4 Axis of Motion

4.1 Properties of rotation matrix

Let us study additional properties of the rotation matrix in three-dimensional space.

§ **9 Eigenvalues of** $\mathtt{R}$ Let $\mathtt{R}$ be a rotation matrix. Then for every $\vec{v} \in \mathbb{C}^3$

$$(\mathtt{R}\,\vec{v})^\top \mathtt{R}\,\vec{v} = \vec{v}^\top \mathtt{R}^\top \mathtt{R}\,\vec{v} = \vec{v}^\top (\mathtt{R}^\top \mathtt{R})\,\vec{v} = \vec{v}^\top \vec{v} \tag{4.1}$$

we see that for all $\vec{v} \in \mathbb{C}^3$ and $\lambda \in \mathbb{C}$ such that

$$\mathtt{R}\,\vec{v} = \lambda\,\vec{v} \tag{4.2}$$

then holds

$$|\lambda|^2 (\vec{v}^\top \vec{v}) = (\vec{v}^\top \vec{v}) \tag{4.3}$$

and hence $|\lambda|^2 = 1$ for all $\vec{v} \neq \vec{0}$. We conclude that the absolute value of eigenvalues of $\mathtt{R}$ is one.

Next, there is a real unit eigenvalue since $\mathtt{R}$ is a real matrix with characteristic polynomial

$$\begin{aligned}
p(\lambda) &= |(\lambda\,\mathtt{I} - \mathtt{R})| = \left| \left(\begin{bmatrix} \lambda - r_{11} & -r_{12} & -r_{13} \\ -r_{21} & \lambda - r_{22} & -r_{23} \\ -r_{31} & -r_{32} & \lambda - r_{33} \end{bmatrix} \right) \right| && (4.4)\\
&= \lambda^3 - (r_{11} + r_{22} + r_{33})\,\lambda^2 && (4.5)\\
&\quad + (r_{11}\,r_{22} - r_{21}\,r_{12} + r_{11}\,r_{33} - r_{31}\,r_{13} + r_{22}\,r_{33} - r_{23}\,r_{32})\,\lambda && (4.6)\\
&\quad + r_{11}\,(r_{23}\,r_{32} - r_{22}\,r_{33}) - r_{21}\,(r_{32}\,r_{13} - r_{12}\,r_{33}) + r_{31}\,(r_{13}\,r_{22} - r_{12}\,r_{23}) && (4.7)\\
&= \lambda^3 - \operatorname{trace}\mathtt{R}\,\lambda^2 + (\mathtt{R}_{11} + \mathtt{R}_{22} + \mathtt{R}_{33})\,\lambda - |\mathtt{R}| && (4.8)
\end{aligned}$$

It follows from the Fundamental theorem of algebra [6] the $p(\lambda) = 0$ has always a solution in $\mathbb{C}$ and since coefficients of $p(\lambda)$ are all real, the solutions must come in complex conjugated pairs. The degree of $p(\lambda)$ is three and thus at least one solution must be real and hence equal to ± 1. Now, since $p(0) = -|(\mathtt{R})| = -1$, $\lim_{\lambda \to \infty} p(\lambda) = \infty$, and $p(\lambda)$ is a continuous function, it must (by the mean value theorem [2]) cross the positive side of the real axis and hence one eigenvalue has to be one. Let us denote the eigenvalues as $\lambda_1 = 1$, $\lambda_2 = x + yi$ and $\lambda_3 = x - yi$ with real x, y. It follows from above that $x^2 + y^2 = 1$. We see that there is either one real or three real solutions since if $y = 0$, then $x^2 = 1$ and hence $\lambda_2 = \lambda_3 = \pm 1$. We conclude that we can encounter only two situations when all eigenvalues are real. Either $\lambda_1 = \lambda_2 = \lambda_3 = 1$, or $\lambda_1 = 1$ and $\lambda_2 = \lambda_3 = -1$.

§ **10 Eigenvectors of R.** Let us now look at eigenvectors of R and let's first investigate the situation when all eigenvalues of R are real.

Let $\lambda_1 = \lambda_2 = \lambda_3 = 1$. Then $p(\lambda) = (\lambda - 1)^3 = \lambda^3 - 3\,\lambda^2 + 3\,\lambda - 1$. It means that $r_{11} + r_{22} + r_{33} = 1$ and since $r_{11} \leqslant 1$, $r_{22} \leqslant 1$, $r_{33} \leqslant 1$, it leads to $r_{11} = r_{22} = r_{33} = 3$, which implies $\mathtt{R} = \mathtt{I}$. Then $\mathtt{I} - \mathtt{R} = 0$ and all non-zero vectors of $\mathbb{R}^3$ are eigenvectors of R. Notice that rank of $\mathtt{R} - \mathtt{I}$ is zero in this case.

Next consider $\lambda_1 = 1$ and $\lambda_2 = \lambda_3 = -1$. The eigenvectors $\vec{v}$ corresponding to $\lambda_2 = \lambda_3 = -1$ are solutions to

$$\mathtt{R}\,\vec{v} \;=\; -\vec{v} \tag{4.9}$$

There is always at least one one-dimensional space of such vectors. We also see that there is a rotation matrix

$$\mathtt{R} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{4.10}$$

with eigenvectors

$$r \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\; r \neq 0, \qquad \text{and} \qquad s \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},\; s^2 + t^2 \neq 0, \tag{4.11}$$

which means that there is a one-dimensional space of eigenvectors corresponding to 1 and a two-dimensional space of eigenvectors corresponding to -1. Notice that rank of $\mathtt{R} - \mathtt{I}$ is two here.

How does the situation look for a general R with eigenvalues $1, -1, -1$? Consider an eigenvector $\vec{v}_1$ corresponding to 1 and an eigenvector $\vec{v}_2$ corresponding to -1. They are linearly independent. Otherwise there has to be $s \in \mathbb{R}$ such that $\vec{v}_2 = s\,\vec{v}_1 \neq 0$ and then

$$\vec{v}_2 \;=\; s\,\vec{v}_1 \tag{4.12}$$

$$\mathtt{R}\,\vec{v}_2 \;=\; s\,\mathtt{R}\,\vec{v}_1 \tag{4.13}$$

$$-\vec{v}_2 \;=\; s\,\vec{v}_1 \tag{4.14}$$

leading to $s = -s$ and therefore $s = 0$ which contradicts $\vec{v}_2 \neq 0$. Now, let us look at vectors $\vec{v}_3 \in \mathbb{R}^3$ defined by

$$\begin{bmatrix} \vec{v}_1^\top \\ \vec{v}_2^\top \end{bmatrix} \vec{v}_3 \;=\; 0 \tag{4.15}$$

The above linear system has a one-dimensional space of solutions since the rows of its matrix are independent. Chose a fixed solution $\vec{v}_3 \neq 0$. Then

$$\begin{bmatrix} \vec{v}_1^\top \\ \vec{v}_2^\top \end{bmatrix} \mathtt{R}^\top\,\vec{v}_3 = \begin{bmatrix} \vec{v}_1^\top \mathtt{R}^\top \\ \vec{v}_2^\top \mathtt{R}^\top \end{bmatrix} \vec{v}_3 = \begin{bmatrix} \vec{v}_1^\top \\ -\vec{v}_2^\top \end{bmatrix} \vec{v}_3 = 0 \tag{4.16}$$

We see that $\mathtt{R}\,\vec{v}_3$ and $\vec{v}_3$ are in the same one-dimensional space, i.e. they are linearly dependent and we can write

$$\mathtt{R}^\top\vec{v}_3 = s\,\vec{v}_3 \tag{4.17}$$

for some $s \in \mathbb{C}$. Multiplying equation 4.18 by $\mathtt{R}$ form the left and dividing both sides by s gives

$$\frac{1}{s}\vec{v}_3 = \mathtt{R}\,\vec{v}_3 \tag{4.18}$$

Clearly $\vec{v}_3$ is an eigenvector of $\mathtt{R}$. Since it is not a multiple of $\vec{v}_1$, it must correspond to eigenvalue -1. Moreover, $\vec{v}_2^\top \vec{v}_3 = 0$ and hence they are linearly independent. We have shown that if -1 is an eigenvalue of $\mathtt{R}$, then there are always two linearly independent vectors corresponding to the eigenvalue -1 and therefore there is a two-dimensional space of eigenvectors corresponding to -1. Notice that the rank of $\mathtt{R}-\mathtt{I}$ is two in this case since the two-dimensional subspace corresponding to -1 can be complemented only by a one-dimensional subspace corresponding to 1 to avoid intersecting the subspaces in a non-zero vector.

Finally, let us look at arbitrary (even non-real) eigenvalues. Assume $\lambda = x + yi$. Then we have

$$\mathtt{R}\,\vec{v} \;=\; (x+yi)\,\vec{v} \tag{4.19}$$

If $y \neq 0$, vector $\vec{v}$ must be non-real in this case else we would have a real vector on the left and a non-real vector on the right. Furthermore, the eigenvalues are pairwise distinct and hence there are three one-dimensional subspaces of eigenvectors (we now understand the space as $\mathbb{C}^3$ over $\mathbb{C}$). In particular, there is exactly one dimensional subspace corresponding to eigenvalue 1. Notice that, again, the rank of $\mathtt{R}-\mathtt{I}$ is two.

Let $\vec{v}$ be an eigenvector of a rotation matrix $\mathtt{R}$. Then

$$\begin{aligned}
\mathtt{R}\,\vec{v} &= (x+yi)\,\vec{v} && (4.20)\\
\mathtt{R}^\top\mathtt{R}\,\vec{v} &= (x+yi)\,\mathtt{R}^\top\vec{v} && (4.21)\\
\vec{v} &= (x+yi)\,\mathtt{R}^\top\vec{v} && (4.22)\\
\frac{1}{(x+yi)}\,\vec{v} &= \mathtt{R}^\top\vec{v} && (4.23)\\
(x-yi)\,\vec{v} &= \mathtt{R}^\top\vec{v} && (4.24)
\end{aligned}$$

We see that the eigenvector $\vec{v}$ of $\mathtt{R}$ corresponding to eigenvalue $x + yi$ is the eigenvector of $\mathtt{R}^\top$ corresponding to eigenvalue $x - yi$ and vice versa. Thus there is the following interesting correspondence between eigenvalues and eigenvectors of $\mathtt{R}$ and $\mathtt{R}^\top$. Considering eigenvalue-eigenvector pairs $(1, \vec{v}_1)$, $(x + yi, \vec{v}_2)$, $(x - yi, \vec{v}_3)$ of $\mathtt{R}$ we have $(1, \vec{v}_1)$, $(x - yi, \vec{v}_2)$, $(x + yi, \vec{v}_3)$ pairs of $\mathtt{R}^\top$.

§ **11 Matrix** $(\mathtt{R}-\mathtt{I})$. We have seen that $\operatorname{rank}(\mathtt{R}-\mathtt{I}) = 0$ for $\mathtt{R} = \mathtt{I}$ and $\operatorname{rank}(\mathtt{R}-\mathtt{I}) = 2$ for all rotation matrices $\mathtt{R} \neq \mathtt{I}$.

Let us next investigate the relationship between the range and the null space of $(\mathtt{R} - \mathtt{I})$. The null space of $(\mathtt{R} - \mathtt{I})$ is generated by eigenvectors corresponding to 1 since $(\mathtt{R} - \mathtt{I})\,\vec{v} = 0$ implies $\mathtt{R}\,\vec{v} = \vec{v}$.

Now assume that vector $\vec{v}$ is also in the range of $(\mathtt{R} - \mathtt{I})$. Then, there is a vector $\vec{a} \in \mathbb{R}^3$ such that $\vec{v} = (\mathtt{R} - \mathtt{I})\,\vec{a}$. Let us evaluate the square of the length of $\vec{v}$

$$\vec{v}^\top\vec{v} \;=\; \vec{v}^\top(\mathtt{R}-\mathtt{I})\,\vec{a} = (\vec{v}^\top\mathtt{R} - \vec{v}^\top)\,\vec{a} = (\vec{v}^\top - \vec{v}^\top)\,\vec{a} = 0 \tag{4.25}$$

which implies $\vec{v} = \vec{0}$. We have used result 4.24 with $x = 1$ and $y = 0$.

We conclude that in this case the range and the null space intersect only in the zero vector.

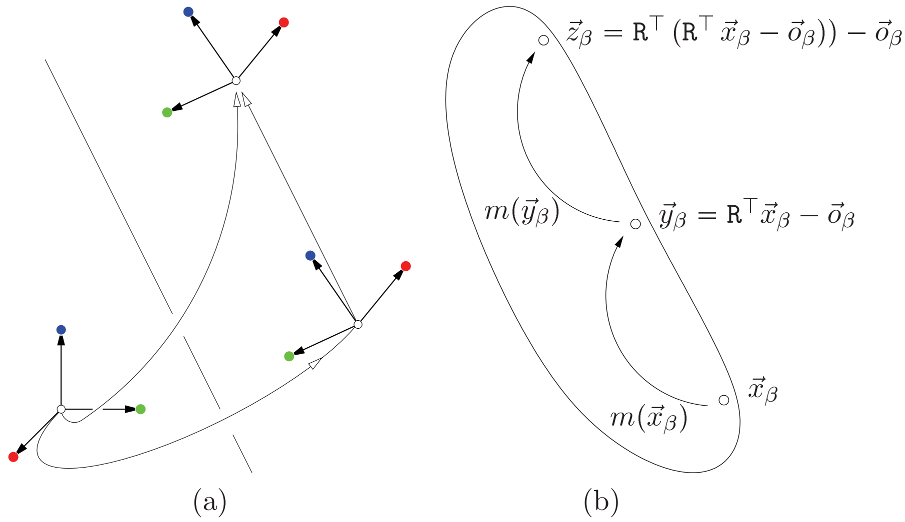

Figure 4.1: Axis of motion.

4.2 Axis of motion

We will study motion and show that every motion in three dimensional space has an axis of motion. *Axis of motion* is a line of points that remain in the line after the motion. The existence of such an axis will allow us to decompose every motion into a sequence of a rotation around the axis followed by a translation along the axis as shown in Figure 4.1(a).

§ **12 Algebraic characterization of the axis of motion.** Consider Equation 3.9 and denote the motion so defined as $m(\vec{x}_\beta) = \mathtt{R}^\top\vec{x}_\beta - \vec{o}_\beta$ w.r.t. a fixed coordinate system (O, β). Now let us study the sets of points that remain fixed by the motion, i.e. sets F such that for all $\vec{x}_\beta \in F$ motion m leaves the $m(\vec{x}_\beta)$ in the set, i.e. $m(\vec{x}_\beta) \in F$. Obviously, complete space and the empty set are fixed sets. How do look other, non-trivial, fixed sets?

A nonempty F contains at least one $\vec{x}_\beta$. Then, both $\vec{y} = m(\vec{x}_\beta)$ and $\vec{z}_\beta = m(\vec{y}_\beta)$ must be in F, see Figure 4.1(b). Let us investigate such fixed points $\vec{x}_\beta$ for which

$$\vec{z}_\beta - \vec{y}_\beta = \vec{y}_\beta - \vec{x}_\beta \tag{4.26}$$

holds. We do not yet know whether such equality has to necessary hold for points of all fixed sets F but we see that it holds for the identity motion id that leaves all points unchanged, i.e. $id(\vec{x}_\beta) = \vec{x}_\beta$. We will find later that it holds for all motions

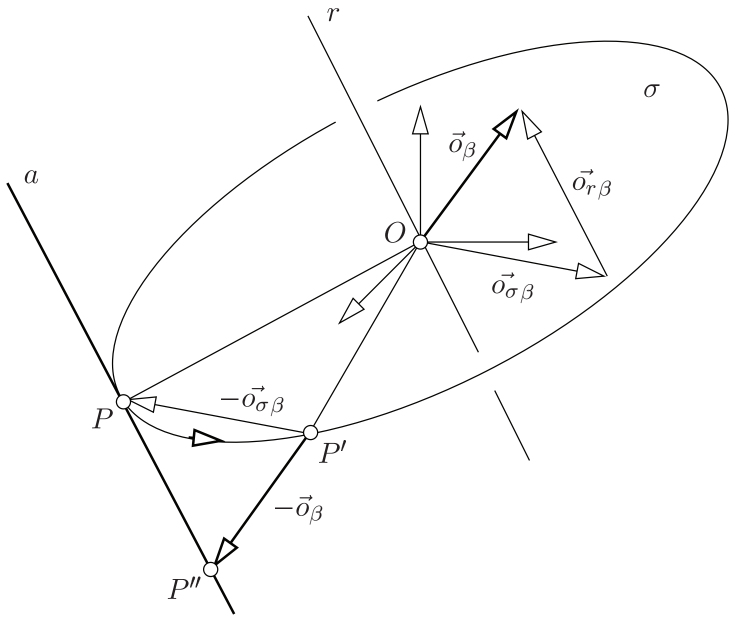

Figure 4.2: Axis a of motion is parallel to the axis of rotation r and intersects the perpendicular plane σ passing through the origin O at a point P, which is first rotated in σ away from a to P' and then returned back to P'' on a by translation $-\vec{o}_\beta$. Point P is determined by the component $-\vec{o}_{\sigma\beta}$ of $-\vec{o}_\beta$, which is in the plane σ.

and all their fixed sets. Consider the following sequence of equatities

$$
\begin{aligned}
\vec{z}_\beta - \vec{y}_\beta &= \vec{y}_\beta - \vec{x}_\beta \\
\mathtt{R}^\top(\mathtt{R}^\top\vec{x}_\beta - \vec{o}_\beta)) - \vec{o}_\beta - \mathtt{R}^\top\vec{x}_\beta + \vec{o}_\beta &= \mathtt{R}^\top\vec{x}_\beta - \vec{o}_\beta - \vec{x}_\beta \\
(\mathtt{R}^\top)^2\vec{x}_\beta - \mathtt{R}^\top\vec{o}_\beta - \mathtt{R}^\top\vec{x}_\beta &= \mathtt{R}^\top\vec{x}_\beta - \vec{o}_\beta - \vec{x}_\beta \\
(\mathtt{R}^\top)^2\vec{x}_\beta - 2\,\mathtt{R}^\top\vec{x}_\beta + \vec{x}_\beta &= \mathtt{R}^\top\vec{o}_\beta - \vec{o}_\beta \\
\left((\mathtt{R}^\top)^2 - 2\,\mathtt{R}^\top + \mathtt{I}\right)\vec{x}_\beta &= (\mathtt{R}^\top - \mathtt{I})\vec{o}_\beta \\
(\mathtt{R}^\top - \mathtt{I})(\mathtt{R}^\top - \mathtt{I})\,\vec{x}_\beta &= (\mathtt{R}^\top - \mathtt{I})\,\vec{o}_\beta \qquad (4.27)\\
(\mathtt{R}^\top - \mathtt{I})\left((\mathtt{R}^\top - \mathtt{I})\,\vec{x}_\beta - \vec{o}_\beta\right) &= 0 \qquad (4.28)
\end{aligned}
$$

Equation 4.28 has always a solution. Let us see why. Recall that rank$\,(\mathtt{R}^\top - \mathtt{I})$ is either two or zero. If it is zero, then $\mathtt{R}^\top - \mathtt{I} = 0$ and (i) Equation 4.28 holds for every $\vec{x}_\beta$. Let rank$\,(\mathtt{R}^\top - \mathtt{I})$ be two. Vector $\vec{o}_\beta$ either is zero or it is not zero. If it is zero, then Equation 4.28 becomes $(\mathtt{R}^\top - \mathtt{I})^2\,\vec{x}_\beta = 0$, which has (ii) a one-dimensional space of solutions. Let $\vec{o}_\beta$ be non-zero. Vector $\vec{o}_\beta$ either is in the span of $(\mathtt{R}^\top - \mathtt{I})$ or it is not. If $\vec{o}_\beta$ is in the span of $(\mathtt{R}^\top - \mathtt{I})$, then $(\mathtt{R}^\top - \mathtt{I})\,\vec{x}_\beta - \vec{o}_\beta = 0$ has (iii) one-dimensional affine space of solutions. If $\vec{o}_\beta$ is not in the span of $(\mathtt{R}^\top - \mathtt{I})$, then $(\mathtt{R}^\top - \mathtt{I})\,\vec{x}_\beta - \vec{o}_\beta$ for $\vec{x}_\beta \in \mathbb{R}^3$ generates a vector in all one-dimensional subspaces of $\mathbb{R}^3$ which are not in the span of $(\mathtt{R}^\top - \mathtt{I})$. Therefore, it generates a vector $\vec{z}_\beta = (\mathtt{R}^\top - \mathtt{I})\,\vec{y}_\beta - \vec{o}_\beta$ in the null space of $(\mathtt{R}^\top - \mathtt{I})$, because the null space and the span of $(\mathtt{R}^\top - \mathtt{I})$ are disjoint for $\mathtt{R} \neq \mathtt{I}$ as we have shown before. Equation $(\mathtt{R}^\top - \mathtt{I})\,\vec{x}_\beta = \vec{z}_\beta + \vec{o}_\beta$ is satisfied by (iv) a one-dimensional affine set of vectors. We can conclude that every motion has a fixed line of points for which Equation 4.26 holds. Therefore, every motion has a fixed line of points, every motion has an axis.

§ **13 Geometrical characterization of the axis of motion** We now understand the algebraic description of motion. Can we also understand the situation geometrically? Figure 4.2 gives the answer. We shall concentrate on the general situation with $\mathtt{R} \neq \mathtt{I}$ and $\vec{o}_\beta \neq 0$. The main idea of the figure is that the axis of motion a consists of points that are first rotated away from a by the pure rotation $\mathtt{R}^\top$ around r and then returned back to a by the pure translation $-\vec{o}_\beta$.

Figure 4.2 shows axis a of motion, which is parallel to the axis of rotation r and intersects the perpendicular plane σ passing through the origin O at a point P, which is first rotated in σ away from a to P' and then returned back to P'' on a by translation $-\vec{o}_\beta$. Point P is determined by the component $-\vec{o_\sigma}_\beta$ of $-\vec{o}_\beta$, which is in the plane σ. Notice that every vector $\vec{o}_\beta$ can be written as a sum of its component $\vec{o_r}_\beta$ parallel to r and component $\vec{o_\sigma}_\beta$ perpendicular to r.

§ **14 Motion axis is parallel to rotation axis.** Let us verify algebraically that the rotation axis r is parallel to the motion axis a. Consider Equation 4.27, which we can rewrite as

$$(\mathtt{R}^\top - \mathtt{I})^2\, \vec{x}_\beta \quad = \quad (\mathtt{R}^\top - \mathtt{I})\, \vec{o}_\beta \tag{4.29}$$

Define axis r of motion as the set of points that are left fixed by the pure rotation $\mathtt{R}^\top$, i.e.

$$(\mathtt{R}^\top - \mathtt{I})\, \vec{x}_\beta \quad = \quad 0 \tag{4.30}$$

$$\mathtt{R}^\top\, \vec{x}_\beta \quad = \quad \vec{x}_\beta \tag{4.31}$$

These are eigenvectors of $\mathtt{R}^\top$ and the zero vector. Take any two solutions $\vec{x}_{1_\beta}$, $\vec{x}_{2_\beta}$ of 4.29 and evaluate

$$(\mathtt{R}^\top - \mathtt{I})^2(\vec{x}_{1_\beta} - \vec{x}_{2_\beta}) \quad = \quad (\mathtt{R}^\top - \mathtt{I})\, \vec{o}_\beta - (\mathtt{R}^\top - \mathtt{I})\, \vec{o}_\beta = 0 \tag{4.32}$$

and thus a non-zero $\vec{x}_{1_\beta} - \vec{x}_{2_\beta}$ is an eigenvector of $\mathtt{R}^\top$. We see that the direction vectors of a lie in the subspace of direction vectors of r.

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