\vec{x}$	...	vector
$\mathtt{A}$	...	matrix
$\mathtt{A}^\top$	...	the transpose of $\mathtt{A}$
$\mathtt{I}$	...	the identity matrix
$\mathtt{R}$	...	a rotation matrix
$\beta = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix}$	...	an ordered triple of vectors
$\vec{x}_\beta$	...	the column matrix of coordinates of $\vec{x}$ w.r.t. the basis β

2 Change of coordinates induced by the change of basis

Let us discuss the relationship between the coordinates of a vector in a linear space, which is induced by passing from one basis to another. We shall derive the relationship between the coordinates in a three-dimensional linear space V^3 over real numbers, which is the most important when modeling the geometry around us. The formulas for all other n-dimensional spaces are obtained by passing from 3 to n.

Coordinates Let us consider an ordered basis $\beta = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix}$ of V^3. A vector $\vec{x} \in V^3$ is uniquely expressed as a linear combination of the basic vectors by its *coordinates* x, y, z, i.e. $\vec{x} = x\,\vec{b}_1 + y\,\vec{b}_2 + z\,\vec{b}_3$ and can be represented as an ordered triple of coordinates, as a coordinate vector $\vec{x}_\beta = \begin{bmatrix} x & y & z \end{bmatrix}^\top$.

Two bases Having two ordered bases $\beta = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix}$ and $\beta' = \begin{bmatrix} \vec{b}'_1 & \vec{b}'_2 & \vec{b}'_3 \end{bmatrix}$ leads to expressing one vector $\vec{x}$ in two ways as $\vec{x} = x\,\vec{b}_1 + y\,\vec{b}_2 + z\,\vec{b}_3$ and $\vec{x} = x'\,\vec{b}'_1 + y'\,\vec{b}'_2 + z'\,\vec{b}'_3$. The vectors of the basis β can also be expressed in the basis β' using their coordinates. Let us introduce

$$\begin{aligned} \vec{b}_1 &= a_{11}\,\vec{b}'_1 + a_{21}\,\vec{b}'_2 + a_{31}\,\vec{b}'_3 \\ \vec{b}_2 &= a_{12}\,\vec{b}'_1 + a_{22}\,\vec{b}'_2 + a_{32}\,\vec{b}'_3 \\ \vec{b}_3 &= a_{13}\,\vec{b}'_1 + a_{23}\,\vec{b}'_2 + a_{33}\,\vec{b}'_3 \end{aligned} \tag{2.1}$$

Change of coordinates We will next use the above equations to relate the coordinates of $\vec{x}$ w.r.t. the basis β to the coordinates of $\vec{x}$ w.r.t. the basis β'

$$\begin{aligned} \vec{x} &= x\,\vec{b}_1 + y\,\vec{b}_2 + z\,\vec{b}_3 \\ &= x\,(a_{11}\,\vec{b}'_1 + a_{21}\,\vec{b}'_2 + a_{31}\,\vec{b}'_3) + y\,(a_{12}\,\vec{b}'_1 + a_{22}\,\vec{b}'_2 + a_{32}\,\vec{b}'_3) + z\,(a_{13}\,\vec{b}'_1 + a_{23}\,\vec{b}'_2 + a_{33}\,\vec{b}'_3) \\ &= (a_{11}\,x + a_{12}\,y + a_{13}\,z)\,\vec{b}'_1 + (a_{21}\,x + a_{22}\,y + a_{23}\,z)\,\vec{b}'_2 + (a_{31}\,x + a_{32}\,y + a_{33}\,z)\,\vec{b}'_3 \\ &= x'\,\vec{b}'_1 + y'\,\vec{b}'_2 + z'\,\vec{b}'_3 \end{aligned}$$

Since coordinates are unique, we get

$$\begin{aligned} x' &= a_{11}\,x + a_{12}\,y + a_{13}\,z \\ y' &= a_{21}\,x + a_{22}\,y + a_{23}\,z \\ z' &= a_{31}\,x + a_{32}\,y + a_{33}\,z \end{aligned}$$

Coordinate vectors $\vec{x}_\beta$ and $\vec{x}'_\beta$ are thus related by the following matrix multiplication

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

which we concisely write as

$$\vec{x}_{\beta'} = \mathtt{A}\,\vec{x}_{\beta} \tag{2.2}$$

The columns of matrix $\mathtt{A}$ can be viewed as coordinate vectors representing basic vectors, $\vec{b}_1, \vec{b}_2, \vec{b}_3$ of β in the basis β'

$$\mathtt{A} = \begin{bmatrix} | & | & | \\ \vec{b}_{1_{\beta'}} & \vec{b}_{2_{\beta'}} & \vec{b}_{3_{\beta'}} \\ | & | & | \end{bmatrix}$$

and the matrix multiplication can be interpreted as the linear combination of columns of $\mathtt{A}$ by coordinates of $\vec{x}$ w.r.t. β

$$\vec{x}_{\beta'} = x\,\vec{b}_{1_{\beta'}} + y\,\vec{b}_{2_{\beta'}} + z\,\vec{b}_{3_{\beta'}}$$

Matrix $\mathtt{A}$ plays such an important role here that it deserves its own name. Matrix $\mathtt{A}$ is very often called the *change of basis matrix from basis β to β'* or the *transition matrix from basis β to basis β'* [1, 2] since it can be used to pass from coordinates w.r.t. β to coordinates w.r.t. β' by Equation 2.2.

However, literature [3, 4] calls $\mathtt{A}$ the *change of basis matrix from basis β' to β*, i.e. it (seemingly illogically) swaps the bases. This choice is motivated by the fact that $\mathtt{A}$ can also be used to obtain the vectors of β from vectors of β' using Equation 2.1 as

$$\begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix} = \begin{bmatrix} a_{11}\,\vec{b}'_1 + a_{21}\,\vec{b}'_2 + a_{31}\,\vec{b}'_3 & a_{12}\,\vec{b}'_1 + a_{22}\,\vec{b}'_2 + a_{32}\,\vec{b}'_3 & a_{13}\,\vec{b}'_1 + a_{23}\,\vec{b}'_2 + a_{33}\,\vec{b}'_3 \end{bmatrix} \tag{2.3}$$

$$= \begin{bmatrix} \vec{b}'_1 & \vec{b}'_2 & \vec{b}'_3 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$= \begin{bmatrix} \vec{b}'_1 & \vec{b}'_2 & \vec{b}'_3 \end{bmatrix} \mathtt{A} \tag{2.4}$$

where the multiplication of a row of column vectors by a matrix from the right in Equation 2.4 has the meaning given by Equation 2.3 above. Yet another variation of the naming appeared in [5, 6] where $\mathtt{A}^{-1}$ was named the *change of basis matrix from basis β to β'*.

We have to conclude that the meaning associated with the *change of basis matrix* varies in the literature and hence we will avoid this confusing name and talk about $\mathtt{A}$ as about the *matrix transforming coordinates of a vector from basis β to basis β'*.

There is the following interesting variation of Equation 2.4

$$\begin{bmatrix} \vec{b}'_1 \\ \vec{b}'_2 \\ \vec{b}'_3 \end{bmatrix} = \mathtt{A}^{-\top} \begin{bmatrix} \vec{b}_1 \\ \vec{b}_2 \\ \vec{b}_3 \end{bmatrix} \tag{2.5}$$

where the basic vectors of β and β' are understood as elements of column vectors. The following questions arises: When are the coordinates of a vector $\vec{x}$ (Equation 2.2) and the basic vectors themselves (Equation 2.5) transformed in the same way? In other words, when $\mathtt{A} = \mathtt{A}^{-\top}$.

3 Motion

Let us introduce a mathematical model of rigid motion in three-dimensional Euclidean space. The important property of motion is that it only relocates objects without changing their shape. Distances between points on moving objects remain equal.

3.1 Change of position vector coordinates induced by motion

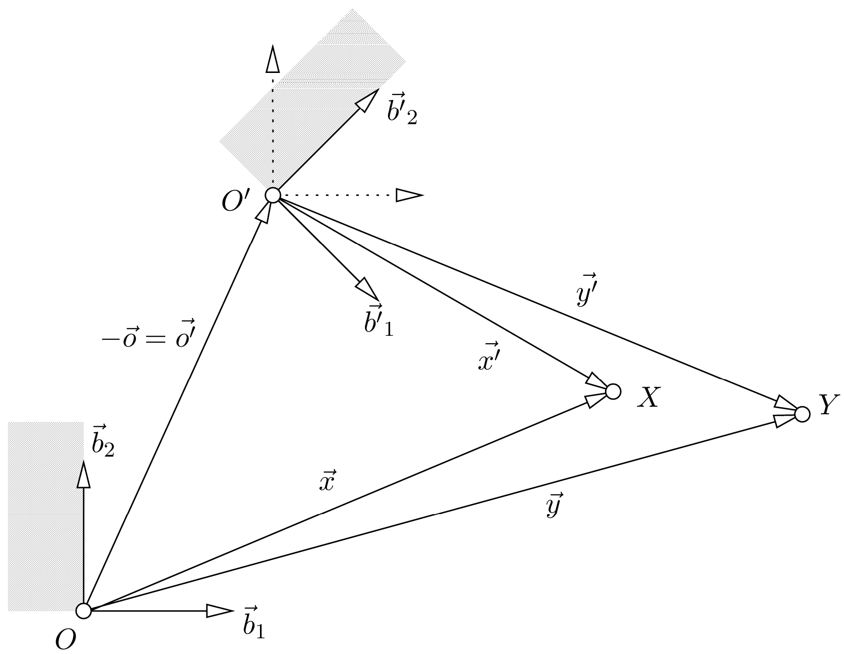


Figure 3.1: Representation of rigid motion.

Figure 3.1 illustrates the model of rigid motion using coordinate systems, points and their position vectors. A coordinate system (O, β) with origin O and basis β is attached to a moving rigid body. As the body moves to a new position, a new coordinate system (O', β') is constructed. Assume a point X in a general position w.r.t. the body, which is represented in the frame (O, β) by its position vector $\vec{x}$. The same point X is represented in the frame (O', β') by its position vector $\vec{x'}$. The motion induces a mapping $\vec{x}_\beta \mapsto \vec{x'}_{\beta'}$. Such a mapping also determines the motion itself and provides its convenient mathematical model.

Let us derive the formula for the mapping $\vec{x}_\beta \mapsto \vec{x'}_{\beta'}$ between the coordinates $\vec{x'}_{\beta'}$ of vector $\vec{x'}$ and coordinates $\vec{x}_\beta$ of vector $\vec{x}$. Consider the following equations:

$$\vec{x} = \vec{x'} + \vec{o'} \tag{3.1}$$

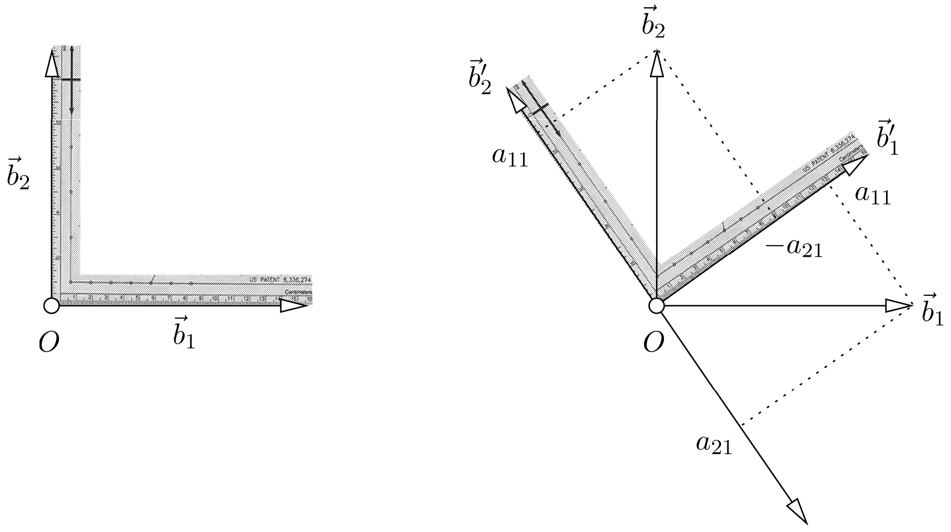
$$\vec{x'} = \vec{x} - \vec{o'} \tag{3.2}$$
$$\vec{x'}_{\beta'} = \vec{x}_{\beta'} - \vec{o'}_{\beta'} \tag{3.3}$$
$$\vec{x'}_{\beta'} = \begin{bmatrix} \vec{b}_{1_{\beta'}} & \vec{b}_{2_{\beta'}} & \vec{b}_{3_{\beta'}} \end{bmatrix} \left(\vec{x}_{\beta} - \vec{o'}_{\beta}\right) \tag{3.4}$$
$$\vec{x'}_{\beta'} = \mathtt{R} \left(\vec{x}_{\beta} - \vec{o'}_{\beta}\right) \tag{3.5}$$
$$\vec{x'}_{\beta'} = \mathtt{R}\,\vec{x}_{\beta} - \mathtt{R}\,\vec{o'}_{\beta} \tag{3.6}$$
$$\vec{x'}_{\beta'} = \mathtt{R}\,\vec{x}_{\beta} - \vec{o'}_{\beta'} \tag{3.7}$$
$$\vec{x'}_{\beta'} = \mathtt{R}\,\vec{x}_{\beta} + \vec{o}_{\beta'} \tag{3.8}$$

Vector $\vec{x}$ is the sum of vectors $\vec{x'}$ and $\vec{o}$, Equation 3.1, 3.2. We can express all vectors in (the same) basis β', Equation 3.3. To pass to the basis β of the frame (O, β) we introduce matrix $\mathtt{R} = \begin{bmatrix} \vec{b}_{1_{\beta'}} & \vec{b}_{2_{\beta'}} & \vec{b}_{3_{\beta'}} \end{bmatrix}$, which transforms the coordinates of vectors from β to β', Eq. 3.4. Columns of matrix $\mathtt{R}$ are coordinates $\vec{b}_{1_{\beta'}}, \vec{b}_{2_{\beta'}}, \vec{b}_{3_{\beta'}}$ of basic vectors $\vec{b}_1, \vec{b}_2, \vec{b}_3$ of basis β in basis β'. Equations 3.6, 3.7, 3.8 present frequent variations of the mapping. Notice that equation 3.8 introduced vector $\vec{o} = -\vec{o'}$.

Alternative model of the rigid motion can be developed from the relationship between the points X and Y and their position vectors in Figure 3.1. The point Y is obtained by moving point X altogether with the moving object. It means that the coordinates $\vec{y'}_{\beta'}$ of the position vector $\vec{y'}$ of Y in the coordinate system (O', β') equal the coordinates $\vec{x}_{\beta}$ of the position vector $\vec{x}$ of X in the coordinate system (O, β), i.e.

$$\vec{y'}_{\beta'} = \vec{x}_{\beta}$$
$$\vec{y}_{\beta'} + \vec{o}_{\beta'} = \vec{x}_{\beta}$$
$$\mathtt{R}\left(\vec{y}_{\beta} + \vec{o}_{\beta}\right) = \vec{x}_{\beta}$$
$$\vec{y}_{\beta} = \mathtt{R}^{\top}\vec{x}_{\beta} - \vec{o}_{\beta} \tag{3.9}$$

Equation 3.9 describes how is the point X moved to point Y w.r.t. the coordinate system (O, β).



(a) (b)

Figure 3.2: Rotation in two-dimensional space.

3.2 Rotation matrix

Motion that leaves at least one point fixed is called rotation. Choosing such a fixed point as the origin leads to $O = O'$ and hence to $\vec{o} = \vec{0}$. The motion is then fully described by matrix $\mathtt{R}$, which is called *rotation matrix*

To understand the matrix $\mathtt{R}$, we shall start with an experiment in two-dimensional plane. Imagine a right-angled triangle ruler as shown in Figure 3.2(a) with arms of equal length and let us define a coordinate system as in the figure. Next, rotate the triangle ruler around its tip, i.e. around the origin O of the coordinate system. We know, and we can verify it by direct physical measurement, that thanks to the symmetry of the situation, the parallelograms through the tips of $\vec{b}_1$ and $\vec{b}_2$ and along $\vec{b}'_1$ and $\vec{b}'_2$ will be rotated by 90 degrees. We see that

$$\begin{aligned}\vec{b}_1 &= a_{11}\,\vec{b}'_1 + a_{21}\,\vec{b}'_2 \\ \vec{b}_2 &= -a_{21}\,\vec{b}'_1 + a_{11}\,\vec{b}'_2\end{aligned}$$

for some real numbers a_{11} and a_{21}. By comparing Equations 2.1, 2.2, and 3.8 we conclude that coordinates $\vec{x}_\beta$ are transformed to coordinates $\vec{x}'_{\beta'}$ as

$$\vec{x}'_{\beta'} = \begin{bmatrix} a_{11} & -a_{21} \\ a_{21} & a_{11} \end{bmatrix} \vec{x}_\beta$$

and thus

$$\mathtt{R} = \begin{bmatrix} a_{11} & -a_{21} \\ a_{21} & a_{11} \end{bmatrix}$$

We immediately see that

$$\mathtt{R}^\top \mathtt{R} = \begin{bmatrix} a_{11} & a_{21} \\ -a_{21} & a_{11} \end{bmatrix} \begin{bmatrix} a_{11} & -a_{21} \\ a_{21} & a_{11} \end{bmatrix} = \begin{bmatrix} a_{11}^2 + a_{21}^2 & 0 \\ 0 & a_{11}^2 + a_{21}^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

since $(a_{11}^2 + a_{21}^2)$ is the squared length of the basic vector of b_1, which is one. We derived an interesting result

$$\begin{aligned}\mathtt{R}^{-1} &= \mathtt{R}^\top \\ \mathtt{R} &= \mathtt{R}^{-\top}\end{aligned}$$

Next important observation is that for coordinates $\vec{x}_\beta$ and $\vec{x'}_{\beta'}$, related by rotation, there holds

$$(x')^2 + (y')^2 = \vec{x'}^\top_{\beta'} \vec{x'}_{\beta'} = (\mathtt{R}\,\vec{x}_\beta)^\top \mathtt{R}\,\vec{x}_\beta = \vec{x}_\beta^\top \left(\mathtt{R}^\top \mathtt{R}\right) \vec{x}_\beta = \vec{x}_\beta^\top \vec{x}_\beta = x^2 + y^2$$

Now, if the basis β was constructed as in Figure 3.2, in which case it is called *orthonormal*, then the parallelogram used to measure coordinates x, y of $\vec{x}$ is a rectangle and hence $x^2 + y^2$ is the squared length of $\vec{x}$ by the Pythagoras theorem. If β' is related by rotation, then also $(x')^2 + (y')^2$ is the squared length of $\vec{x}$, again thanks to the Pythagoras theorem.

We see that $\vec{x}_\beta^\top \vec{x}_\beta$ is the squared length of $\vec{x}$ when β is orthonormal and that this length is preserved by computing it in the same way from the new coordinates of $\vec{x}$ in the new coordinate system after motion. The change of coordinates induced by motion is modeled by rotation matrix $\mathtt{R}$, which has the desired property $\mathtt{R}^\top \mathtt{R} = \mathtt{I}$, when the bases β, β' are both orthonormal.

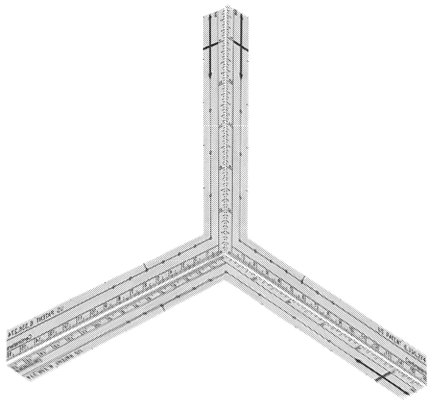


Figure 3.3: A three-dimensional coordinate system.

Let us now think about three dimensions. It would be possible to generalize Figure 3.2 to three dimensions, construct orthonormal bases and use rectangular parallelograms to establish the relationship between elements of $\mathtt{R}$ in three dimensions. However, the figure and the derivations would become much more complicated.

We shall follow a more intuitive path instead. Consider that we have found that with two-dimensional orthonormal bases, the length of vectors could be computed by the Pythagoras theorem since the parallelograms determining the coordinates were rectangular. To achieve this in three dimensions, we need (and can!) use bases consisting from three orthogonal vectors. Then again the parallelograms will be rectangular and hence the Pythagoras theorem for three dimensions can be used analogically as in two dimensions, Figure 3.3.

Considering orthonormal bases β, β' we require the following to hold for all vectors $\vec{x}$ with $\vec{x}_\beta = \begin{bmatrix} x & y & z \end{bmatrix}^\top$ and $\vec{x}'_{\beta'} = \begin{bmatrix} x' & y' & z' \end{bmatrix}^\top$

$$\begin{aligned}
(x')^2 + (y')^2 + (z')^2 &= x^2 + y^2 + z^2 \\
\vec{x'}^\top_{\beta'} \vec{x'}_{\beta'} &= \vec{x}^\top_\beta \vec{x}_\beta \\
(\mathtt{R}\,\vec{x}_\beta)^\top \mathtt{R}\,\vec{x}_\beta &= \vec{x}^\top_\beta \vec{x}_\beta \\
\vec{x}^\top_\beta \left(\mathtt{R}^\top \mathtt{R}\right) \vec{x}_\beta &= \vec{x}^\top_\beta \vec{x}_\beta \\
\vec{x}^\top_\beta \mathtt{C}\,\vec{x}_\beta &= \vec{x}^\top_\beta \vec{x}_\beta
\end{aligned} \tag{3.10}$$

Equation 3.10 must hold for all vectors $\vec{x}$ and hence also for special vectors such as those with coordinates

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Let us see what that implies, e.g., for the first vector

$$\begin{aligned}
\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \mathtt{C} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} &= 1 \\
c_{11} &= 1
\end{aligned}$$

Taking the second and the third vector similarly leads to $c_{22} = c_{33} = 1$. Now, let's try the fourth vector

$$\begin{aligned}
\begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \mathtt{C} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} &= 2 \\
1 + c_{12} + c_{21} + 1 &= 2 \\
c_{12} + c_{21} &= 0
\end{aligned}$$

Again, taking the fifth and the sixth vector leads to $c_{13} + c_{31} = c_{23} + c_{32} = 0$. This brings us to the following form of $\mathtt{C}$

$$\mathtt{C} = \begin{bmatrix} 1 & c_{12} & c_{13} \\ -c_{12} & 1 & c_{23} \\ -c_{13} & -c_{23} & 1 \end{bmatrix}$$

Moreover, we see that $\mathtt{C}$ is symmetric since

$$\mathtt{C}^\top = \left(\mathtt{R}^\top \mathtt{R}\right)^\top = \mathtt{R}^\top \mathtt{R} = \mathtt{C}$$

which leads to $-c_{12} = c_{12}$, $-c_{13} = c_{13}$ and $-c_{23} = c_{23}$, i.e. $c_{12} = c_{13} = c_{23} = 0$ and allows us to conclude that

$$\mathtt{R}^\top \mathtt{R} = \mathtt{C} = \mathtt{I} \tag{3.11}$$

Interestingly, not all matrices $\mathtt{R}$ satisfying Equation 3.11 represent motions in three-dimensional space.

Consider, e.g., matrix

$$\mathtt{S} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Matrix S does not correspond to any rotation of the space since it keeps the plane xy fixed and reflects all other points w.r.t. this xy plane. We see that some matrices satisfying Equation 3.11 are rotations but are such matrices also are not rotations. Can we somehow distinguish them?

Notice that $\det \mathtt{S} = -1$ while $\det \mathtt{I} = 1$. It might be therefore interesting to study the determinant of $\mathtt{C}$ in general. Consider that

$$1 = \det \mathtt{I} = \det \left(\mathtt{R}^\top \mathtt{R}\right) = \det \mathtt{R}^\top \det \mathtt{R} = \det \mathtt{R} \det \mathtt{R} = (\det \mathtt{R})^2$$

which gives that $\det \mathtt{R} = \pm 1$. We see that the sign of the determinant splits all matrices satisfying Equation 3.11 into two groups – rotations, which have positive determinant, and reflections, which have negative determinant. Product of any two rotations will again be rotation, product of a rotation and a reflection will be a reflection and product of two reflections will be a rotation.

To summarize, rotation in three-dimensional space is represented by a 3×3 matrix $\mathtt{R}$ with $\mathtt{R}^\top \mathtt{R} = \mathtt{I}$ and $\det \mathtt{R} = 1$.

3.3 Coordinate vectors

We see that the matrix $\mathtt{R}$ induced by motion has the property that coordinates and the basic vectors are transformed in the same way. This is particularly useful observation when β is formed by the standard basis, i.e.

$$\beta = \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

For motion Equation 2.5 becomes

$$\begin{bmatrix} \vec{b}'_1 \\ \vec{b}'_2 \\ \vec{b}'_3 \end{bmatrix} = \mathtt{R} \begin{bmatrix} \vec{b}_1 \\ \vec{b}_2 \\ \vec{b}_3 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \begin{bmatrix} \vec{b}_1 \\ \vec{b}_2 \\ \vec{b}_3 \end{bmatrix} = \begin{bmatrix} r_{11}\,\vec{b}_1 + r_{12}\,\vec{b}_2 + r_{13}\,\vec{b}_3 \\ r_{21}\,\vec{b}_1 + r_{22}\,\vec{b}_2 + r_{23}\,\vec{b}_3 \\ r_{31}\,\vec{b}_1 + r_{32}\,\vec{b}_2 + r_{33}\,\vec{b}_3 \end{bmatrix}$$

and hence

$$\vec{b}'_1 = r_{11}\,\vec{b}_1 + r_{12}\,\vec{b}_2 + r_{13}\,\vec{b}_3 = r_{11} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + r_{12} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + r_{13} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} r_{11} \\ r_{12} \\ r_{13} \end{bmatrix}$$

and similarly for $\vec{b}'_2$ and $\vec{b}'_3$. We conclude that

$$\begin{bmatrix} \vec{b}'_1 & \vec{b}'_2 & \vec{b}'_3 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{21} & r_{31} \\ r_{12} & r_{22} & r_{32} \\ r_{13} & r_{23} & r_{33} \end{bmatrix} = \mathtt{R}^\top$$

This also corresponds to solving for $\mathtt{R}$ in Equation 2.4 with $\mathtt{A} = \mathtt{R}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \vec{b}'_1 & \vec{b}'_2 & \vec{b}'_3 \end{bmatrix} \mathtt{R}$$

3.4 Properties of rotation matrix

Let us study additional properties of the rotation matrix in three-dimensional space.

Eigenvalues of $\mathtt{R}$ Let $\mathtt{R}$ be a rotation matrix. Then for every $\vec{v} \in \mathbb{C}^3$

$$(\mathtt{R}\,\vec{v})^\top \mathtt{R}\,\vec{v} = \vec{v}^\top \mathtt{R}^\top \mathtt{R}\,\vec{v} = \vec{v}^\top (\mathtt{R}^\top \mathtt{R})\,\vec{v} = \vec{v}^\top \vec{v}$$

we see that for all $\vec{v} \in \mathbb{C}^3$ and $\lambda \in \mathbb{C}$ such that

$$\mathtt{R}\,\vec{v} = \lambda\,\vec{v}$$

then holds

$$|\lambda|^2 (\vec{v}^\top \vec{v}) = (\vec{v}^\top \vec{v})$$

and hence $|\lambda|^2 = 1$ for all $\vec{v} \neq \vec{0}$. We conclude that the absolute value of eigenvalues of $\mathtt{R}$ is one.

Next, there is a real unit eigenvalue since $\mathtt{R}$ is a real matrix with characteristic polynomial

$$\begin{aligned} p(\lambda) &= \det(\lambda\,\mathtt{I} - \mathtt{R}) = \det \left(\begin{bmatrix} \lambda - r_{11} & -r_{12} & -r_{13} \\ -r_{21} & \lambda - r_{22} & -r_{23} \\ -r_{31} & -r_{32} & \lambda - r_{33} \end{bmatrix} \right) \\ &= \lambda^3 - (r_{11} + r_{22} + r_{33})\,\lambda^2 \\ &\quad + (r_{11}\,r_{22} - r_{21}\,r_{12} + r_{11}\,r_{33} - r_{31}\,r_{13} + r_{22}\,r_{33} - r_{23}\,r_{32})\,\lambda \\ &\quad + r_{11}\,(r_{23}\,r_{32} - r_{22}\,r_{33}) - r_{21}\,(r_{32}\,r_{13} - r_{12}\,r_{33}) + r_{31}\,(r_{13}\,r_{22} - r_{12}\,r_{23}) \\ &= \lambda^3 - \operatorname{trace} \mathtt{R}\,\lambda^2 + (\mathtt{R}_{11} + \mathtt{R}_{22} + \mathtt{R}_{33})\,\lambda - \det \mathtt{R} \end{aligned}$$

It follows from the Fundamental theorem of algebra [7] the $p(\lambda) = 0$ has always a solution in $\mathbb{C}$ and since coefficients of $p(\lambda)$ are all real, the solutions must come in complex conjugated pairs. The degree of $p(\lambda)$ is three and thus at least one solution must be real and hence equal to ± 1. Now, since $p(0) = -\det(\mathtt{R}) = -1$, $\lim_{\lambda\to\infty} p(\lambda) = \infty$, and $p(\lambda)$ is a continuous finction, it must cross the positive side of the real axis and hence one eigenvalue has to be one. Let us denote the solutions as $\lambda_1 = 1$, $\lambda_2 = x + yi$ and $\lambda_3 = x - yi$ with real x, y. It follows from above that $x^2 + y^2 = 1$. We see that there is either one real or three real solutions since if $y = 0$, then $x^2 = 1$ and hence $\lambda_2 = \lambda_3 = \pm 1$. We conclude that we can encounter only two situations when all eigenvalues are real. Either $\lambda_1 = \lambda_2 = \lambda_3 = 1$, or $\lambda_1 = 1$ and $\lambda_2 = \lambda_3 = -1$.

Eigenvectors of $\mathtt{R}$. Let us now look at eigenvectors of $\mathtt{R}$ and let's first investigate the situation when all eigenvalues of $\mathtt{R}$ are real.

Let $\lambda_1 = \lambda_2 = \lambda_3 = 1$. Then $p(\lambda) = (\lambda - 1)^3 = \lambda^3 - 3\,\lambda^2 + 3\,\lambda - 1$. It means that $r_{11} + r_{22} + r_{33} = 1$ and since $r_{11} \le 1$, $r_{22} \le 1$, $r_{33} \le 1$, it leads to $r_{11} = r_{22} = r_{33} = 1$, which implies $\mathtt{R} = \mathtt{I}$. Then $\mathtt{I} - \mathtt{R} = 0$ and all non-zero vectors of $\mathbb{R}^3$ are eigenvectors of $\mathtt{R}$. Notice that rank of $\mathtt{R} - \mathtt{I}$ is zero in this case.

Next consider $\lambda_1 =$ and $\lambda_2 = \lambda_3 = -1$. The eigenvectors $\vec{v}$ corresponding to $\lambda_2 = \lambda_3 = -1$ are solutions to

$$\mathtt{R}\,\vec{v} \;=\; -\vec{v}$$

there is always at least one one-dimensional space of such vectors. We also see that there is a rotation matrix

$$\mathtt{R} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

with eigenvectors

$$r \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\; r \neq 0, \qquad \text{or} \qquad s \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},\; s\,t \neq 0,$$

which means that there is a one-dimensional space of eigenvectors corresponding to 1 and a two-dimensional space of eigenvectors corresponding to -1. Notice that rank of $\mathtt{R} - \mathtt{I}$ is two here.

How does the situation look for a general $\mathtt{R}$ with such eigenvalues. Consider an eigenvector $\vec{v}_1$ corresponding to 1 and an eigenvector $\vec{v}_2$ corresponding to -1. They are linearly independent, otherwise there has to be $s \in \mathbb{R}$ such that $\vec{v}_2 = s\,\vec{v}_1 \neq 0$ and then

$$\begin{aligned} \vec{v}_2 &= s\,\vec{v}_1 \\ \mathtt{R}\,\vec{v}_2 &= s\,\mathtt{R}\,\vec{v}_1 \\ -\vec{v}_2 &= s\,\vec{v}_1 \end{aligned}$$

leading to $s = -s$ and therefore $s = 0$ which contradicts $\vec{v}_2 \neq 0$. Now, let us look at vectors $\vec{v}_3 \in \mathbb{R}^3$ defined by

$$\begin{bmatrix} \vec{v}_1^\top \\ \vec{v}_2^\top \end{bmatrix} \vec{v}_3 \;=\; 0$$

The above linear system has a one-dimensional space of solutions since the rows of its matrix are independent. Choose a fixed solution $\vec{v}_3 \neq 0$. Then

$$\begin{bmatrix} \vec{v}_1^\top \\ \vec{v}_2^\top \end{bmatrix} \mathtt{R}\,\vec{v}_3 = \begin{bmatrix} \vec{v}_1^\top \mathtt{R}^\top \\ \vec{v}_2^\top \mathtt{R}^\top \end{bmatrix} \vec{v}_3 = \begin{bmatrix} \vec{v}_1^\top \\ -\vec{v}_2^\top \end{bmatrix} \vec{v}_3 = 0$$

We see that $\mathtt{R}\,\vec{v}_3$ and $\vec{v}_3$ are in the same one-dimensional space, i.e. they are linearly dependent and we can write $\mathtt{R}\,\vec{v}_3 = s\,\vec{v}_3$ for some s. Clearly $\vec{v}_3$ is an eigenvector of $\mathtt{R}$. Since it is not a multiple of $\vec{v}_1$, it must correspond to eigenvalue -1. Next, $\vec{v}_2^\top \vec{v}_3 = 0$ and hence they are linearly independent. We have shown that if -1 is an eigenvalue of $\mathtt{R}$, then there are always two linearly independent vectors corresponding to the eigenvalue -1 and therefore there is a two-dimensional space of eigenvectors corresponding to -1. Notice that the rank of $\mathtt{R} - \mathtt{I}$ is two in this case since the two-dimensional subspace corresponding to -1 can be complemented only by a one-dimensional subspace corresponding to 1 to avoid intersecting the subspaces in a non-zero vector.

Finally, let us look at non-real eigenvalues. Assume $y \neq 0$, i.e. $\lambda_1 = 1$, $\lambda_2 = x+yi$ and $\lambda_3 = x - yi$. Then for a non-real eigenvalue, e.g. λ_2, we have

$$\mathtt{R}\,\vec{v} = (x + yi)\,\vec{v}$$

Vector $\vec{v}$ must be non-real in this case else we would have a real vector on the left and a non-real vector on the right. Since $y \neq 0$, the eigenvalues are pairwise distinct and hence there are three one-dimensional subspaces of eigenvectors. In particular, there is exactly one dimensional subspace corresponding to eigenvalue 1. Notice that, again, the rank of $\mathtt{R} - \mathtt{I}$.

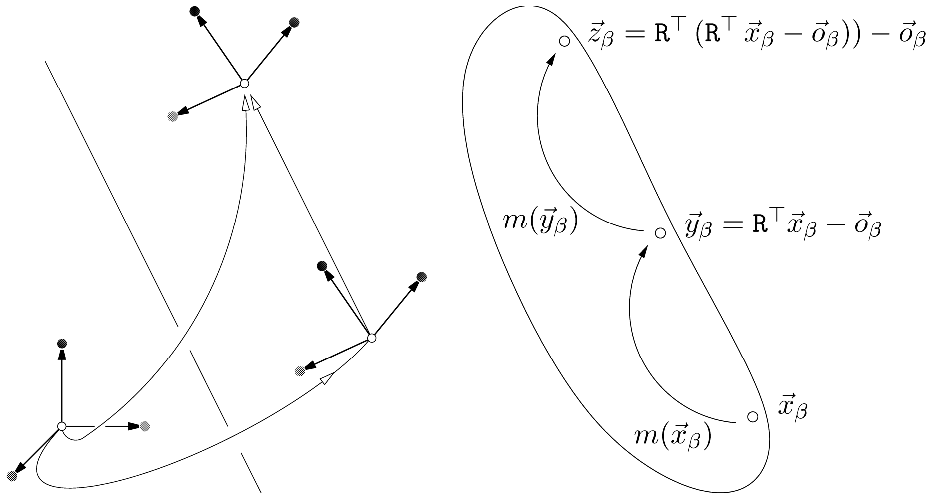
Let $\vec{v}$ be an eigenvector of a rotation matrix $\mathtt{R}$. Then

$$\begin{aligned} \mathtt{R}\,\vec{v} &= (x + yi)\,\vec{v} \\ \mathtt{R}^\top \mathtt{R}\,\vec{v} &= (x + yi)\,\mathtt{R}^\top \vec{v} \\ \vec{v} &= (x + yi)\,\mathtt{R}^\top \vec{v} \\ \frac{1}{(x + yi)}\,\vec{v} &= \mathtt{R}^\top \vec{v} \\ (x - yi)\,\vec{v} &= \mathtt{R}^\top \vec{v} \end{aligned}$$

We see that the eigenvector $\vec{v}$ of $\mathtt{R}$ corresponding to eigenvalue $x + yi$ is the eigenvector of $\mathtt{R}^\top$ corresponding to eigenvalue $x - yi$ and vice versa. Thus there is the following interesting correspondence between eigenvalues and eigenvectors of $\mathtt{R}$ and $\mathtt{R}^\top$. Considering eigenvalue-eigenvector pairs $(1, \vec{v}_1)$, $(x + yi, \vec{v}_2)$, $(x - yi, \vec{v}_3)$ of $\mathtt{R}$ we have $(1, \vec{v}_1)$, $(x - yi, \vec{v}_2)$, $(x + yi, \vec{v}_3)$ pairs of $\mathtt{R}^\top$.

Matrix $(\mathtt{R} - \mathtt{I})$. We have seen that $\operatorname{rank}(\mathtt{R} - \mathtt{I}) = 0$ for $\mathtt{R} = \mathtt{I}$ and $\operatorname{rank}(\mathtt{R} - \mathtt{I}) = 2$ for all rotation matrices $\mathtt{R} \neq \mathtt{I}$. Notice also that $\operatorname{rank}(\mathtt{R}^\top - \mathtt{I}) = \operatorname{rank}(\mathtt{R}^\top - \mathtt{I})^\top = \operatorname{rank}(\mathtt{R} - \mathtt{I})$ since rank of a matrix equals the rank of its transpose [3, 7].

Let us next investigate the relationship between the range and the null space of $(\mathtt{R} - \mathtt{I})$. We shall again consider two cases. When $\mathtt{R} = \mathtt{I}$, the range of $\mathtt{R}$ is $\mathbb{R}^3$ and the null space of $\mathtt{R}$ is also $\mathbb{R}^3$. Otherwise, when $\mathtt{R} \neq \mathtt{I}$, we have a one-dimensional null space of vectors $\vec{v}$ solving $(\mathtt{R} - \mathtt{I})\,\vec{v} = 0$ since this is exactly the space of eigenvectors corresponding to the eigenvalue 1. Now denote columns of $(\mathtt{R} - \mathtt{I}) = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \end{bmatrix}$ and consider a vector $\vec{v}$ from the range of $(\mathtt{R} - \mathtt{I})$. Then, there are $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ such that $\vec{v} = \alpha_1\,\vec{a}_1 + \alpha_2\,\vec{a}_2 + \alpha_3\,\vec{a}_3$. Assuming that $\vec{v}$ is also in the null space of $(\mathtt{R}-\mathtt{I})$ we get $0^\top = -\vec{v}^\top(\mathtt{R}-\mathtt{I})^\top = \vec{v}^\top(\mathtt{R}-\mathtt{I}) = \vec{v}^\top \begin{bmatrix}\vec{a}_1 & \vec{a}_2 & \vec{a}_3\end{bmatrix}$ leading to $\vec{v}^\top\vec{a}_1 = \vec{v}^\top\vec{a}_2 = \vec{v}^\top\vec{a}_3 = 0$ and hence $\vec{v}^\top\vec{v} = \alpha_1\vec{v}^\top\vec{a}_1 + \alpha_2\vec{v}^\top\vec{a}_2 + \alpha_3\vec{v}^\top\vec{a}_3 = 0$ implies $\vec{v} = 0$. We conclude that in this case the range and the null space intersect only in the zero vector.



(a) (b)

Figure 3.4: Axis of motion.

3.5 Axis of motion

We will study the motion and show that every motion in three dimensional space has an axis of motion. *Axis of motion* is a line of points that remain in the line after the motion. The existence of such an axis will allow us to decompose every motion into a sequence of a rotation around the axis followed by a translation along the axis as shown in Figure 3.4(a).

Algebraic characterization Consider Equation 3.9 and denote the motion so defined as $m(\vec{x}_\beta) = \mathtt{R}^\top\vec{x}_\beta - \vec{o}_\beta$ w.r.t. a fixed coordinate system (O, β). Now let us study the sets of points that remain fixed by the motion, i.e. sets F such that for all $\vec{x_\beta} \in F$ motion m leaves the $m(\vec{x_\beta})$ in the set, i.e. $m(\vec{x_\beta}) \in F$. Obviously, complete space na the empty sets are fixed sets. How do look other, non-trivial, fixed sets?

A nonempty F contains at least one $\vec{x_\beta}$. Then, both $\vec{y} = m(\vec{x_\beta})$ and $\vec{z_\beta} = m(\vec{y_\beta})$ must be in F, see Figure 3.4(b). Let us investigate such fixed points $\vec{x_\beta}$ for which

$$\vec{z}_\beta - \vec{y}_\beta = \vec{y}_\beta - \vec{x}_\beta \tag{3.12}$$

holds. We do not yet know whether such equality has to necessary hold for points of all fixed sets F but we see that it holds for the identity motion id that leaves all points unchanged, i.e. $id(\vec{x_\beta}) = \vec{x_\beta}$. We will find later that it holds for all motions and all their fixed sets.

$$\begin{aligned}
\vec{z}_\beta - \vec{y}_\beta &= \vec{y}_\beta - \vec{x}_\beta \\
\mathtt{R}^\top(\mathtt{R}^\top\vec{x}_\beta - \vec{o}_\beta)) - \vec{o}_\beta - \mathtt{R}^\top\vec{x}_\beta + \vec{o}_\beta &= \mathtt{R}^\top\vec{x}_\beta - \vec{o}_\beta - \vec{x}_\beta \\
(\mathtt{R}^\top)^2\vec{x}_\beta - \mathtt{R}^\top\vec{o}_\beta - \mathtt{R}^\top\vec{x}_\beta &= \mathtt{R}^\top\vec{x}_\beta - \vec{o}_\beta - \vec{x}_\beta
\end{aligned}$$

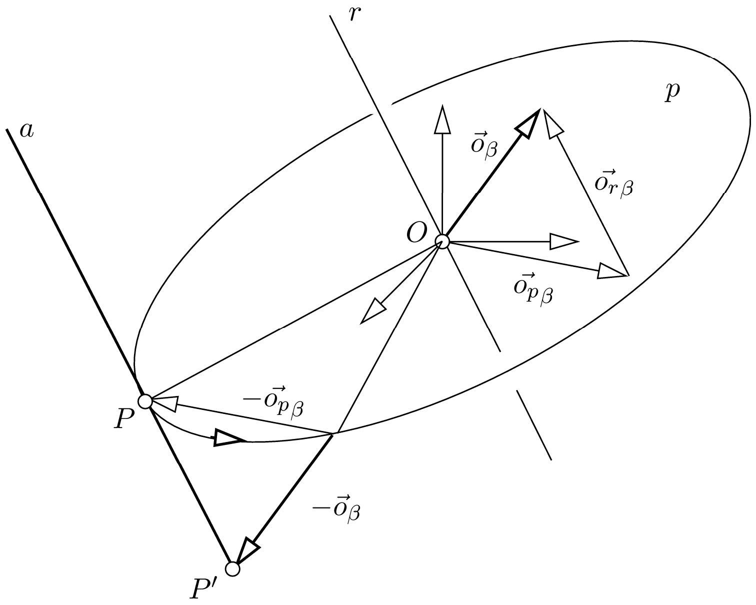


Figure 3.5: Axis a of motion is parallel to the axis of rotation r and intersects the perpendicular plane p passing through the origin O at a point P, which is first rotated in p away from a and then returned back to P' on a by translation $-\vec{o}_\beta$. Point P is determined by the component of $-\vec{o}_\beta$, which is in the plane p.

$$\begin{aligned}
(\mathtt{R}^\top)^2\vec{x}_\beta - 2\,\mathtt{R}^\top\vec{x}_\beta + \vec{x}_\beta &= \mathtt{R}^\top\vec{o}_\beta - \vec{o}_\beta \\
\left((\mathtt{R}^\top)^2 - 2\,\mathtt{R}^\top + \mathtt{I}\right)\vec{x}_\beta &= (\mathtt{R}^\top - \mathtt{I})\vec{o}_\beta \\
(\mathtt{R}^\top - \mathtt{I})(\mathtt{R}^\top - \mathtt{I})\,\vec{x}_\beta &= (\mathtt{R}^\top - \mathtt{I})\,\vec{o}_\beta \qquad (3.13)\\
(\mathtt{R}^\top - \mathtt{I})\left((\mathtt{R}^\top - \mathtt{I})\,\vec{x}_\beta - \vec{o}_\beta\right) &= 0 \qquad (3.14)
\end{aligned}$$

Equation 3.14 has always a solution. Let us see why. Recall that rank$\,(\mathtt{R}^\top - \mathtt{I})$ is either two or zero. If it is zero, then $\mathtt{R}^\top - \mathtt{I} = 0$ and (i) Equation 3.14 holds for every $\vec{x}_\beta$. Let rank$\,(\mathtt{R}^\top - \mathtt{I})$ be two. Vector $\vec{o}_\beta$ either is zero or it is not zero. If it is zero, then Equation 3.14 becomes $(\mathtt{R}^\top - \mathtt{I})^2\,\vec{x}_\beta = 0$, which has (ii) a one-dimensional space of solutions. Let $\vec{o}_\beta$ be non-zero. Vector $\vec{o}_\beta$ either is in the span of $(\mathtt{R}^\top - \mathtt{I})$ or it is not. If $\vec{o}_\beta$ is in the span of $(\mathtt{R}^\top - \mathtt{I})$, then $(\mathtt{R}^\top - \mathtt{I})\,\vec{x}_\beta - \vec{o}_\beta = 0$ has (iii) one-dimensional affine space of solutions. If $\vec{o}_\beta$ is not in the span of $(\mathtt{R}^\top - \mathtt{I})$, then $(\mathtt{R}^\top - \mathtt{I})\,\vec{x}_\beta - \vec{o}_\beta$ for $\vec{x}_\beta \in \mathbb{R}^3$ generates a vector in all one-dimensional subspaces of $\mathbb{R}^3$ which are not in the span of $(\mathtt{R}^\top - \mathtt{I})$. Therefore, it generates a vector $\vec{z}_\beta = (\mathtt{R}^\top - \mathtt{I})\,\vec{y}_\beta - \vec{o}_\beta$ in the null space of $(\mathtt{R}^\top - \mathtt{I})$, because the null space and the span of $(\mathtt{R}^\top - \mathtt{I})$ are disjoint for $\mathtt{R} \neq \mathtt{I}$ as we have shown before. Equation $(\mathtt{R}^\top - \mathtt{I})\,\vec{x}_\beta = \vec{z}_\beta + \vec{o}_\beta$ is satisfied by (iv) a one-dimensional affine set of vectors. We can conclude that every motion has a fixed line of points for which Equation 3.12 holds. Therefore, every motion has a fixed line of points, every motion has an axis.

We have encountered four different cases above. Let us look at them in more detail.

Geometrical characterization We now understand the algebraic description of motion. Can we also understand the situation geometrically? Figure 3.5 gives the an-

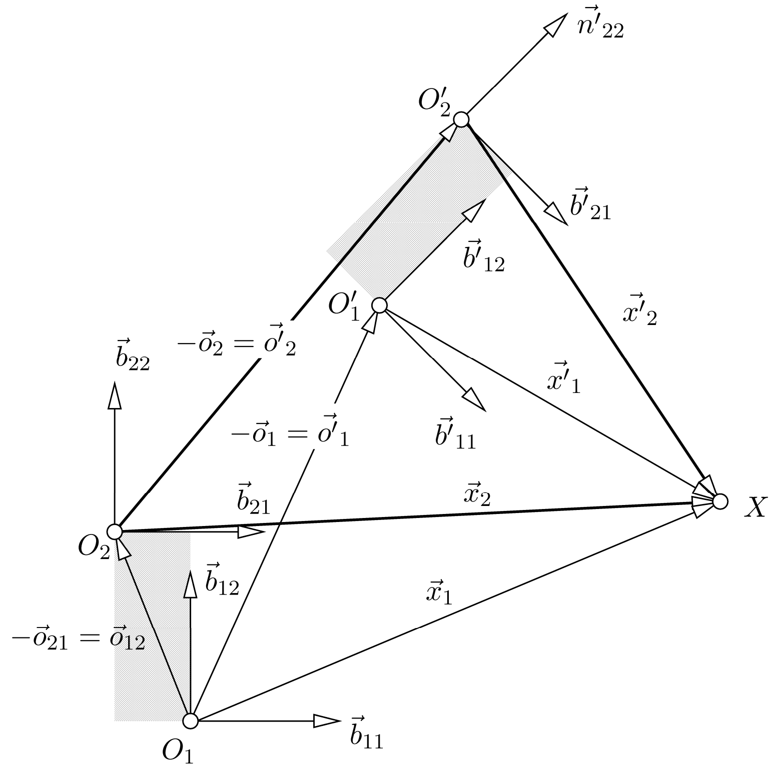


Figure 3.6: Representation of rigid motion under a change of the coordinate system attached to the moving body.

swer. We shall concentrate on the general situation with $\mathtt{R} \neq \mathtt{I}$ and $\vec{o}_\beta \neq 0$. The main idea of the figure is that the axis of rotation consists of points that are first rotated away from a by the pure rotation around r and then returned back to a by the pure translation $-\vec{o}_\beta$.

Figure shows axis a of motion, which is parallel to the axis of rotation r and intersects the perpendicular plane p passing through the origin O at a point P, which is first rotated in p away from a and then returned back to P' on a by translation $-\vec{o}_\beta$. Point P is determined by the component of $-\vec{o}_\beta$, which is in the plane p. Notice that every vector $\vec{o}_\beta$ can be written as a sum of its component $\vec{o_r}_\beta$ parallel to r and component $\vec{o_n}_\beta$ perpendicular to r.

3.6 Change of the reference coordinate system

Until now, we have been studying motion from the point of view of one reference coordinate system attached to the moving body. Let us next discover the relationship between motion descriptions when passing from one coordinate system to another.

Figure 3.6 shows a body before motion with two coordinates systems $S_1 = (O_1, \beta_1 = (\vec{b}_{11}, \vec{b}_{12}))$ and $S_2 = (O_2, \beta_2 = (\vec{b}_{21}, \vec{b}_{22}))$ and the same body after motion with moved coordinate systems $S'_1 = (O'_1, \beta'_1 = (\vec{b}'_{11}, \vec{b}'_{12}))$ and $S'_2 = (O'_2, \beta'_2 = (\vec{b}'_{21}, \vec{b}'_{22}))$.

Considering Equation 3.8, we can modeled the motion w.r.t. the two coordinate systems as

$$\vec{x'_1}_{\beta'_1} = \mathtt{R}_1 \, \vec{x_1}_{\beta_1} + \vec{o_1}_{\beta'_1}$$

$$\vec{x'}_{2\beta'_2} = \mathtt{R}_2\, \vec{x}_{2\beta_2} + \vec{o}_{2\beta'_2}$$

Similarly, the change from S_1 to S_2 and from S'_1 to S'_2 can be modeled as

$$\begin{aligned}
\vec{x}_{2\beta_2} &= \mathtt{R}_{12}\, \vec{x}_{1\beta_1} + \vec{o}_{21\beta_2} \\
\vec{x'}_{2\beta'_2} &= \mathtt{R}'_{12}\, \vec{x'}_{1\beta'_1} + \vec{o'}_{21\beta'_2}
\end{aligned}$$

The goal is now to find the relationship between $\mathtt{R}_1$, $\vec{o}_{1\beta'_1}$ and $\mathtt{R}_2$, $\vec{o}_{2\beta'_2}$.

$$\begin{aligned}
\mathtt{R}'_{12}\, \vec{x'}_{1\beta'_1} + \vec{o'}_{21\beta'_2} &= \mathtt{R}_2\, (\mathtt{R}_{12}\, \vec{x}_{1\beta_1} + \vec{o}_{21\beta_2}) + \vec{o}_{2\beta'_2} \\
\vec{x'}_{1\beta'_1} + (\mathtt{R}'_{12})^\top\, \vec{o'}_{21\beta'_2} &= (\mathtt{R}'_{12})^\top\, \mathtt{R}_2\, (\mathtt{R}_{12}\, \vec{x}_{1\beta_1} + \vec{o}_{21\beta_2}) + (\mathtt{R}'_{12})^\top\, \vec{o}_{2\beta'_2} \\
\vec{x'}_{1\beta'_1} &= (\mathtt{R}'_{12})^\top\, \mathtt{R}_2\, (\mathtt{R}_{12}\, \vec{x}_{1\beta_1} + \vec{o}_{21\beta_2}) + (\mathtt{R}'_{12})^\top\, \vec{o}_{2\beta'_2} - (\mathtt{R}'_{12})^\top\, \vec{o'}_{21\beta'_2} \\
\vec{x'}_{1\beta'_1} &= (\mathtt{R}'_{12})^\top\, \mathtt{R}_2\, \mathtt{R}_{12}\, \vec{x}_{1\beta_1} + (\mathtt{R}'_{12})^\top\, \mathtt{R}_2\, \vec{o}_{21\beta_2} + (\mathtt{R}'_{12})^\top\, \vec{o}_{2\beta'_2} - (\mathtt{R}'_{12})^\top\, \vec{o'}_{21\beta'_2} \\
\vec{x'}_{1\beta'_1} &= (\mathtt{R}'_{12})^\top\, \mathtt{R}_2\, \mathtt{R}_{12}\, \vec{x}_{1\beta_1} + (\mathtt{R}'_{12})^\top\, (\mathtt{R}_2\, \vec{o}_{21\beta_2} + \vec{o}_{2\beta'_2} - \vec{o'}_{21\beta'_2})
\end{aligned}$$

We conclude that

$$\begin{aligned}
\mathtt{R}_1 &= (\mathtt{R}'_{12})^\top\, \mathtt{R}_2\, \mathtt{R}_{12} \\
\vec{o}_{1\beta'_1} &= (\mathtt{R}'_{12})^\top\, (\mathtt{R}_2\, \vec{o}_{21\beta_2} + \vec{o}_{2\beta'_2} - \vec{o'}_{21\beta'_2})
\end{aligned}$$

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