Advanced Robotics

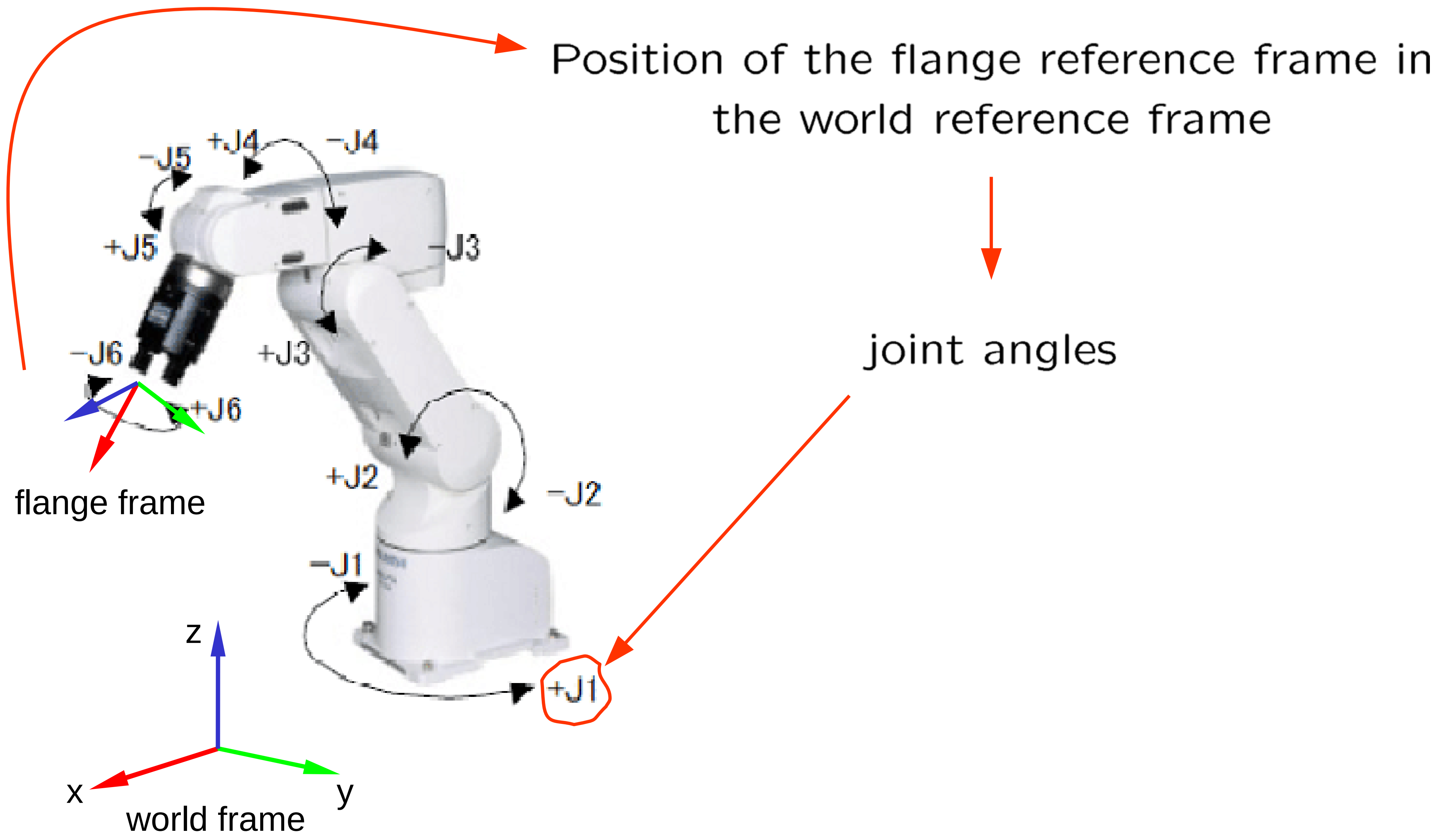
Lecture 5

pajdla@cmp.felk.cvut.cz

Inverse kinematic task

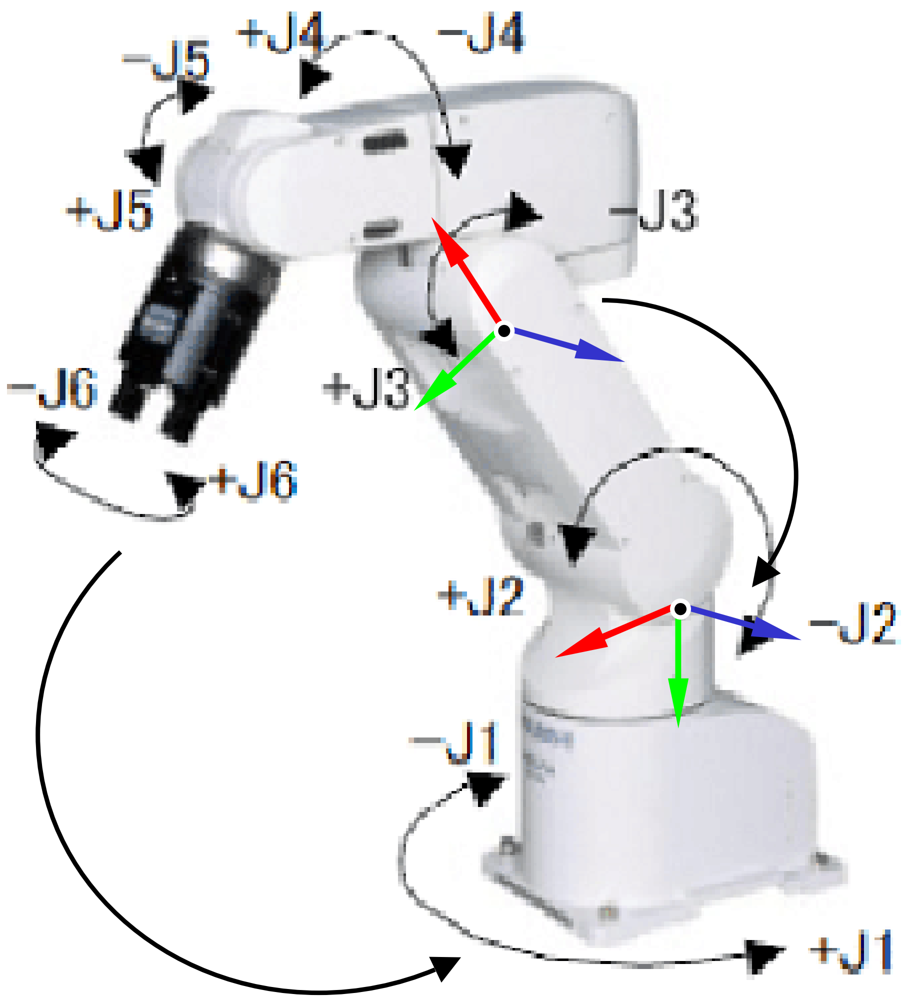
pajdla@cmp.felk.cvut.cz

Inverse kinematic task



pajdla@cmp.felk.cvut.cz

Two consecutive bodies are related by a transform



4 parameters

$$\alpha_i \mid a_i \mid \theta_i \mid d_i$$

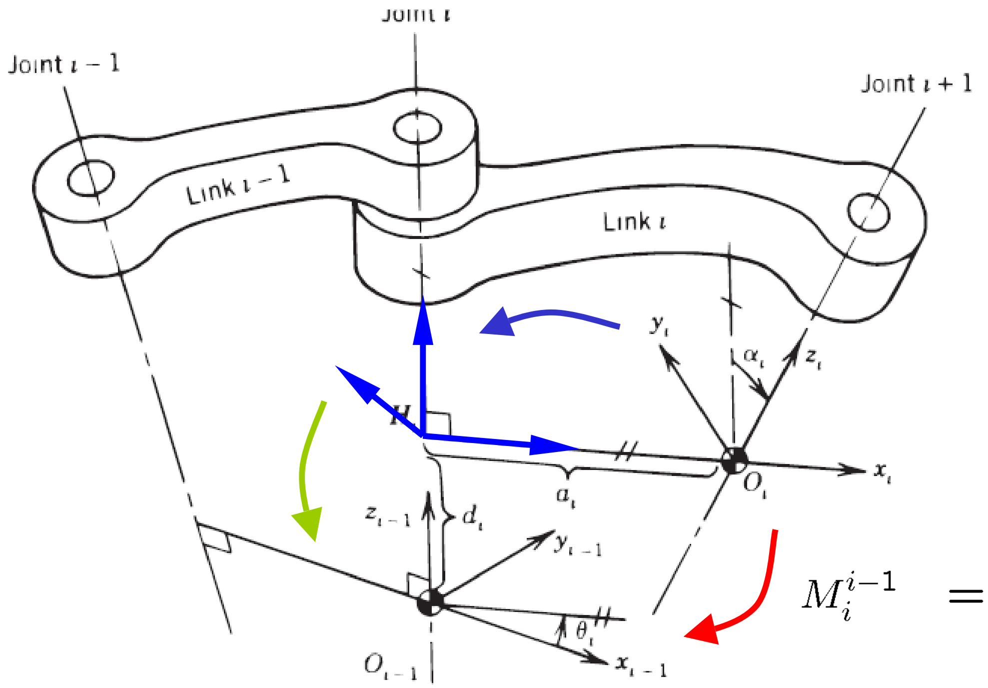
$$M_i^{i-1} = \begin{bmatrix} \cos\theta_i & -\sin\theta_i\cos\alpha_i & \sin\theta_i\sin\alpha_i & a_i\cos\theta_i \\ \sin\theta_i & \cos\theta_i\cos\alpha_i & -\cos\theta_i\sin\alpha_i & a_i\sin\theta_i \\ 0 & \sin\alpha_i & \cos\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$M = M_1^0 M_2^1 M_3^2 M_4^3 M_5^4 M_6^5$$

Serial manipulator with 6 motions

pajdla@cmp.felk.cvut.cz

Denavit-Hartenberg motion decomposition will be useful



$$M_i^{i-1} = \begin{bmatrix} \cos\theta_i & -\sin\theta_i\cos\alpha_i & \sin\theta_i\sin\alpha_i & a_i\cos\theta_i \\ \sin\theta_i & \cos\theta_i\cos\alpha_i & -\cos\theta_i\sin\alpha_i & a_i\sin\theta_i \\ 0 & \sin\alpha_i & \cos\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$M_i^{i-1} = M_{int}^{i-1}\, M_i^{int}$$

$$M_{int}^{i-1} = \begin{bmatrix} \cos\theta_i & -\sin\theta_i & 0 & 0 \\ \sin\theta_i & \cos\theta_i & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad M_i^{int} = \begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & \cos\alpha_i & -\sin\alpha_i & 0 \\ 0 & \sin\alpha_i & \cos\alpha_i & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Inverse kinematics – formulation 1

Given the position of the flange, i.e. the matrix M

and parameters of the mechanism, e.g. $\alpha_i,\ a_i,\ d_i$

compute the control variables $\theta_1,\ \theta_2,\ \theta_3,\ \theta_4,\ \theta_5,\ \theta_6$

pajdla@cmp.felk.cvut.cz

Matrix motion equation

$$\underset{\text{known}}{M} = \underbrace{M_1^0 M_2^1 M_3^2 M_4^3 M_5^4 M_6^5}_{\text{function of}}$$

known

function of

$\alpha_i,\ a_i,\ d_i$

and

$\theta_1,\ \theta_2,\ \theta_3,\ \theta_4,\ \theta_5,\ \theta_6$

pajdla@cmp.felk.cvut.cz

Change of variables – from trigonometry to algebra

$$M_{int}^{i-1} = \begin{bmatrix} \cos\theta_i & -\sin\theta_i & 0 & 0 \\ \sin\theta_i & \cos\theta_i & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad M_i^{int} = \begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & \cos\alpha_i & -\sin\alpha_i & 0 \\ 0 & \sin\alpha_i & \cos\alpha_i & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\cos\theta_i \longrightarrow c_i$$

$$\sin\theta_i \longrightarrow s_i$$

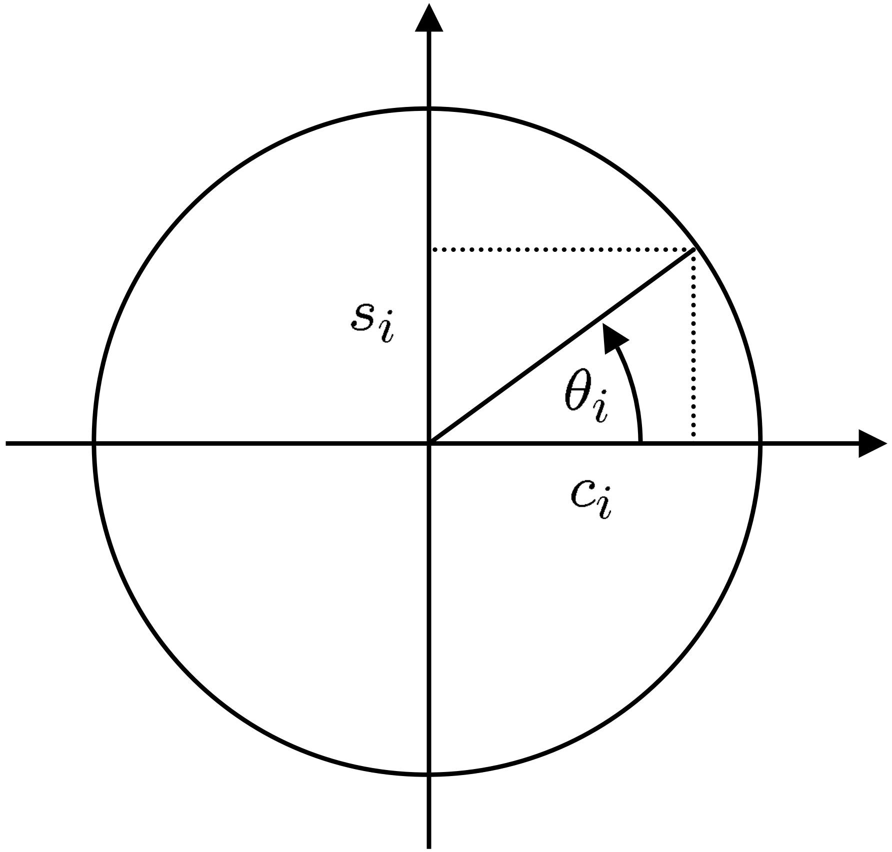
$$\cos\alpha_i \longrightarrow p_i$$

$$\sin\alpha_i \longrightarrow q_i$$

$$M_{int}^{i-1} = \begin{bmatrix} c_i & -s_i & 0 & 0 \\ s_i & c_i & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad M_i^{int} = \begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & p_i & -q_i & 0 \\ 0 & q_i & p_i & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Algebraic identity

1 unknown θ_i $\longrightarrow$ 2 unknowns $c_i,\ s_i$ + 1 algebraic identity



$$c_i^2 + s_i^2 = 1$$

Change of variables – from trigonometry to algebra

$$M_{int}^{i-1} = \begin{bmatrix} \cos\theta_i & -\sin\theta_i & 0 & 0 \\ \sin\theta_i & \cos\theta_i & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad M_i^{int} = \begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & \cos\alpha_i & -\sin\alpha_i & 0 \\ 0 & \sin\alpha_i & \cos\alpha_i & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\cos\theta_i \longrightarrow c_i$$

$$\sin\theta_i \longrightarrow s_i$$

$$\cos\alpha_i \longrightarrow p_i$$

$$\sin\alpha_i \longrightarrow q_i$$

$$M_{int}^{i-1} = \begin{bmatrix} c_i & -s_i & 0 & 0 \\ s_i & c_i & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad M_i^{int} = \begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & p_i & -q_i & 0 \\ 0 & q_i & p_i & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$c_i^2 + s_i^2 = 1 \qquad p_i^2 + q_i^2 = 1$$

pajdla@cmp.felk.cvut.cz

Given the position of the arm, i.e. the matrix M

and parameters of the mechanisn, e.g. $\alpha_i,\ a_i,\ d_i$

compute the control variables

$$s_1,\ c_1\ ;\ s_2,\ c_2\ ;\ s_3,\ c_3\ ;\ s_4,\ c_4\ ;\ s_5,\ c_5\ ;\ s_6,\ c_6$$

subject to the constraint

$$M = M_1^0(c_1, s_1)\, M_2^1(c_2, s_2)\, M_3^2(c_3, s_3)\, M_4^3(c_4, s_4)\, M_5^4(c_5, s_5)\, M_6^5(c_6, s_6)$$

and

$$\begin{array}{ll} c_1^2 + s_1^2 = 1 & c_4^2 + s_4^2 = 1 \\ c_2^2 + s_2^2 = 1 & c_5^2 + s_5^2 = 1 \\ c_3^2 + s_3^2 = 1 & c_6^2 + s_6^2 = 1 \end{array}$$

Counting unknowns and equations

12 unknowns

$s_1,\ c_1\ ;\ s_2,\ c_2\ ;\ s_3,\ c_3\ ;\ s_4,\ c_4\ ;\ s_5,\ c_5\ ;\ s_6,\ c_6$

12 equations (3 x 4 matrix) but only 6 independent (M constains rotation)

$$M = M_1^0(c_1, s_1)\, M_2^1(c_2, s_2)\, M_3^2(c_3, s_3)\, M_4^3(c_4, s_4)\, M_5^4(c_5, s_5)\, M_6^5(c_6, s_6)$$

6 equations

$$c_1^2 + s_1^2 = 1 \qquad c_4^2 + s_4^2 = 1$$
$$c_2^2 + s_2^2 = 1 \qquad c_5^2 + s_5^2 = 1$$
$$c_3^2 + s_3^2 = 1 \qquad c_6^2 + s_6^2 = 1$$

There is 12 unknowns and 12 equations $\longrightarrow$ can be solved

Decomposition to elementary motions

$$M = M_1^0 M_2^1 M_3^2 M_4^3 M_5^4 M_6^5$$

$$M = \overbrace{M_{int}^0 \, M_1^{int}} \, M_{int}^1 \, M_2^{int} \, M_{int}^2 \, M_3^{int} \, M_{int}^3 \, M_4^{int} \, M_{int}^4 \, M_5^{int} \, M_{int}^5 \, M_6^{int}$$

and rename matrices to make it shorter

$$M_i^{i-1} \longrightarrow M_i$$

$$M = M_1^0 M_2^1 M_3^2 M_4^3 M_5^4 M_6^5 \longrightarrow M = M_1 M_2 M_3 M_4 M_5 M_6$$

$$M_{int}^{i-1}\, M_i^{int} \longrightarrow M_{i1}\, M_{i2}$$

$$M = M_{int}^0\, M_1^{int}\, M_{int}^1\, M_2^{int}\, M_{int}^2\, M_3^{int}\, M_{int}^3\, M_4^{int}\, M_{int}^4\, M_5^{int}\, M_{int}^5\, M_6^{int}$$

$$\downarrow$$

$$M = M_{11}\, M_{12}\, M_{21}\, M_{22}\, M_{31}\, M_{32}\, M_{41}\, M_{42}\, M_{51}\, M_{52}\, M_{61}\, M_{62}$$

Inversion of D-H motion matrix preserves "linearity"

$$M_i^{i-1} = \begin{bmatrix} \cos\theta_i & -\sin\theta_i\cos\alpha_i & \sin\theta_i\sin\alpha_i & a_i\cos\theta_i \\ \sin\theta_i & \cos\theta_i\cos\alpha_i & -\cos\theta_i\sin\alpha_i & a_i\sin\theta_i \\ 0 & \sin\alpha_i & \cos\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} c_i & -s_i\,p_i & s_i\,q_i & a_i\,c_i \\ s_i & c_i\,p_i & -c_i\,q_i & a_i\,s_i \\ 0 & q_i & p_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{linear in } c_i,\ s_i}$$

$$= \begin{bmatrix} \cos\theta_i & -\sin\theta_i & 0 & 0 \\ \sin\theta_i & \cos\theta_i & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & \cos\alpha_i & -\sin\alpha_i & 0 \\ 0 & \sin\alpha_i & \cos\alpha_i & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Inversion of D-H motion matrix preserves “linearity”

$$\mathrm{inv}(M_i^{i-1}) = \mathrm{inv}\left(\begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & \cos\alpha_i & -\sin\alpha_i & 0 \\ 0 & \sin\alpha_i & \cos\alpha_i & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right) \mathrm{inv}\left(\begin{bmatrix} \cos\theta_i & -\sin\theta_i & 0 & 0 \\ \sin\theta_i & \cos\theta_i & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 1 & 0 & 0 & -a_i \\ 0 & \cos\alpha_i & \sin\alpha_i & 0 \\ 0 & -\sin\alpha_i & \cos\alpha_i & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_i & \sin\theta_i & 0 & 0 \\ -\sin\theta_i & \cos\theta_i & 0 & 0 \\ 0 & 0 & 1 & -d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta_i & \sin\theta_i & 0 & -a_i \\ -\sin\theta_i\cos\alpha_i & \cos\theta_i\cos\alpha_i & \sin\alpha_i & -d_i\sin\alpha_i \\ \sin\theta_i\sin\alpha_i & -\cos\theta_i\sin\alpha_i & \cos\alpha_i & -d_i\cos\alpha_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

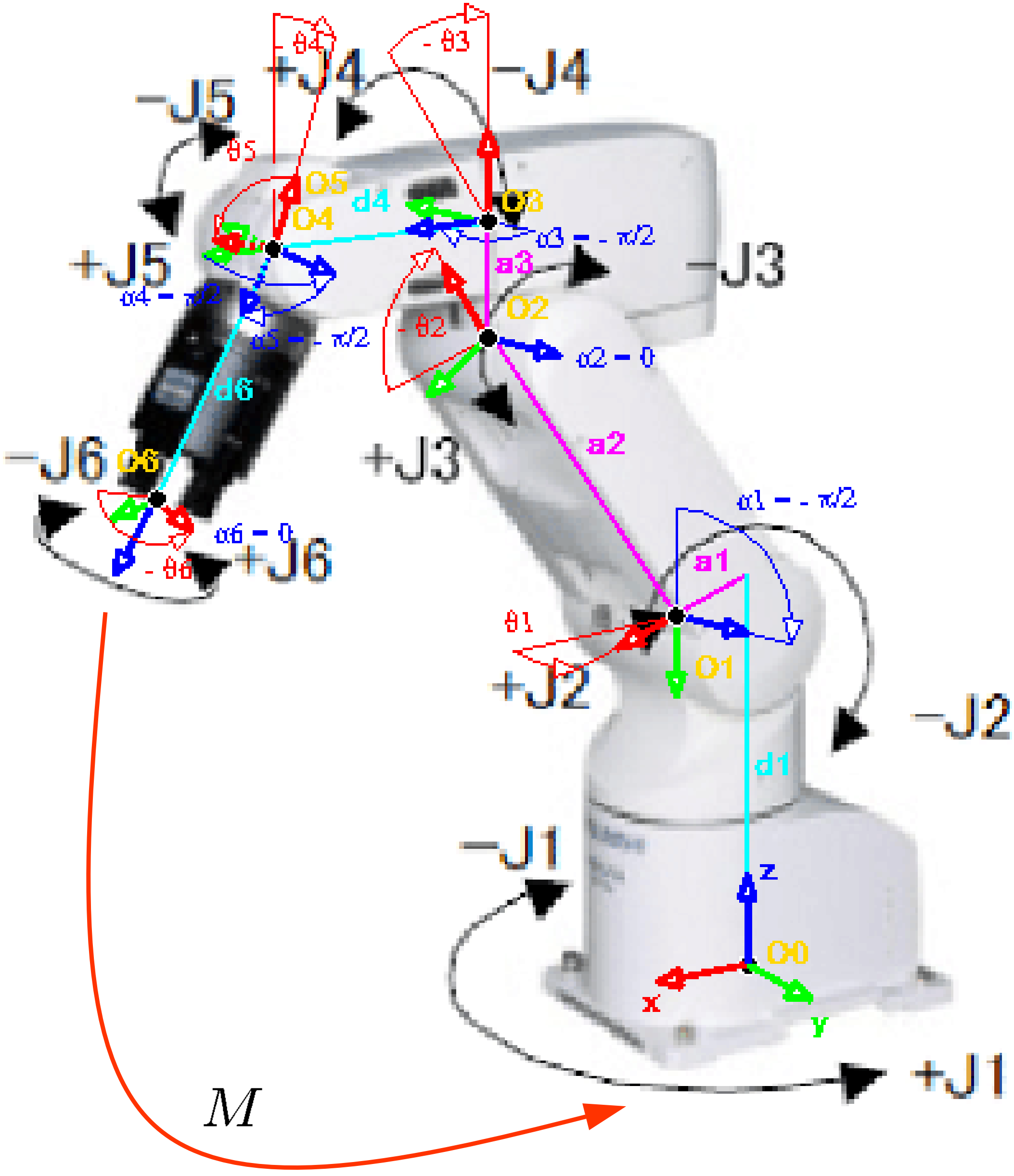
$$= \underbrace{\begin{bmatrix} c_i & s_i & 0 & -a_i \\ -s_i\,p_i & c_i\,p_i & q_i & -d_i\,q_i \\ s_i\,q_i & -c_i\,q_i & p_i & -d_i\,p_i \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{linear in } c_i,\ s_i}$$

Separate unknowns as much as possible

$$M = \overbrace{M_1(c_1, s_1)\, M_2(c_2, s_2)\, M_3(c_3, s_3)\, M_4(c_4, s_4)\, M_5(c_5, s_5)\, M_6(c_6, s_6)}^{\text{products of 6 unknowns}}$$

$$\underbrace{\underbrace{M_2^{-1}(c_2, s_2)\, M_1^{-1}(c_1, s_1)\, M\, M_6^{-1}(c_6, s_6)}_{\text{product of 3 unknowns}} = \underbrace{M_3(c_3, s_3)\, M_4(c_4, s_4)\, M_5(c_5, s_5)}_{\text{product of 3 unknowns}}}_{\text{algebraic equations of degree 3}}$$

3-2-1 mechanism – the most common & simpler



$$M_1 = \begin{bmatrix} \cos\theta_1 & 0 & -\sin\theta_1 & a_1 cos\theta_1 \\ \sin\theta_1 & 0 & \cos\theta_1 & a_1 sin\theta_1 \\ 0 & -1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$M_2 = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 & a_2\cos\theta_2 \\ \sin\theta_2 & \cos\theta_2 & 0 & a_2\sin\theta_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$M_3 = \begin{bmatrix} \cos\theta_3 & 0 & -\sin\theta_3 & a_3\cos\theta_3 \\ \sin\theta_3 & 0 & \cos\theta_3 & a_3\sin\theta_3 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$M_4 = \begin{bmatrix} \cos\theta_4 & 0 & \sin\theta_4 & 0 \\ \sin\theta_4 & 0 & -\cos\theta_4 & 0 \\ 0 & 1 & 0 & d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$M_5 = \begin{bmatrix} \cos\theta_5 & 0 & -\sin\theta_5 & 0 \\ \sin\theta_5 & 0 & \cos\theta_5 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$M_6 = \begin{bmatrix} \cos\theta_6 & -\sin\theta_6 & 0 & 0 \\ \sin\theta_6 & \cos\theta_6 & 0 & 0 \\ 0 & 0 & 1 & d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$M = M_1 M_2 M_3 M_4 M_5 M_6$$

pajdla@cmp.felk.cvut.cz

Inverse Kinematics — General Mechanism with 6 Rotations

pajdla@cmp.felk.cvut.cz

Click to open PDF presentation

T.Pajdla: Inverse Kinematics of a 6-DOF Manipulator.
[1] D.Manocha, J.F.Canny. Efficient Inverse Kinematics for General 6R Manipulators. IEEE Trans. on Robotics and Automation, 10(5), pp. 648-657, Oct. 2004

General Mechanism - Explanation
13 Nov 2006

Packages & settings

DH-Kinematics

6-DOF Robot IK formulation

Given ai, di, i = 1...6, and Mh, find parameters ci, si, pi, ri subject to

(1) M1 * M2 * M3 * M4 * M5 * M6 = Mh

(2) (M11*M12)*(M21*M22)*(M31*M32)*(M41*M42)*(M51*M52)*(M61*M62) = Mh

(3) ci^2 + si^2 = 1 i = 1...6

(4) pi^2 + ri^2 = 1 i = 1...6

[2] M. Raghavan, B. Roth. Kinematic Analysis of the 6R Manipulator of General Geometry.
Int. Symposium on Robotic Research. pp. 264-269, Tokyo 1990.

Write (1) equivalently as

(5) M3 * M4 * M5 = M2^{-1} * M1^{-1} * Mh * M6^{-1}

(6) M31*M32*M41*M42*M51*M52

= M22^{-1}*M21^{-1}*M12^{-1}*M11^{-1}* Ah * M62^{-1}*M61^{-1}

Formulation & Solution

The manipulator matrices

```
> M31 :=dhTs(3)[1]:
  M32 :=dhTs(3)[2]:
  M41 :=dhTs(4)[1]:
  M42 :=dhTs(4)[2]:
  M51 :=dhTs(5)[1]:
  M52 :=dhTs(5)[2]:
```


pajdla@cmp.felk.cvut.cz