T.Pajdla: Inverse Kinematics of a 6-DOF Manipulator - 2

[1] D.Manocha, J.F.Canny. Efficient Inverse Kinematics for General 6R Manipulators. IEEE Trans. on Robotics and Automation, 10(5), pp. 648-657, Oct. 2004
[2] M. Raghavan, B. Roth. Kinematic Analysis of the 6R Manipulator of General Geometry. Int. Symposium on Robotic Research. pp. 264-269, Tokyo 1990

General Mechanism - Explanation
2 Nov 2009

Packages & settings

```
> restart:
  with(ListTools):
  with(LinearAlgebra):
  with(PolynomialTools):
  with(combinat, choose):
  with(Groebner):
  with(MatrixPolynomialAlgebra):
  interface(rtablesize=24):
  interface(warnlevel=0):
  Digits:=30:
  eps:=1e-6:
```

DH-Kinematics functions

Joint transformations:

```
> # Two one-parametric motions transformatin in DH-convention(phi, theta, a,d)
  indexed by i
  # c = cos(phi), s = sin(phi), lambda = cos(alpha), mu = sin(alpha)
  dhTs := proc(i)
  local M1, M2;
     M1:=Matrix(4,4,[[+cat(`c`,i),-cat(`s`,i),0,         0],
                     [ +cat(`s`,i),+cat(`c`,i),0,         0],
                     [           0,          0,1,cat(`d`,i)],
                     [           0,          0,0,         1]]);
     M2:=Matrix(4,4,[[1,        0,           0,cat(`a`,i)],
                     [ 0,+cat(`lambda`,i),-cat(`mu`,i),         0],
                     [ 0,+cat(`mu`,i),+cat(`lambda`,i),         0],
                     [ 0,          0,           0,         1]]);
     [M1,M2];
  end proc:
  #
  # Inverse of the DH-convention for one-aprametric DH rigid motion transformations
  dhInvs := proc(M)
     local M1, M2;
     M1 := M[1];
     M2 := M[2];
     [simplify(MatrixInverse(M2),{M2[3,2]^2+M2[3,3]^2=1}),
      simplify(MatrixInverse(M1),{M1[1,1]^2+M1[2,1]^2=1})];
  end proc:
  #
  # Rigid motion transformatin in DH-convention(phi, theta, a,d) indexed by i
  # c = cos(phi), s = sin(phi), P = cos(alpha), R = sin(alpha)
  dhT := proc(i)
  local M;
     M:=dhTs(i);
     M[1].M[2];
  end proc:
  #
  # Inverse of the DH-convention rigid motion transformation
  dhInv := proc(M)
     simplify(MatrixInverse(M),{M[1,1]^2+M[2,1]^2=1,M[3,2]^2+M[3,3]^2=1});
  end proc:
```

```
#
# Simplify using trigonometric indentities c^2+s^2=1 & lambda^2+mu^2=1
dhSimpl := proc(M,i)
   simplify(M,{cat(`c`,i)^2+cat(`s`,i)^2=1,cat(`lambda`,i)^2+cat(`mu`,i)^2=1});
end proc:
#
## Direct Kinematic Task
#
dhDKT := proc(p)
   subs(p,dhT(1).dhT(2).dhT(3).dhT(4).dhT(5).dhT(6));
end proc:
#
# Simplify using Rotation matrin in Mh
MhSimpl := proc(M)
 simplify(
  simplify(
   simplify(
    simplify(M,
            {lx^2+ly^2+lz^2=1,mx^2+my^2+mz^2=1,nx^2+ny^2+nz^2=1}),
           {lx*mx+ly*my+lz*mz=0,lx*nx+ly*ny+lz*nz=0,mx*nx+my*ny+mz*nz=0}),
          {lx^2+mx^2+nx^2=1,ly^2+my^2+ny^2=1,lz^2+mz^2+nz^2=1}),
         {lx*ly+mx*my+nx*ny=0,lx*lz+mx*mz+nx*nz=0,lz*ly+mz*my+nz*ny=0});
end proc:
#
# Simplify a general motion matrix using rotation matrix identities in columns
rcSimp := proc(M,R)
       simplify(
        simplify(
         simplify(
          simplify(
           simplify(
            simplify(M,{R[1,1]*R[1,1]+R[2,1]*R[2,1]+R[3,1]*R[3,1]=1}),
                {R[1,1]*R[1,2]+R[2,1]*R[2,2]+R[3,1]*R[3,2]=0}),
               {R[1,1]*R[1,3]+R[2,1]*R[2,3]+R[3,1]*R[3,3]=0}),
              {R[1,2]*R[1,2]+R[2,2]*R[2,2]+R[3,2]*R[3,2]=1}),
             {R[1,2]*R[1,3]+R[2,2]*R[2,3]+R[3,2]*R[3,3]=0}),
            {R[1,3]*R[1,3]+R[2,3]*R[2,3]+R[3,3]*R[3,3]=1});
end proc:
#
# Simplify a general motion matrix using rotation matrix identities in rows
rrSimp := proc(M,R)
       simplify(
        simplify(
         simplify(
          simplify(
           simplify(
            simplify(M,{R[1,1]*R[1,1]+R[1,2]*R[1,2]+R[1,3]*R[1,3]=1}),
                       {R[1,1]*R[2,1]+R[1,2]*R[2,2]+R[1,3]*R[2,3]=0}),
                       {R[1,1]*R[3,1]+R[1,2]*R[3,2]+R[1,3]*R[3,3]=0}),
                       {R[2,1]*R[2,1]+R[2,2]*R[2,2]+R[2,3]*R[2,3]=1}),
                       {R[2,1]*R[3,1]+R[2,2]*R[3,2]+R[2,3]*R[3,3]=0}),
                       {R[3,1]*R[3,1]+R[3,2]*R[3,2]+R[3,3]*R[3,3]=1});
end proc:
#
# Matrix representation of a set of polynomials
PolyCoeffMatrix:=proc(S,m,Ord::{ShortTermOrder, TermOrder})
local A,v,i,j,k,c,q;
        A:=Matrix(nops(S),nops(m),storage=sparse);
        v:=indets(m);
        for i from 1 to nops(S) do
                c:=[coeffs(expand(S[i]),v,'q')];
                q:=[q];
```

```
                  for j from 1 to nops(m) do
                           for k from 1 to nops(q) do
                                   if (m[j]=q[k]) then A[i,j]:=c[k] end if
                           end do
                   end do
          end do;
          Matrix(A);
end proc:
#
## Cartesian product of a two lists
#
LxL:=proc(X::list,Y::list)
     Flatten(map(x->(map(y->Flatten([x,y]),Y)),X),1);
end proc:
#
## n x 1 matrix to a list conversion
#
M2L:=proc(M)
 convert(convert(M,Vector),list);
end proc:
#
## Highlit non-zero entries
#
spy:=proc(A)
   map(x->`if`(simplify(x)=0,0, `if`(simplify(x)=1,1,`*`)) ,A):
end proc:
#
# Monomials of a set of polynomial in all indeterminates
#
PolyMonomials:=proc(S::list(ratpoly),Ord::{ShortTermOrder, TermOrder}) # Monomials
of a set of polynomials
local v,m,i,c,q;
        v:=indets(S);
        m:=[];
        for i from 1 to nops(S) do
                c:=[coeffs(expand(S[i]),v,'q')];
                m:=[op(m),q];
        end do;
        m:=MakeUnique(m);
        sort(m,(t1,t2)->testorder(t2,t1,Ord));
end proc:
#
## Monomias of a set of polynomials in given indeterminates
#
PolyVarsMonomials:=proc(S::list(ratpoly),Ord::{ShortTermOrder, TermOrder}) #
Monomials of a set of polynomials in variavbles of Ord
local v,m,i,c,q;
        v:={op(Ord)};
        m:=[];
        for i from 1 to nops(S) do
                c:=[coeffs(expand(S[i]),v,'q')];
                m:=[op(m),q];
        end do;
        m:=MakeUnique(m);
        sort(m,(t1,t2)->testorder(t2,t1,Ord));
end proc:
```

6-DOF Robot IK formulation

Given ai, di, i = 1...6, and Mh, find parameters ci, si, pi, ri subject to

(1) M1 * M2 * M3 * M4 * M5 * M6 = Mh

(2) (M11*M12)*(M21*M22)*(M31*M32)*(M41*M42)*(M51*M52)*(M61*M62) = Mh

(3) ci^2 + si^2 = 1 i = 1...6

(4) pi^2 + ri^2 = 1 i = 1...6

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Write (1) equivalently as

(5) M3 * M4 * M5 = M2^{-1} * M1^{-1} * Mh * M6^{-1}

(6) M31*M32*M41*M42*M51*M52

= M22^{-1}*M21^{-1}*M12^{-1}*M11^{-1}* Mh * M62^{-1}*M61^{-1}

Solution

Symbolically from 12 equations to Z p = 0

The manipulator matrices

```
> M31 :=dhTs(3)[1]:
  M32 :=dhTs(3)[2]:
  M41 :=dhTs(4)[1]:
  M42 :=dhTs(4)[2]:
  M51 :=dhTs(5)[1]:
  M52 :=dhTs(5)[2]:
  iM22:=dhInvs(dhTs(2))[1]:
  iM21:=dhInvs(dhTs(2))[2]:
  iM12:=dhInvs(dhTs(1))[1]:
  iM11:=dhInvs(dhTs(1))[2]:
  Mh  :=Matrix(4,4,[[lx,mx,nx,rx],[ly,my,ny,ry],[lz,mz,nz,rz],[0,0,0,1]]):
  iM62:=dhInvs(dhTs(6))[1]:
  iM61:=dhInvs(dhTs(6))[2]:
```

Let us first inspect the matrices.

```
> M31,M32,M41,M42,M51,M52,"=",iM22,iM21,iM12,iM11,Mh,iM62,iM61;
```

$$\begin{bmatrix} c3 & -s3 & 0 & 0 \\ s3 & c3 & 0 & 0 \\ 0 & 0 & 1 & d3 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & a3 \\ 0 & \lambda 3 & -\mu 3 & 0 \\ 0 & \mu 3 & \lambda 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} c4 & -s4 & 0 & 0 \\ s4 & c4 & 0 & 0 \\ 0 & 0 & 1 & d4 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & a4 \\ 0 & \lambda 4 & -\mu 4 & 0 \\ 0 & \mu 4 & \lambda 4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} c5 & -s5 & 0 & 0 \\ s5 & c5 & 0 & 0 \\ 0 & 0 & 1 & d5 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 & 0 & a5 \\ 0 & \lambda 5 & -\mu 5 & 0 \\ 0 & \mu 5 & \lambda 5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, "=", \begin{bmatrix} 1 & 0 & 0 & -a2 \\ 0 & \lambda 2 & \mu 2 & 0 \\ 0 & -\mu 2 & \lambda 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} c2 & s2 & 0 & 0 \\ -s2 & c2 & 0 & 0 \\ 0 & 0 & 1 & -d2 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & -a1 \\ 0 & \lambda 1 & \mu 1 & 0 \\ 0 & -\mu 1 & \lambda 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} c1 & s1 & 0 & 0 \\ -s1 & c1 & 0 & 0 \\ 0 & 0 & 1 & -d1 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} lx & mx & nx & rx \\ ly & my & ny & ry \\ lz & mz & nz & rz \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & -a6 \\ 0 & \lambda 6 & \mu 6 & 0 \\ 0 & -\mu 6 & \lambda 6 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} c6 & s6 & 0 & 0 \\ -s6 & c6 & 0 & 0 \\ 0 & 0 & 1 & -d6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Notice that the the two last columns of iM61 are free of c6, s6 and so we can get six equations without the sixth variable. Thus, take only those 6 equations to liminate c6, s6.

```
> M31,M32,M41,M42,M51,M52[1..4,3..4],"=",iM22,iM21,iM12,iM11,Mh,iM62,iM61[1..4,3..
  4];
```

$$\begin{bmatrix} c3 & -s3 & 0 & 0 \\ s3 & c3 & 0 & 0 \\ 0 & 0 & 1 & d3 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & a3 \\ 0 & \lambda 3 & -\mu 3 & 0 \\ 0 & \mu 3 & \lambda 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} c4 & -s4 & 0 & 0 \\ s4 & c4 & 0 & 0 \\ 0 & 0 & 1 & d4 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & a4 \\ 0 & \lambda 4 & -\mu 4 & 0 \\ 0 & \mu 4 & \lambda 4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} c5 & -s5 & 0 & 0 \\ s5 & c5 & 0 & 0 \\ 0 & 0 & 1 & d5 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & a5 \\ -\mu 5 & 0 \\ \lambda 5 & 0 \\ 0 & 1 \end{bmatrix},$$

$$"=", \begin{bmatrix} 1 & 0 & 0 & -a2 \\ 0 & \lambda 2 & \mu 2 & 0 \\ 0 & -\mu 2 & \lambda 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} c2 & s2 & 0 & 0 \\ -s2 & c2 & 0 & 0 \\ 0 & 0 & 1 & -d2 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & -a1 \\ 0 & \lambda 1 & \mu 1 & 0 \\ 0 & -\mu 1 & \lambda 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} c1 & s1 & 0 & 0 \\ -s1 & c1 & 0 & 0 \\ 0 & 0 & 1 & -d1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} lx & mx & nx & rx \\ ly & my & ny & ry \\ lz & mz & nz & rz \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix}1 & 0 & 0 & -a6\\ 0 & \lambda 6 & \mu 6 & 0\\ 0 & -\mu 6 & \lambda 6 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}, \begin{bmatrix}0 & 0\\ 0 & 0\\ 1 & -d6\\ 0 & 1\end{bmatrix}$$

Do two following manipulations to "simplify" the set of equations:

1) Multiply both sides from the left by M22

```
> dhInv(iM22),M31,M32,M41,M42,M51,M52[1..4,3..4],"=",iM21,iM12,iM11,Mh,iM62,iM61[1..4,3..4];
```

$$\begin{bmatrix}1 & 0 & 0 & a2\\ 0 & \lambda 2 & -\mu 2 & 0\\ 0 & \mu 2 & \lambda 2 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}, \begin{bmatrix}c3 & -s3 & 0 & 0\\ s3 & c3 & 0 & 0\\ 0 & 0 & 1 & d3\\ 0 & 0 & 0 & 1\end{bmatrix}, \begin{bmatrix}1 & 0 & 0 & a3\\ 0 & \lambda 3 & -\mu 3 & 0\\ 0 & \mu 3 & \lambda 3 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}, \begin{bmatrix}c4 & -s4 & 0 & 0\\ s4 & c4 & 0 & 0\\ 0 & 0 & 1 & d4\\ 0 & 0 & 0 & 1\end{bmatrix}, \begin{bmatrix}1 & 0 & 0 & a4\\ 0 & \lambda 4 & -\mu 4 & 0\\ 0 & \mu 4 & \lambda 4 & 0\\ 0 & 0 & 0 & 1\end{bmatrix},$$

$$\begin{bmatrix}c5 & -s5 & 0 & 0\\ s5 & c5 & 0 & 0\\ 0 & 0 & 1 & d5\\ 0 & 0 & 0 & 1\end{bmatrix}, \begin{bmatrix}0 & a5\\ -\mu 5 & 0\\ \lambda 5 & 0\\ 0 & 1\end{bmatrix}, \text{"="}, \begin{bmatrix}c2 & s2 & 0 & 0\\ -s2 & c2 & 0 & 0\\ 0 & 0 & 1 & -d2\\ 0 & 0 & 0 & 1\end{bmatrix}, \begin{bmatrix}1 & 0 & 0 & -a1\\ 0 & \lambda 1 & \mu 1 & 0\\ 0 & -\mu 1 & \lambda 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}, \begin{bmatrix}c1 & s1 & 0 & 0\\ -s1 & c1 & 0 & 0\\ 0 & 0 & 1 & -d1\\ 0 & 0 & 0 & 1\end{bmatrix},$$

$$\begin{bmatrix}lx & mx & nx & rx\\ ly & my & ny & ry\\ lz & mz & nz & rz\\ 0 & 0 & 0 & 1\end{bmatrix}, \begin{bmatrix}1 & 0 & 0 & -a6\\ 0 & \lambda 6 & \mu 6 & 0\\ 0 & -\mu 6 & \lambda 6 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}, \begin{bmatrix}0 & 0\\ 0 & 0\\ 1 & -d6\\ 0 & 1\end{bmatrix}$$

2) Multiply both sides from the left by

```
> <<1,0,0,0>|<0,1,0,0>|<0,0,1,0>|<0,0,d2,1>>;
```

$$\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & d2\\ 0 & 0 & 0 & 1\end{bmatrix}$$

```
> <<1,0,0,0>|<0,-1,0,0>|<0,0,1,0>|<0,0,d2,1>>.dhInv(iM22),M31,M32,M41,M42,M51,M52[1..4,3..4],
  "=",
  <<1,0,0,0>|<0,-1,0,0>|<0,0,1,0>|<0,0,d2,1>>.iM21,iM12,iM11,Mh,iM62,iM61[1..4,3..4];
```

$$\begin{bmatrix}1 & 0 & 0 & a2\\ 0 & -\lambda 2 & \mu 2 & 0\\ 0 & \mu 2 & \lambda 2 & d2\\ 0 & 0 & 0 & 1\end{bmatrix}, \begin{bmatrix}c3 & -s3 & 0 & 0\\ s3 & c3 & 0 & 0\\ 0 & 0 & 1 & d3\\ 0 & 0 & 0 & 1\end{bmatrix}, \begin{bmatrix}1 & 0 & 0 & a3\\ 0 & \lambda 3 & -\mu 3 & 0\\ 0 & \mu 3 & \lambda 3 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}, \begin{bmatrix}c4 & -s4 & 0 & 0\\ s4 & c4 & 0 & 0\\ 0 & 0 & 1 & d4\\ 0 & 0 & 0 & 1\end{bmatrix}, \begin{bmatrix}1 & 0 & 0 & a4\\ 0 & \lambda 4 & -\mu 4 & 0\\ 0 & \mu 4 & \lambda 4 & 0\\ 0 & 0 & 0 & 1\end{bmatrix},$$

$$\begin{bmatrix}c5 & -s5 & 0 & 0\\ s5 & c5 & 0 & 0\\ 0 & 0 & 1 & d5\\ 0 & 0 & 0 & 1\end{bmatrix}, \begin{bmatrix}0 & a5\\ -\mu 5 & 0\\ \lambda 5 & 0\\ 0 & 1\end{bmatrix}, \text{"="}, \begin{bmatrix}c2 & s2 & 0 & 0\\ s2 & -c2 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}, \begin{bmatrix}1 & 0 & 0 & -a1\\ 0 & \lambda 1 & \mu 1 & 0\\ 0 & -\mu 1 & \lambda 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}, \begin{bmatrix}c1 & s1 & 0 & 0\\ -s1 & c1 & 0 & 0\\ 0 & 0 & 1 & -d1\\ 0 & 0 & 0 & 1\end{bmatrix},$$

$$\begin{bmatrix}lx & mx & nx & rx\\ ly & my & ny & ry\\ lz & mz & nz & rz\\ 0 & 0 & 0 & 1\end{bmatrix}, \begin{bmatrix}1 & 0 & 0 & -a6\\ 0 & \lambda 6 & \mu 6 & 0\\ 0 & -\mu 6 & \lambda 6 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}, \begin{bmatrix}0 & 0\\ 0 & 0\\ 1 & -d6\\ 0 & 1\end{bmatrix}$$

Do the multiplications, construct the equations, and take the 6 ones we are interested in.
Denote the left hand side by ee1 and the right hand side by ee2.

```
> ee1:=dhSimpl((<<1,0,0,0>|<0,-1,0,0>|<0,0,1,0>|<0,0,d2,1>>.dhInv(iM22).M31.M32.M41.M42.M51.M52)[1..3,3..4],2):
  ee2:=dhSimpl((<<1,0,0,0>|<0,-1,0,0>|<0,0,1,0>|<0,0,d2,1>>.iM21.iM12.iM11.Mh.iM62.iM61)[1..3,3..4],2):
```

We have 6 equations in 10 unknowns s1, c1, s2, c2, s3, c3, s4, c4, s5, c5.
Let us generate more equations.

To do so, denote the columns of the left and right hand side as:

[l2 p2] = [l1 p1]

```
> l2:=ee1[1..3,1..1]:
  p2:=ee1[1..3,2..2]:
  l1:=ee2[1..3,1..1]:
  p1:=ee2[1..3,2..2]:
> expand(p1[1,1]);
```

$-d6\ \mu 6\ mx\ c2\ c1 + d6\ \mu 6\ mx\ s2\ \lambda 1\ s1 - d6\ \mu 6\ my\ c2\ s1 - d6\ \mu 6\ my\ s2\ \lambda 1\ c1 - d6\ \mu 6\ s2\ \mu 1\ mz - d6\ \lambda 6\ nx\ c2\ c1 + d6\ \lambda 6\ nx\ s2\ \lambda 1\ s1 - d6\ \lambda 6\ ny\ c2\ s1 - d6\ \lambda 6\ ny\ s2\ \lambda 1\ c1 - d6\ \lambda 6\ s2\ \mu 1\ nz - a6\ lx\ c2\ c1 + a6\ lx\ s2\ \lambda 1\ s1 - a6\ ly\ c2\ s1 - a6\ ly\ s2\ \lambda 1\ c1 - a6\ s2\ \mu 1\ lz + rx\ c2\ c1 - rx\ s2\ \lambda 1\ s1 + ry\ c2\ s1 + ry\ s2\ \lambda 1\ c1 + s2\ \mu 1\ rz - s2\ \mu 1\ d1 - c2\ a1$

```
> expand(p2[1,1]);
```

$a5\ c5\ c3\ c4 - a5\ c5\ s3\ \lambda 3\ s4 - a5\ s5\ \lambda 4\ c3\ s4 - a5\ s5\ \lambda 4\ s3\ \lambda 3\ c4 + a5\ s5\ s3\ \mu 3\ \mu 4 + d5\ \mu 4\ c3\ s4 + d5\ \mu 4\ s3\ \lambda 3\ c4 + d5\ s3\ \mu 3\ \lambda 4 + a4\ c3\ c4 - a4\ s3\ \lambda 3\ s4 + s3\ \mu 3\ d4 + c3\ a3 + a2$

New equations can be now generated by scalar and vector operations on the columns.

1) p2 . p2 = p1 . p1 ... 1 new equation

2) p2 . l2 = p1 . l1 ... 1 new equation

3) p2 x l2 = p1 x l1 ... 3 new equations

4) (p1 . p1) l1 - 2 (p1 . l1) p1 = (p2 . p2) l2 - 2 (p2 . l2) p2 ... 3 new equations

which can be derived from

A x (B x C) = (A . C) B - (A . B) C

using the substitution p = A = C, l = B

0 = A x (B x C) = (A . C) B - (A . B) C

and thus

- (A . B) C = (A . C) B - 2 (A . B) C

we get

- (p1 . l1) p1 = (p1 . p1) l1 - 2 (p1 . l1) p1

- (p2 . l2) p2 = (p2 . p2) l2 - 2 (p2 . l2) p2

and using

p1 = p2 & l1 = l2

we get

- (p1 . l1) p1 = - (p2 . l2) p2

(p1 . p1) l1 - 2 (p1 . l1) p1 = (p2 . p2) l2 - 2 (p2 . l2) p2

ad 1) p2 . p2 = p1 . p1 ... 1 new equation

```
> pp1:=MhSimpl(dhSimpl(dhSimpl(dhSimpl(Transpose(p1).p1,2),1),6));
  pp2:=dhSimpl(dhSimpl(dhSimpl(dhSimpl(Transpose(p2).p2,2),3),4),5);
```

$pp1 :=$

$[-2\ rz\ d1 + rx^2 + a6^2 + ry^2 + d6^2 + a1^2 + rz^2 - 2\ rx\ c1\ a1 - 2\ ry\ s1\ a1 + d1^2 + (2\ a6\ c1\ a1 - 2\ a6\ rx)\ lx + (-2\ a6\ ry + 2\ a6\ s1\ a1)\ ly + (-2\ a6\ rz + 2\ a6\ d1)\ lz + (-2\ d6\ \mu 6\ rx + 2\ d6\ \mu 6\ c1\ a1)\ mx + (-2\ d6\ \mu 6\ ry + 2\ d6\ \mu 6\ s1\ a1)\ my + (-2\ d6\ \mu 6\ rz + 2\ d6\ \mu 6\ d1)\ mz + (-2\ d6\ \lambda 6\ rx + 2\ d6\ \lambda 6\ c1\ a1)\ nx + (2\ d6\ \lambda 6\ s1\ a1 - 2\ d6\ \lambda 6\ ry)\ ny + (-2\ d6\ \lambda 6\ rz + 2\ d6\ \lambda 6\ d1)\ nz]$

$pp2 :=$

$[2\ d2\ \lambda 2\ d3 + 2\ c3\ a3\ a2 + a2^2 + 2\ a4\ c3\ c4\ a2 + 2\ d5\ s3\ \mu 3\ \lambda 4\ a2 + d2^2 + 2\ d5\ \mu 4\ c3\ s4\ a2 + a3^2 + 2\ s3\ \mu 3\ d4\ a2 - 2\ a4\ s3\ \lambda 3\ s4\ a2 + d4^2 + 2\ d2\ \lambda 2\ a4\ s4\ \mu 3 + 2\ d2\ \lambda 2\ d5\ \lambda 4\ \lambda 3 + 2\ d2\ \mu 2\ a4\ s4\ c3\ \lambda 3 - 2\ d2\ \mu 2\ d4\ c3\ \mu 3 + 2\ d2\ \mu 2\ a4\ s3\ c4 + 2\ d5\ \lambda 4\ \lambda 3\ d3 + 2\ d2\ \lambda 2\ d4\ \lambda 3 + 2\ d2\ \mu 2\ s3\ a3 + 2\ d5\ \mu 4\ s3\ \lambda 3\ c4\ a2 - 2\ d2\ \mu 2\ d5\ \lambda 4\ c3\ \mu 3 + 2\ d4\ \lambda 3\ d3 + 2\ d2\ \mu 2\ d5\ \mu 4\ s3\ s4 + 2\ d3\ a4\ s4\ \mu 3 + a4^2 + d5^2 + d3^2 + (-2\ a5\ s3\ \lambda 3\ s4\ a2 + 2\ a5\ c3\ c4\ a2$

$+ 2\, d2\, \mu 2\, a5\, s3\, c4 + 2\, a5\, s4\, \mu 3\, d3 + 2\, a5\, a4 + 2\, d2\, \lambda 2\, a5\, s4\, \mu 3 + 2\, d2\, \mu 2\, a5\, s4\, c3\, \lambda 3 + 2\, a5\, c4\, a3)\, c5 + ($
$-2\, a5\, \lambda 4\, s3\, \lambda 3\, c4\, a2 - 2\, a5\, \lambda 4\, c3\, s4\, a2 + 2\, d2\, \lambda 2\, a5\, \lambda 4\, c4\, \mu 3 - 2\, d2\, \mu 2\, a5\, \lambda 4\, s3\, s4 + 2\, a5\, s3\, \mu 3\, \mu 4\, a2$
$- 2\, a5\, \lambda 4\, s4\, a3 + 2\, d2\, \mu 2\, a5\, \lambda 4\, c4\, c3\, \lambda 3 - 2\, d2\, \mu 2\, a5\, \mu 4\, c3\, \mu 3 + 2\, d4\, a5\, \mu 4 + 2\, d3\, a5\, \mu 4\, \lambda 3 + 2\, d2\, \lambda 2\, a5\, \mu 4\, \lambda 3$
$+ 2\, a5\, \lambda 4\, c4\, \mu 3\, d3)\, s5 + a5^2 + 2\, d4\, d5\, \lambda 4 - 2\, d2\, \mu 2\, d5\, \mu 4\, c4\, c3\, \lambda 3 + 2\, d5\, \mu 4\, s4\, a3 + 2\, a4\, c4\, a3$
$- 2\, d3\, d5\, \mu 4\, c4\, \mu 3 - 2\, d2\, \lambda 2\, d5\, \mu 4\, c4\, \mu 3\,]$

ad 2) p2 . l2 = p1 . l1 ... 1 new equation

```
> pl1:=MhSimpl(dhSimpl(dhSimpl(dhSimpl(Transpose(p1).l1,2),1),6)):
  pl2:=dhSimpl(dhSimpl(dhSimpl(dhSimpl(Transpose(p2).l2,2),3),4),5):
```

ad 3) p2 x l2 = p1 x l1 ... 3 new equations

```
> pxl1:=map(x->expand(x),convert(CrossProduct(convert(p1,Vector),convert(l1,Vector
  )),Matrix)):
  pxl2:=map(x->expand(x),convert(CrossProduct(convert(p2,Vector),convert(l2,Vector
  )),Matrix)):
  m1x:=map(x->expand(x),MhSimpl(dhSimpl(dhSimpl(dhSimpl(pxl1,2),1),6))):
  m2x:=map(x->expand(x),dhSimpl(dhSimpl(dhSimpl(dhSimpl(pxl2,2),3),4),5)):
```

ad 4) (p1 . p1) l1 - 2 (p1 . l1) p1 = (p2 . p2) l2 - 2 (p2 . l2) p2 ... 3 new equations

```
> plpl1:=map(x->expand(x),ScalarMultiply(l1,pp1[1,1]) -
  ScalarMultiply(p1,2*pl1[1,1])):
  plpl2:=map(x->expand(x),ScalarMultiply(l2,pp2[1,1]) -
  ScalarMultiply(p2,2*pl2[1,1])):
  mp1:=MhSimpl(dhSimpl(dhSimpl(dhSimpl(simplify(plpl1),2),1),6)):
  mp2:=dhSimpl(dhSimpl(dhSimpl(dhSimpl(dhSimpl(simplify(plpl2),2),3),4),5),1):
```

Gather all 14 equations together:

```
> E1:=<p1,l1,pp1,pl1,m1x,mp1>:
  E2:=<p2,l2,pp2,pl2,m2x,mp2>:
```

and construct its linear representation.

Let us look at c3, s3 as on parameters and consider the monomials of the 8 remaining unknowns s1,c1,s2,c2,s4,c4,s5,c5 in the 6 equations.

1) Notice that on the right hand side, there are the following monomials in s1,c1,s2,c2:

```
> t1:=<<s1*s2,s1*c2,c1*s2,c1*c2,s1,c1,s2,c2,1>>:
```

2) Notice that on the left hand side, there are the following monomials in s4,c4,s5,c5:

```
> t2:=<<s4*s5,s4*c5,c4*s5,c4*c5,s4,c4,s5,c5,1>>:
```

Construct the linear representation in the above monomials:

```
> M1:=PolyCoeffMatrix(M2L(E1),M2L(t1),plex(op(indets(t1)))):
  M2:=PolyCoeffMatrix(M2L(E2),M2L(t2),plex(op(indets(t2)))):
```

Check it.

```
> Transpose(simplify(E1-M1.t1));
  Transpose(simplify(E2-M2.t2));
```

[0 0 0 0 0 0 0 0 0 0 0 0 0 0]

[0 0 0 0 0 0 0 0 0 0 0 0 0 0]

OK.

Move the constants from the right to the left and denote the left hand side of the equations P and the right hand side of the equations Q.

```
> P:= <M2[1..14,1..8]|M2[1..14,9]-M1[1..14,9]>:
  Q:=  M1[1..14,1..8]:
```

Modify the corresponding monomial vectors and name them pp, qq.

```
> p:= t2:
  q:= <t1[1..8,1]>:
```

We have 14 equations in 17 monomials of 10 unknowns constructed form 5 angles:

```
> Dimensions(P),`...`,Dimensions(p);
  Dimensions(Q),`...`,Dimensions(q);
```

14, 9, ..., 9, 1

14, 8, ..., 8, 1

Check it.

```
> Transpose((-P.p+Q.q)-(M1.t1-M2.t2));
```

[0 0 0 0 0 0 0 0 0 0 0 0 0 0]

OK.

The matrices PP, QQ are semi-sparse:

```
> <Transpose(p),spy(P)>,<Transpose(q),spy(Q)>;
```

s4 s5	s4 c5	c4 s5	c4 c5	s4	c4	s5	c5	1
*	*	*	*	*	*	*	0	*
*	*	*	*	*	*	*	0	*
*	*	*	*	*	*	*	0	*
*	*	*	*	*	*	0	*	*
*	*	*	*	*	*	0	*	*
*	*	*	*	*	*	0	*	*
*	*	*	*	*	*	*	*	*
*	*	*	*	*	*	*	*	*
*	*	*	*	*	*	*	*	*
*	*	*	*	*	*	*	*	*
*	*	*	*	*	*	*	*	*
*	*	*	*	*	*	*	*	*
*	*	*	*	*	*	*	*	*
*	*	*	*	*	*	*	*	*

,

s1 s2	s1 c2	c1 s2	c1 c2	s1	c1	s2	c2
*	*	*	*	0	0	*	*
*	*	*	*	0	0	*	*
0	0	0	0	*	*	0	0
*	*	*	*	0	0	*	0
*	*	*	*	0	0	0	*
0	0	0	0	*	*	0	0
0	0	0	0	*	*	0	0
0	0	0	0	*	*	0	0
*	*	*	*	0	0	*	0
*	*	*	*	0	0	0	*
0	0	0	0	*	*	0	0
*	*	*	*	0	0	*	*
*	*	*	*	0	0	*	*
0	0	0	0	*	*	0	0

The set of equations can be written as

```
     P    p   =     Q    q
   14 x 9         14 x 8
```

Notice that P = P(c3, s3) and Q is a constant matrix if the mechanism and its pose are fixed.

Split P and Q to two submatrices 8x8 P8, Q8 and 6x6 P6, Q6

```
   P  =  [  P8 ]             Q  =  [  Q8 ]
         [ Q6  ]                   [ Q6  ]
```

Assume that Q8 has full rank. Then, we can write

```
        P8  p  =  Q8  q

        P6  p  =  Q6  q
```

express q from the first 8 equations

```
        q   =   inv(Q8)  P8  p
```

and substitute into the remaining 6 equations

```
        P6  p   =  Q6  inv(Q8)  P8  p
```

which, after moving all to the left, gives

```
        (P6 - Q6  inv(Q8)  P8)  p = 0
```

or when introducing

```
        Z  =  (P6 - Q6  inv(Q8)  P8)
```

we get

```
                Z  p  =  0
```

This is a system of 6 homogeneous linear equations in 8 monomials.

Constructing Z however, depends on the acrual values in Q. The extreme case happens when the first 8x8 submatrix of Q is singular. Then, we have to select rows from Q to get Q8 regular. It is still better to select the eight rows which give us Q8 with the smallest condition number because inv(Q8) is then computed most robustly w.r.t. the rounding errors.

To proceed further, we have to choose a concrete mechanism to get numerical values in P, Q. We shall choose general parameters of a meachnism that will simulate a real, slightly missaligned 3-2-1 robot.

Random general mechanism

```
> Mechanism := {
  a1=85,  a2=280,  a3=100, a4=0.1, a5=-0.1, a6=0.1,
  d1=350, d2=-0.1, d3=0.1, d4=315, d5=-0.1, d6=85,
  lambda1=cos(-Pi/2-Pi/100), mu1=sin(-Pi/2-Pi/100),
  lambda2=cos(Pi/110),       mu2=sin(Pi/110),
  lambda3=cos(-Pi/2-Pi/90),  mu3=sin(-Pi/2-Pi/90),
  lambda4=cos(Pi/2-Pi/120),  mu4=sin(Pi/2-Pi/120),
  lambda5=cos(-Pi/2-Pi/95),  mu5=sin(-Pi/2-Pi/95),
  lambda6=cos(-Pi/200),      mu6=sin(-Pi/200)}:
```

Set randomly the joint angles and compute the corresponding Mh

```
> rndTh:=Pi*RandomMatrix(1,6)/200;
```

$$rndTh := \begin{bmatrix} \frac{\pi}{4} & \frac{\pi}{20} & -\frac{2\pi}{25} & -\frac{9\pi}{200} & -\frac{\pi}{4} & -\frac{11\pi}{100} \end{bmatrix}$$

```
> thetas :=
  map(x->x[1]=x[2],convert(<convert([theta1,theta2,theta3,theta4,theta5,theta6],Ve
  ctor)|
                           convert(simplify(rndTh),Vector)>,listlist)):
  Position :=
  subs(thetas,{c1=cos(theta1),s1=sin(theta1),c2=cos(theta2),s2=sin(theta2),c3=cos(
  theta3),s3=sin(theta3),

  c4=cos(theta4),s4=sin(theta4),c5=cos(theta5),s5=sin(theta5),c6=cos(theta6),s6=si
  n(theta6)}):
  MP := {op(Mechanism),op(Position)};
  MhV := Matrix(4,4,[[lx,mx,nx,rx],[ly,my,ny,ry],[lz,mz,nz,rz],[0,0,0,1]]):
  MhV :=
  map(x->x[1]=x[2],convert(<convert(MhV[1..3,1..4],Vector)|convert(dhDKT(MP)[1..3,
  1..4],Vector)>,listlist)):
```

$$MP := \{ \mu 3 = -\sin\left(\frac{22\pi}{45}\right), c4 = \cos\left(\frac{9\pi}{200}\right), c2 = \cos\left(\frac{\pi}{20}\right), s2 = \sin\left(\frac{\pi}{20}\right), c1 = \frac{\sqrt{2}}{2}, s1 = \frac{\sqrt{2}}{2}, \mu 6 = -\sin\left(\frac{\pi}{200}\right),$$
$$c6 = \cos\left(\frac{11\pi}{100}\right), a1 = 85, a2 = 280, a3 = 100, a4 = 0.1, a5 = -0.1, a6 = 0.1, d1 = 350, d2 = -0.1, d3 = 0.1, d4 = 315,$$
$$d5 = -0.1, \lambda 2 = \cos\left(\frac{\pi}{110}\right), \mu 2 = \sin\left(\frac{\pi}{110}\right), \lambda 4 = \cos\left(\frac{59\pi}{120}\right), d6 = 85, \mu 4 = \sin\left(\frac{59\pi}{120}\right), s6 = -\sin\left(\frac{11\pi}{100}\right),$$
$$\lambda 5 = -\cos\left(\frac{93\pi}{190}\right), \mu 1 = -\sin\left(\frac{49\pi}{100}\right), \lambda 3 = -\cos\left(\frac{22\pi}{45}\right), s4 = -\sin\left(\frac{9\pi}{200}\right), \mu 5 = -\sin\left(\frac{93\pi}{190}\right), c5 = \frac{\sqrt{2}}{2}, s5 = -\frac{\sqrt{2}}{2},$$
$$c3 = \cos\left(\frac{2\pi}{25}\right), \lambda 1 = -\cos\left(\frac{49\pi}{100}\right), \lambda 6 = \cos\left(\frac{\pi}{200}\right), s3 = -\sin\left(\frac{2\pi}{25}\right) \}$$

Numerically from Z p = 0 to f(x3)=0

Substitute the parameters of the manipulator and Mh into matrice P & Q and truncate by first converting it to Maple software floats (precision given by Digits) and then exactly to rational numbers

```
> MPh := {op(Mechanism),op(MhV)}:
  Qx  := simplify(evalf(subs(MPh,Q))):
  Px  := simplify(evalf(subs(MPh,P))):
  QxR := convert(Qx,rational,exact):
  PxR := convert(Px,rational,exact):
```

Look at the 8 largest singular values of QxR

```
> SingularValues(evalf(QxR));
```

$$\begin{bmatrix} 378480.951865628073302112584649 \\ 378480.951865628073302112584647 \\ 304770.655569428774175857855054 \\ 83357.5565310114917490193163683 \\ 96.6051524679321534738938096784 \\ 96.6051524679321534738938092133 \\ 59.4337182736028838590927746217 \\ 59.4337182736028838590927744287 \\ 0. \\ 0. \\ 0. \\ 0. \\ 0. \\ 0. \end{bmatrix}$$

```
> svQxR:=evalf(SingularValues(evalf(QxR)))[1..8];
```

$$svQxR := \begin{bmatrix} 378480.951865628073302112584649 \\ 378480.951865628073302112584647 \\ 304770.655569428774175857855054 \\ 83357.5565310114917490193163683 \\ 96.6051524679321534738938096784 \\ 96.6051524679321534738938092133 \\ 59.4337182736028838590927746217 \\ 59.4337182736028838590927744287 \end{bmatrix}$$

The condition number of QxR is not so bad

```
> svQxR[1]/svQxR[8];
```

6368.11834863322076468545706392

```
> QxRf:=evalf(QxR):
  U, S, Vt := SingularValues(QxRf, output=['U', 'S', 'Vt']):
```

Use SVD to choose the best equations possible to constinue:

Change the basis using the SVD of Q to get Z

P p = Q q

Make SVD of Q: Q = U S V^T

P p = U S V^T q

Multiply by th einverse of U from the left. U has orthonormal columns => inv(U) = U^T

U^T P p = S V^T q

Denote U^T P by A: A = U^T P

A p = S V^T q

Split A into the first 8 rows, A8, and the rest 6 rows Z

[A8] p = [S8 V^T] q
[Z] [0]

Get the equations from the last 6 rows

Z p = 0

```
> A  := Transpose(U).PxR:
  A8 := A[1..8,1..9]:
  Z  := A[9..14,1..9]:
```

Construct matrix Z

```
> `Z = `, <Transpose(p),spy(Z)>;
```

$$Z = ,\begin{bmatrix} s4\,s5 & s4\,c5 & c4\,s5 & c4\,c5 & s4 & c4 & s5 & c5 & 1 \\ * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * \end{bmatrix}$$

We have 6 equations for 8 monomials but also recall that Z contains s3, c3:

```
> indets(Z);
```

$$\{c3, s3\}$$

```
> evalf(Z[1..6,9]);
```

$$[-23.14390254837427431665759 12865\, c3 + 43.27372041143867198709539 59470\, s3 + 55.20252295412762164450142 19896]$$

$$[8.52865585879576648042176499926\, c3 - 17.6254531697049390174587332137\, s3 - 22.0351865097081846862772420950]$$

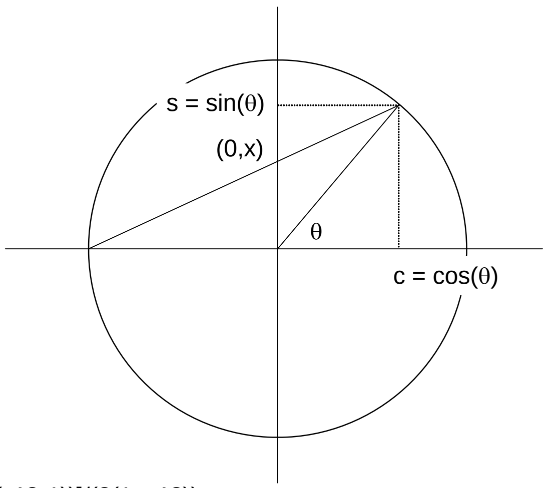
$$[-30.6238402056738192926565560526\, c3 + 97.1708255572899749764500381440\, s3 + 45.2983722865500019522611188548]$$

$$[-0.270528288358414831025540111910\, c3 + 0.857483717889941179625104494392\, s3 - 1.17130256956595328358358210952]$$

$$[2.53874557335572477551923859760\, c3 - 3.06931966798848983398459559096\, s3 - 4.20335467900547641539041211177]$$

$$[43.8455576056752250059309786716\, c3 - 138.216889316769289751762781980\, s3 + 457.521299735815002292038143946]$$

We will now proceed by choosing a different circle parameterization.



x/1 = s/(1+c)

s = x(1+c)

s^2+c^2=1

x^2(1+c)^2+c^2=1

x^2 + 2x^2c +x^2c^2 + c^2 - 1=0

(1+x^2)c^2 + 2 x^2 c + x^2 - 1 = 0

c = [-2x^2 + sqrt(4 x^4 - 4 (1+x^2)(x^2-1))]/(2(1+x^2))

c = [-2x^2 + 2 sqrt(1)]/[2(1+x^2)]

c = (- x^2 + 1) / (1 + x^2)

s = x (1 + c) = 2 x / (1 + x^2)

which then gives

si = 2 x / (1 + xi^2)

ci = (1 - xi^2) / (1 + xi^2)

for i = 4, 5

```
> x:=subs([s4=(2*x4)/(1+x4^2),c4=(1-x4^2)/(1+x4^2),s5=(2*x5)/(1+x5^2),c5=(1-x5^2)/
  (1+x5^2)],p):
  Transpose(x);
```

$$\left[\frac{4\,x4\,x5}{(1+x4^2)\,(1+x5^2)},\ \frac{2\,x4\,(1-x5^2)}{(1+x4^2)\,(1+x5^2)},\ \frac{2\,(1-x4^2)\,x5}{(1+x4^2)\,(1+x5^2)},\ \frac{(1-x4^2)\,(1-x5^2)}{(1+x4^2)\,(1+x5^2)},\ \frac{2\,x4}{1+x4^2},\ \frac{1-x4^2}{1+x4^2},\ \frac{2\,x5}{1+x5^2},\ \frac{1-x5^2}{1+x5^2},\ 1\right]$$

None of the denominators can be zero, thus one can clear them

```
> y:=ScalarMultiply(x,(1+x4^2)*(1+x5^2)):
  Transpose(y);
```

$$[4\,x4\,x5,\ 2\,x4\,(1-x5^2),\ 2\,(1-x4^2)\,x5,\ (1-x4^2)\,(1-x5^2),\ 2\,(1+x5^2)\,x4,\ (1+x5^2)\,(1-x4^2),\ 2\,(1+x4^2)\,x5,\ (1+x4^2)\,(1-x5^2),\ (1+x4^2)\,(1+x5^2)]$$

Use Z and y to get equations in x4, x5 and then express then in a matrix form

```
> qy:=simplify(Z.y):
  mMy:=PolyVarsMonomials(M2L(qy),plex(x4,x5)):
  My:=PolyCoeffMatrix(M2L(qy),mMy,plex(x4,x5)):
```

There are 6 equations for 8 monomials and 2 variables in My:

```
> `My = `, <Transpose(<mMy>),spy(My)>;
```

$$My = ,\ \begin{bmatrix} x4^2\,x5^2 & x5\,x4^2 & x4^2 & x4\,x5^2 & x4\,x5 & x4 & x5^2 & x5 & 1 \\ * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * \end{bmatrix}$$

We can add three more equations as follows.

We see that when multiplying all 6 equations by x4 adds 6 equations but only 3 monomials

```
> mxy:=convert(map(f->f*x4,mMy),set) union convert(mMy,set);
  nops(mxy)-nops(mMy);
```

$$mxy := \{1,\ x4\,x5,\ x5\,x4^2,\ x4\,x5^2,\ x4^3,\ x4^2\,x5^2,\ x5,\ x4,\ x4^2,\ x5^2,\ x4^3\,x5^2,\ x5\,x4^3\}$$

$$3$$

We thus get a system that generates a constraint

```
> qyy:=simplify(<simplify(x4*qy),qy>):
  myy:=PolyVarsMonomials(M2L(qyy),plex(x4,x5));
  Myy:=PolyCoeffMatrix(M2L(qyy),myy,plex(x4,x5)):
```

$$myy := [x4^3\,x5^2,\ x5\,x4^3,\ x4^3,\ x4^2\,x5^2,\ x5\,x4^2,\ x4^2,\ x4\,x5^2,\ x4\,x5,\ x4,\ x5^2,\ x5,\ 1]$$

Check it.

```
> Transpose(simplify(simplify(<Myy.<myy>>)-qyy));
```

$$[0.,\ 0.200000000000000000000000000000\ 10^{-28}\,x4\,s3 + 0.200000000000000000000000000000\ 10^{-28}\,x4^3 + 0.200000000000000000000000000000\ 10^{-28}\,x4\,s3\,x5^2 + 0.200000000000000000000000000000\ 10^{-27}\,x5\,c3\,x4^2 - 0.200000000000000000000000000000\ 10^{-28}\,s3\,x4^3\,x5^2,\ 0.,\ 0.,\ 0.,\ 0.,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0]$$

OK.

There are 12 equations for 11 monomials in My

```
> `Myy = `, <Transpose(<myy>),spy(Myy)>;
```

$$Myy = ,\ \begin{bmatrix} x4^3\,x5^2 & x5\,x4^3 & x4^3 & x4^2\,x5^2 & x5\,x4^2 & x4^2 & x4\,x5^2 & x4\,x5 & x4 & x5^2 & x5 & 1 \\ * & * & * & * & * & * & * & * & * & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & * & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & * & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & * & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & * & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & * & 0 & 0 & 0 \\ 0 & 0 & 0 & * & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & * & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & * & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & * & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & * & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & * & * & * & * & * & * & * & * & * \end{bmatrix}$$

Since myy = [x4^3 x5^2 , . , . , . , ... , 1]
is definitely not a zero vector, this system has a solution if and only if

$$\det(\mathit{Myy}) = 0$$

Now, recall that s3, c3 apear in elements of P, thus P6, and therefore also Myy:

```
> indets(Myy);
```

$$\{c3, s3\}$$

```
> Myy[1,1];
```

$$56.652370696026131477578978424 4 - 27.264040621262874968635458685 8\, c3 + 41.961001520214392706819855310 0\, s3$$

We will again reparameterize c3, s3 into x3 as above:

```
> Transpose(simplify(simplify(<Myy.<myy>>)-qyy));
```

$$[0.,\ 0.2000000000000000000000000000\ 10^{-28}\, x4\, s3 + 0.2000000000000000000000000000\ 10^{-28}\, x4^3 + 0.2000000000000000000000000000\ 10^{-28}\, x4\, s3\, x5^2 + 0.2000000000000000000000000000\ 10^{-27}\, x5\, c3\, x4^2 - 0.2000000000000000000000000000\ 10^{-28}\, s3\, x4^3\, x5^2,\ 0.,\ 0.,\ 0.,\ 0.,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0]$$

Parametrize s3, c3 by x3

```
> Mx3:=simplify((1+x3^2)*subs([s3=(2*x3)/(1+x3^2),c3=(1-x3^2)/(1+x3^2)],Myy)):
> Mx3[1,1];
```

$$29.388330074763256508943519738 6 + 83.916411317289006446214437110 2\, x3^2 + 83.922003040428785413639710620 0\, x3$$

Notice that Mx3 contains second order polynomials in x3:

```
> mx3:=PolyVarsMonomials(M2L(Mx3),plex(x3));
```

$$mx3 := [x3^2, x3, 1]$$

Compute the deteminant giving a plynomial in x3.

```
> dx3:=sort(Determinant(evalf(Mx3)));
```

$$dx3 := 63990.13250180031241785384200 10\, x3^{24} + 652301.82018783172527256907639 3\, x3^{23} + 0.30647130609221123605395511032 3\ 10^{7}\, x3^{22} + 0.94473729508620925639267732074 8\ 10^{7}\, x3^{21} + 0.22837831978500912616043328737 7\ 10^{8}\, x3^{20} + 0.46690473759443412515598952174 3\ 10^{8}\, x3^{19} + 0.82969217862491643765136570873 3\ 10^{8}\, x3^{18} + 0.13176080238426584616654547358 2\ 10^{9}\, x3^{17} + 0.19017678715055149967314091191 2\ 10^{9}\, x3^{16} + 0.25086210136364193220789354427 6\ 10^{9}\, x3^{15} + 0.30486440388315447159926300725 9\ 10^{9}\, x3^{14} + 0.34223105314768679898429041141 2\ 10^{9}\, x3^{13} + 0.35506078329801169720267267992 1\ 10^{9}\, x3^{12} + 0.34076294331050796492177433810\ 10^{9}\, x3^{11} + 0.30183086150467961416680597778\ 10^{9}\, x3^{10} + 0.24591165254380686385748218606\ 10^{9}\, x3^{9} + 0.18293081396948743962880237078 9\ 10^{9}\, x3^{8} + 0.12312478379451576563919322984 2\ 10^{9}\, x3^{7} + 0.73801962380924003518918152660\ 10^{8}\, x3^{6} + 0.38471606549020092003413665905\ 10^{8}\, x3^{5} + 0.16945583856025436181551255053\ 10^{8}\, x3^{4} + 0.59229294034986197140350447 68\ 10^{7}\, x3^{3} + 0.14937956766941289511332565 3\ 10^{7}\, x3^{2} + 193045.87615383314810917686 6\, x3 + 9163.98378718874716813282$$

Normalize it to get 1 at the x3^24 to maximize the precision

```
> nf:=LeadingCoefficient(dx3,tdeg(x3))^(1/12);
  dx3nf:=sort(Determinant(evalf(Mx3)/nf));
```

$$nf := 2.514834545306000800522207762 90$$

$$dx3nf := 0.9999999999999999999999913580015\, x3^{24} + 10.193787615137063872216029334 9\, x3^{23} + 47.893525784399478331054947925 7\, x3^{22} + 147.63796512839378949495760260 4\, x3^{21} + 356.89615079103622802921973296 0\, x3^{20} + 729.65114985704729537495443995 8\, x3^{19} + 1296.5939375130590579495643517 2\, x3^{18} + 2059.0800055080188943705473647 5\, x3^{17} + 2971.9705166292791196604839483 8\, x3^{16} + 3920.3247681442779141890887105 1\, x3^{15} + 4764.2408597071954745192233972 2\, x3^{14} + 5348.1847867412744535220815356\, x3^{13} + 5548.6802326596579445606765532 0\, x3^{12} + 5325.2420332294337030990542131\, x3^{11} + 4716.8344509395700740550686615 1\, x3^{10} + 3842.9620775810759808611362436 4\, x3^{9} + 2858.7347270196826674862536535\, x3^{8} + 1924.1214071724534400112744979 9\, x3^{7} + 1153.3334827654504845410174673 3\, x3^{6} + 601.21154691995243781901159056\, x3^{5} + 264.81557692584376833366497 95\, x3^{4} + 92.560042805536974307330312 5\, x3^{3} + 23.344156642465182831829721 6\, x3^{2} + 3.0168069451708347861839238 7\, x3 + 0.1432093266400241574594360$$

Solve f(x3)=0 for x3

Normalize to get 1 at the leading coefficient:

```
> dx3n:=sort(dx3nf/LeadingCoefficient(dx3nf,tdeg(x3)));
```

$$
\begin{aligned}
dx3n := {} & 0.99999999999999999999999999995\, x3^{24} + 10.193787615137063872216910281 8\, x3^{23} \\
& + 47.893525784399478331059086883 2\, x3^{22} + 147.63796512839378949497036147 4\, x3^{21} \\
& + 356.89615079103622802925057591 8\, x3^{20} + 729.65114985704729537501749639 6\, x3^{19} \\
& + 1296.5939375130590579496764033 4\, x3^{18} + 2059.0800055080188943707253104 0\, x3^{17} \\
& + 2971.9705166292791196607407860 1\, x3^{16} + 3920.3247681442779141894275049 0\, x3^{15} \\
& + 4764.2408597071954745196351228 2\, x3^{14} + 5348.1847867412744535225437256 2\, x3^{13} \\
& + 5548.6802326596579445611560700 3\, x3^{12} + 5325.2420332294337030995144204 1\, x3^{11} \\
& + 4716.8344509395700740554762902 5\, x3^{10} + 3842.9620775810759808614683523 5\, x3^{9} \\
& + 2858.7347270196826674865007053 0\, x3^{8} + 1924.1214071724534400114407805 2\, x3^{7} \\
& + 1153.3334827654504845411171383 9\, x3^{6} + 601.21154691995243781906354725 0\, x3^{5} \\
& + 264.81557692584376833368786485 7\, x3^{4} + 92.560042805536974307338311537 0\, x3^{3} \\
& + 23.344156642465182831831739001 6\, x3^{2} + 3.0168069451708347861841845824 0\, x3 \\
& + 0.14320932664002415745944837614 7
\end{aligned}
$$

Solve for x3 using the companion matrix:

```
> Cx3:=CompanionMatrix(dx3n,x3):
> x3s:=Eigenvalues(Cx3);
```

$$
x3s := \left[\begin{array}{c}
0.5980076573585494573871089826 91 + 0.89595281213745926287662077631 2\, I \\
0.5980076573585494573871089826 98 - 0.89595281213745926287662077631 3\, I \\
0.5513847458847754765031979556 38 + 0.88237718037234357476912729894 6\, I \\
0.5513847458847754765031979556 45 - 0.88237718037234357476912729895 6\, I \\
0.171787357901776964561348089370\ 10^{-6} + 1.00000066974565024138461522654\, I \\
0.171787357901776964561349690779\ 10^{-6} - 1.00000066974565024138461522655\, I \\
0.669746189124345529417174976115\ 10^{-6} + 0.999999828213175069308745028768\, I \\
0.669746189124345529417174432383\ 10^{-6} - 0.999999828213175069308745028770\, I \\
-0.669746183212488685641044914353\ 10^{-6} + 1.00000017178789678612373784384\, I \\
-0.669746183212488685641044393685\ 10^{-6} - 1.00000017178789678612373784385\, I \\
-0.171787363813633806472606358344\ 10^{-6} + 0.999999330253277903182895563986\, I \\
-0.171787363813633806472606297645\ 10^{-6} - 0.999999330253277903182895563984\, I \\
-0.126849485579164462396458822449 + 0.\, I \\
-0.126329378446108174788185342458 + 0.\, I \\
-0.549440821071121208018465364230 + 0.591496699073449378916977642967\, I \\
-0.549440821071121208018465364229 - 0.591496699073449378916977642966\, I \\
-0.549377258056851732594875134312 + 0.591006449677356703855123330933\, I \\
-0.549377258056851732594875134313 - 0.591006449677356703855123330930\, I \\
-1.03301713303467793790204546764 + 0.0828934271118471608334465811558\, I \\
-1.03301713303467793790204546765 - 0.0828934271118471608334465811561\, I \\
-1.82949450681819693744654660459 + 0.0600864046767260336456400111122\, I \\
-1.82949450681819693744654660460 - 0.0600864046767260336456400111125\, I \\
-2.15855009557149911023027128511 + 0.\, I \\
-2.15818402406524636065874729650 + 0.\, I
\end{array}\right]
$$

Select the real solutions and convet them into the rational form.

```
> x3sR:=<select(proc(x) abs(Im(x))=0 end proc,M2L(x3s))>:
  x3s:=map(x->convert(Re(x),rational),x3sR);evalf(x3s);
```

$$
x3s := \left[\begin{array}{c}
\frac{-93062020015073}{733641288257292} \\
\frac{-103675375355179}{820675100522296} \\
\frac{-70758757005823}{32780687902955} \\
\frac{-976018206479974}{452240492746075}
\end{array}\right]
$$

$$\begin{bmatrix} -0.1268494855791644623964588 22450 \\ -0.12632937844610817478818534 2459 \\ -2.1585500955714991102302712 8510 \\ -2.1581840240652463606587472 9650 \end{bmatrix}$$

```
> map(x->subs(x3=x,evalf([(2*x3)/(1+x3^2),(1-x3^2)/(1+x3^2)])),x3s);
  evalf(subs(Position,[s3,c3]));
```

$$\begin{bmatrix} [-0.24968139969169370915298083 1236, 0.96832804289042290059336944 8699] \\ [-0.24868988716485478824635441 3198, 0.96858316112863111948912320 5768] \\ [-0.76282770144083271979058633 5390, -0.6466018078496964568155174 46505] \\ [-0.76291135697128178198722512 9082, -0.6465031023933585582346828 00357] \end{bmatrix}$$

$$[-0.24868988716485478824228374 6006, 0.96858316112863111949016837 5465]$$

Substitute back and solve for all angles

Solve for x5, x5 from Mx3

```
> ixs:=1;
```

$$ixs := 1$$

```
> Mx3s:=subs(x3=x3s[ixs],Mx3):
  `Mx3s = `, <Transpose(<myy>),spy(Mx3s)>;
```

$$Mx3s = , \begin{bmatrix} x4^3\,x5^2 & x5\,x4^3 & x4^3 & x4^2\,x5^2 & x5\,x4^2 & x4^2 & x4\,x5^2 & x4\,x5 & x4 & x5^2 & x5 & 1 \\ * & * & * & * & * & * & * & * & * & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & * & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & * & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & * & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & * & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & * & 0 & 0 & 0 \\ 0 & 0 & 0 & * & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & * & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & * & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & * & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & * & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & * & * & * & * & * & * & * & * & * \end{bmatrix}$$

Notice that the matrix has numerical dimension 11. The last singular value is much smaller than the last but one singular value.

```
> sv,V:= SingularValues(Mx3s, output=['S','Vt']):
  sv/sv[1];
```

$$\begin{bmatrix} 1.0000000000000000000000000000 \\ 0.99548352652774165338112029 0172 \\ 0.60728657700515247848240705 2209 \\ 0.60632154735143710501532436 8999 \\ 0.42255407463219324077501615 3098 \\ 0.40989119632178334196642357 7500 \\ 0.12030901726121002849534733 7441\ 10^{-5} \\ 0.11978498123163005658036137 2018\ 10^{-5} \\ 0.15300911129162891422481241 9837\ 10^{-6} \\ 0.11972941455232828885960473 3388\ 10^{-6} \\ 0.42244505184319114698715765 7637\ 10^{-7} \\ 0.20516908596455198113196675 3823\ 10^{-24} \end{bmatrix}$$

Therefore, the approximate solution of Mx3 X = 0 is the singular vector corresponding the the smallest singular value.

```
> myys := Transpose(V[12..12,1..12])/V[12,12]:
  Norm(Transpose(Mx3s.(myys/myys[12,1])));
```

$$0.132111243637079780\ 10^{-12}$$

```
> myys[12,1];
```

$$0.99999999999999999999999999 99$$

We see Mx3s maps myys very close to the zero vector.

By comparing

```
> <<myy>|<myys>>;
```

$$\begin{bmatrix} x4^3\,x5^2 & -0.24763698231995413359497203607\,9\,10^8 \\ x5\,x4^3 & -0.59815965903771950487142638124\,2\,10^8 \\ x4^3 & -0.14448366086041195915225806426\,7\,10^9 \\ x4^2\,x5^2 & 47192.8194324290076505834466265 \\ x5\,x4^2 & 113992.831427165111080697049113 \\ x4^2 & 275346.244896164928642903732631 \\ x4\,x5^2 & -89.9365751075127425810625963961 \\ x4\,x5 & -217.239083586592365667078024315 \\ x4 & -524.734451819127464432911394766 \\ x5^2 & 0.171394454488051929052280907877 \\ x5 & 0.413998133430478675856545082910 \\ 1 & 0.999999999999999999999999999999 \end{bmatrix}$$

we see that

```
> x4s:=myys[9,1]/myys[12,1];
  x5s:=myys[11,1]/myys[12,1];
```

$x4s := -524.734451819127464432911394767$

$x5s := 0.413998133430478675856545082910$

Compute sines and cosines from x3s, x3s and x5s:

```
> s3s := subs(x3=x3s[ixs],evalf((2*x3)/(1+x3^2)));
  c3s := subs(x3=x3s[ixs],evalf((1-x3^2)/(1+x3^2)));
  s4s := subs(x4=x4s,evalf((2*x4)/(1+x4^2)));
  c4s := subs(x4=x4s,evalf((1-x4^2)/(1+x4^2)));
  s5s := subs(x5=x5s,evalf((2*x5)/(1+x5^2)));
  c5s := subs(x5=x5s,evalf((1-x5^2)/(1+x5^2)));
```

$s3s := -0.249681399691693709152980831236$

$c3s := 0.968328042890422900593369448699$

$s4s := -0.00381143782252335762955430783214$

$c4s := -0.999992736444482899828095974664$

$s5s := 0.706846667825268542321093114904$

$c5s := 0.707366798898785235656403192594$

Check trigonometric identities:

```
> [c3s^2+s3s^2-1,c4s^2+s4s^2-1,c5s^2+s5s^2-1];
```

$[-0.8\ 10^{-29}, 0., -0.4\ 10^{-29}]$

Check the solutions

```
> [s3s,c3s]-evalf(subs(Position,[s3,c3]));
  [s4s,c4s]-evalf(subs(Position,[s4,c4]));
  [s5s,c5s]-evalf(subs(Position,[s5,c5]));
```

[-0.000991512526838920910697085230, -0.000255118238208218896798926766]

[0.137089794115059303535784812725, -1.99001639416104046707953169114]

[1.41395344901181606672193747701, 0.000260017712237711255558830489]

Store the solution in a form convenient for subst

```
> S345:={s3=s3s,c3=c3s,s4=s4s,c4=c4s,s5=s5s,c5=c5s}:
```

Substitute to the original equations

[A8] p = [S8 V^T] q

q = V inv(S8) A8 p

```
> qs:=Transpose(Vt).DiagonalMatrix(map(s->1/s,S[1..8])).subs(S345,A8.p):
> `<q|qs>=`,`       `,<q|qs>;
```

$$\langle q|qs\rangle =, \quad , \begin{bmatrix} s1\, s2 & 0.11101811222578828606534408820\!9 \\ s1\, c2 & 0.698058000436117533671343235303 \\ c1\, s2 & 0.111104741712675087501428458111 \\ c1\, c2 & 0.698602707438703416253676671768 \\ s1 & 0.706830951025433534814876548608 \\ c1 & 0.707382503793074127329197842419 \\ s2 & 0.157064588439997918448488385980 \\ c2 & 0.987588332760851465844772713835 \end{bmatrix}$$

By comparison, se wee that we get the solutions for s1, c1 and s2, c2 as

```
> s1s:=qs[5,1];
  c1s:=qs[6,1];
  s2s:=qs[7,1];
  c2s:=qs[8,1];
```

$$s1s := 0.706830951025433534814876548608$$

$$c1s := 0.707382503793074127329197842419$$

$$s2s := 0.157064588439997918448488385980$$

$$c2s := 0.987588332760851465844772713835$$

which are correct as shows

```
> [s1s,c1s]-evalf(subs(Position,[s1,c1]));
  [s2s,c2s]-evalf(subs(Position,[s2,c2]));
```

$$[-0.000275830161113989585967813497, 0.000275722606526602928353480314]$$

$$[0.000630123399767049438383066513, -0.000100007834286260345267533858]$$

Store the solutions

```
> S12:={s1=s1s,c1=c1s,s2=s2s,c2=c2s}:
```

Finally, let's get the solution to s6, c6. Let's returm to the original, matrix equation
and let's take the two left columns of the matrices:

```
> e61:=dhSimpl((<<1,0,0,0>|<0,-1,0,0>|<0,0,1,0>|<0,0,d2,1>>.dhInv(iM22).M31.M32.M4
  1.M42.M51.M52)[1..3,1..2],2):
  e62:=dhSimpl((<<1,0,0,0>|<0,-1,0,0>|<0,0,1,0>|<0,0,d2,1>>.iM21.iM12.iM11.Mh.iM62
  .iM61)[1..3,1..2],2):
```

Substitute the known solutions

```
> e61v:=subs(S12,subs(S345,evalf(subs(MPh,e61))));
  e62v:=subs(S12,subs(S345,evalf(subs(MPh,e62))));
```

$$e61v := \begin{bmatrix} -0.508386605780919387456065424534 & -0.0226061239021880614183240818139 \\ -0.860869969380662968124569627010 & 0.0378567698279086004139766922711 \\ 0.0211176438338292317319590480029 & 0.999027441134784680871132509606 \end{bmatrix}$$

$e62v :=$

$$[-0.257206945998259271163405364905\, s6 + 0.439104275244625085662217964516\, c6,$$
$$0.257206945998259271163405364905\, c6 + 0.439104275244625085662217964516\, s6]$$
$$[0.779076839793321179600158212797\, c6 - 0.368197660252756499385775835103\, s6,$$
$$0.779076839793321179600158212797\, s6 + 0.368197660252756499385775835103\, c6]$$
$$[0.447466996679469745494628105975\, c6 + 0.893462405422890633380972151302\, s6,$$
$$0.447466996679469745494628105975\, s6 - 0.893462405422890633380972151302\, c6]$$

and form the equations

```
> e6:=convert(map(p->sort(p),e62v-e61v),Vector);
```

$e6 :=$

$$[0.439104275244625085662217964516\, c6 - 0.257206945998259271163405364905\, s6$$
$$+ 0.508386605780919387456065424534]$$
$$[0.779076839793321179600158212797\, c6 - 0.368197660252756499385775835103\, s6$$
$$+ 0.860869969380662968124569627010]$$
$$[0.447466996679469745494628105975\, c6 + 0.893462405422890633380972151302\, s6$$
$$- 0.0211176438338292317319590480029]$$
$$[0.257206945998259271163405364905\, c6 + 0.439104275244625085662217964516\, s6$$
$$+ 0.0226061239021880614183240818139]$$
$$[0.368197660252756499385775835103\, c6 + 0.779076839793321179600158212797\, s6$$
$$- 0.0378567698279086004139766922711]$$

[−0.89346240542289063338097215130 2 c6 + 0.44746699667946974549462810597 5 s6

− 0.99902744113478468087113250960 6]

Convert them into the matrix form

```
> t6:=[c6,s6,1]:
  M6:=PolyCoeffMatrix(M2L(e6),t6):
  <Transpose(<t6>),M6>;
```

[c6 , s6 , 1]

[0.43910427524462508566221796451 6 , -0.25720694599825927116340536490 5 ,
0.50838660578091938745606542453 4]

[0.77907683979332117960015821279 7 , -0.36819766025275649938577583510 3 ,
0.86086996938066296812456962701 0]

[0.44746699667946974549462810597 5 , 0.89346240542289063338097215130 2 ,
-0.02111764383382923173195904802 9]

[0.25720694599825927116340536490 5 , 0.43910427524462508566221796451 6 ,
0.02260612390218806141832408181 39]

[0.36819766025275649938577583510 3 , 0.77907683979332117960015821279 7 ,
-0.03785676982790860041397669227 11]

[-0.89346240542289063338097215130 2 , 0.44746699667946974549462810597 5 ,
-0.99902744113478468087113250960 6]

and choose the first two to get the solution:

```
> cs6s:=MatrixInverse(-M6[1..2,1..2]).M6[1..2,3..3]:
  c6s:=cs6s[1,1];
  s6s:=cs6s[2,1];
```

c6s := -0.88446913664344970919923066668

s6s := 0.46659869970879539502749362701

and compare them with the ground truth

```
> [s6s,c6s]-evalf(subs(Position,[s6,c6]));
```

[0.80533661995408677624977798197 7, -1.82534990559767518152334908578]

Store the solution

```
> S6:={s6=cs6s[2,1],c6=cs6s[1,1]};
```

S6 := { c6 = -0.88446913664344970919923066668, s6 = 0.46659869970879539502749362701 }

```
>
```