Robotics

Description of Rigid Body Position

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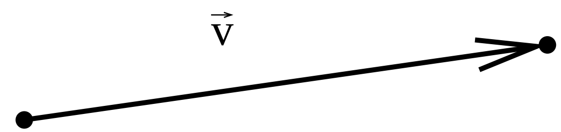
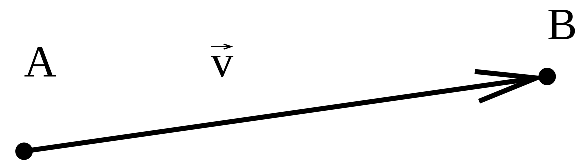
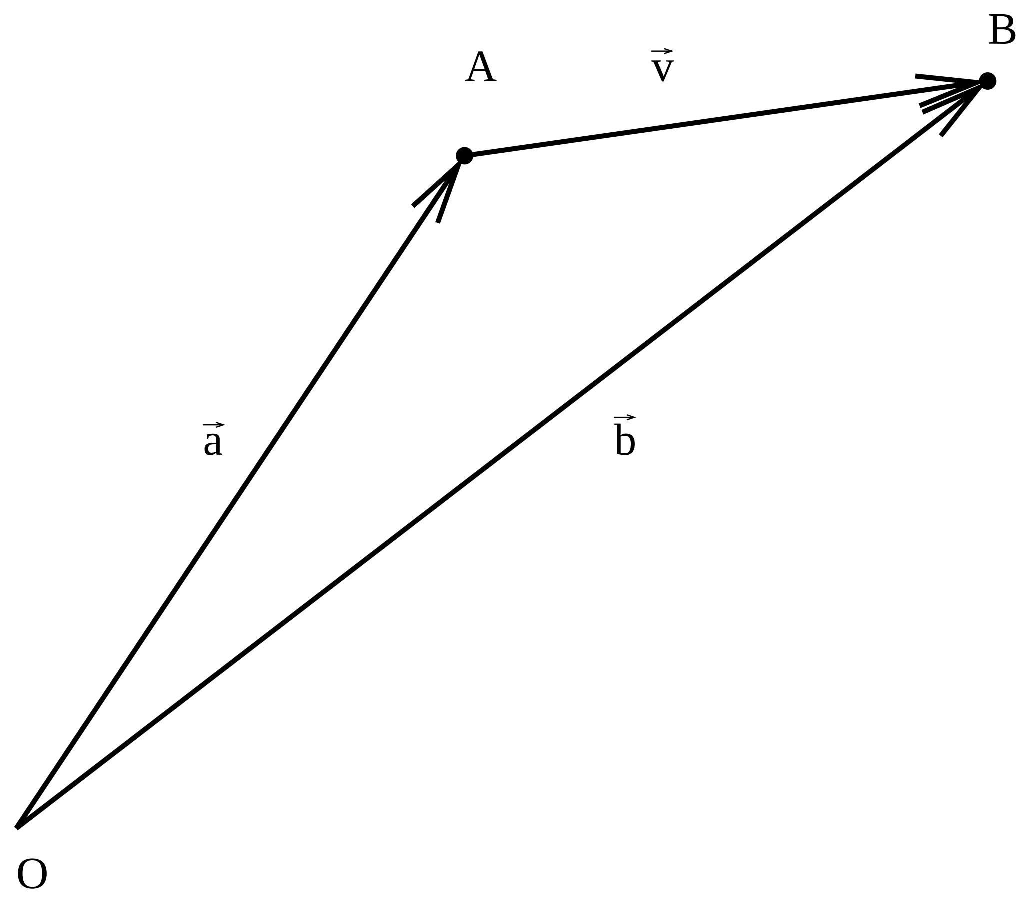
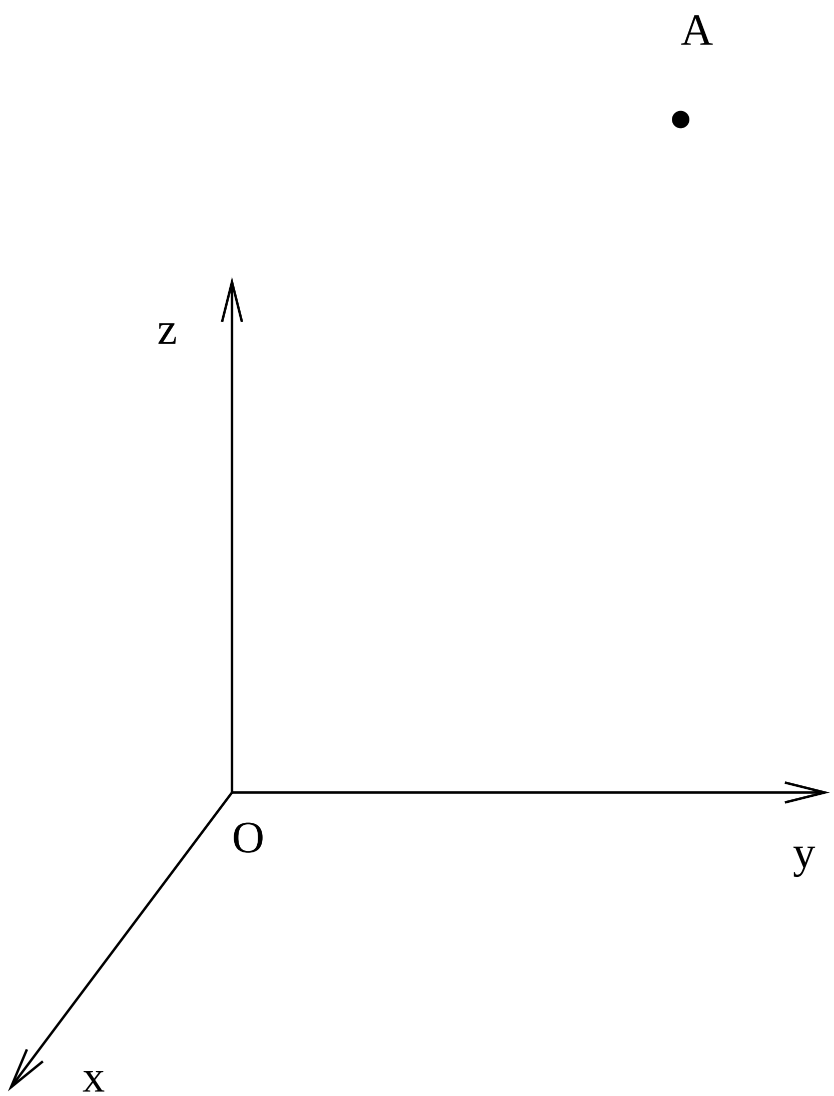
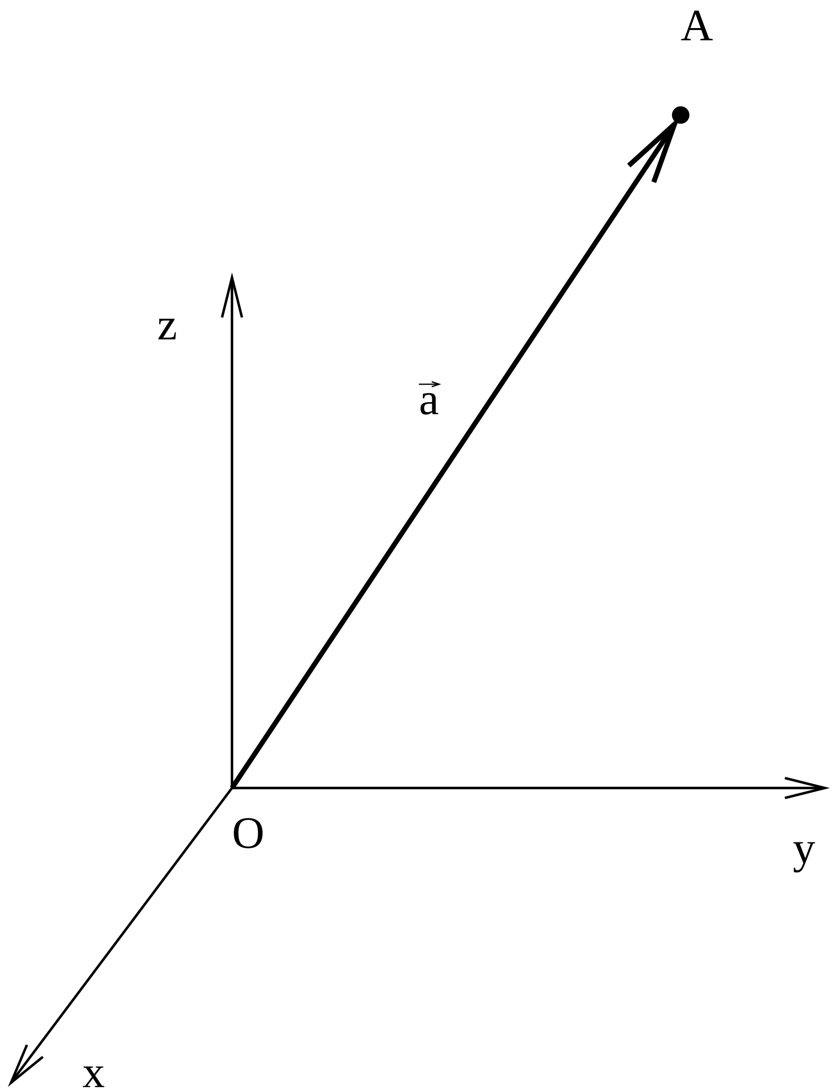
A

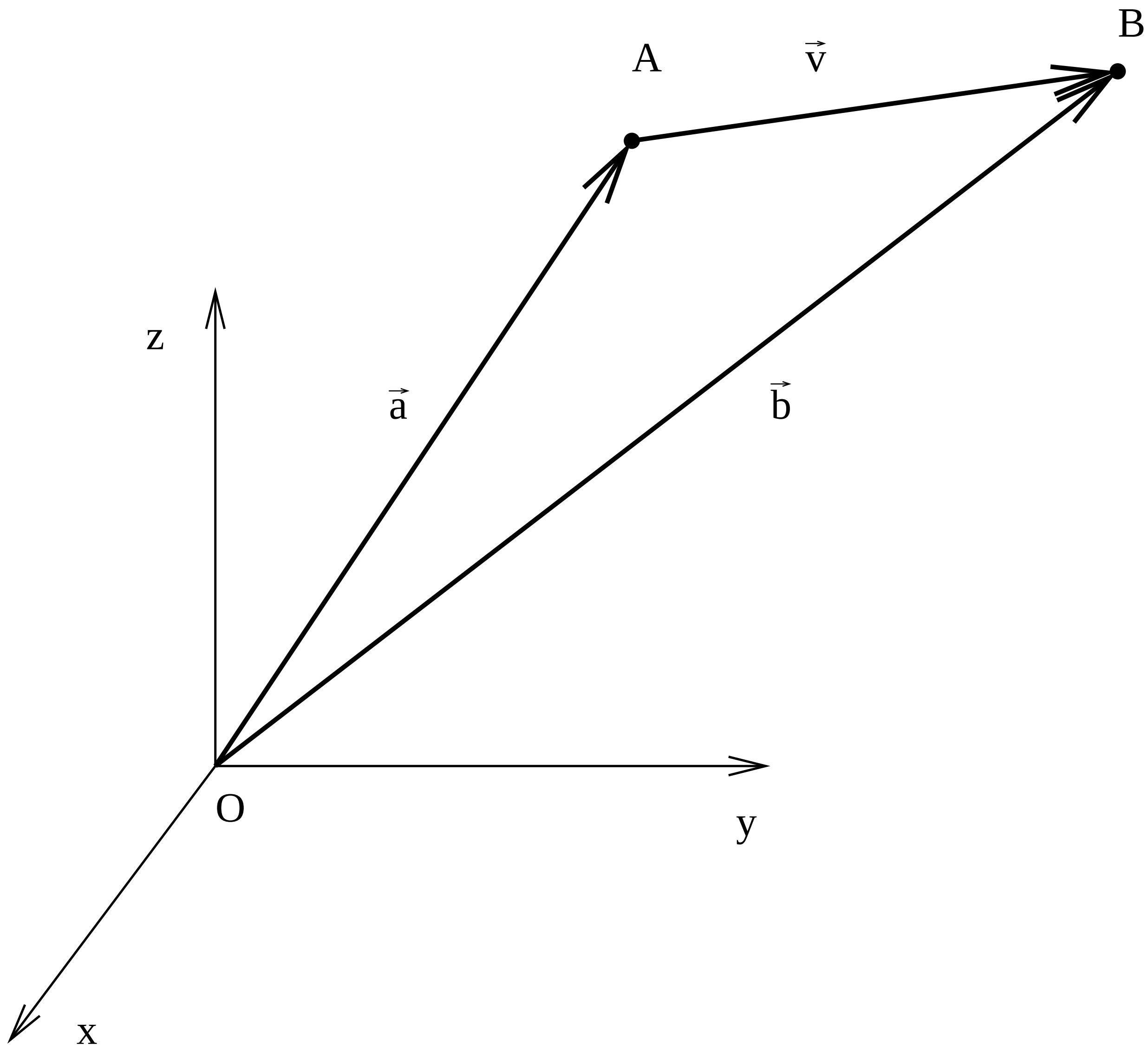

$\vec{v}$

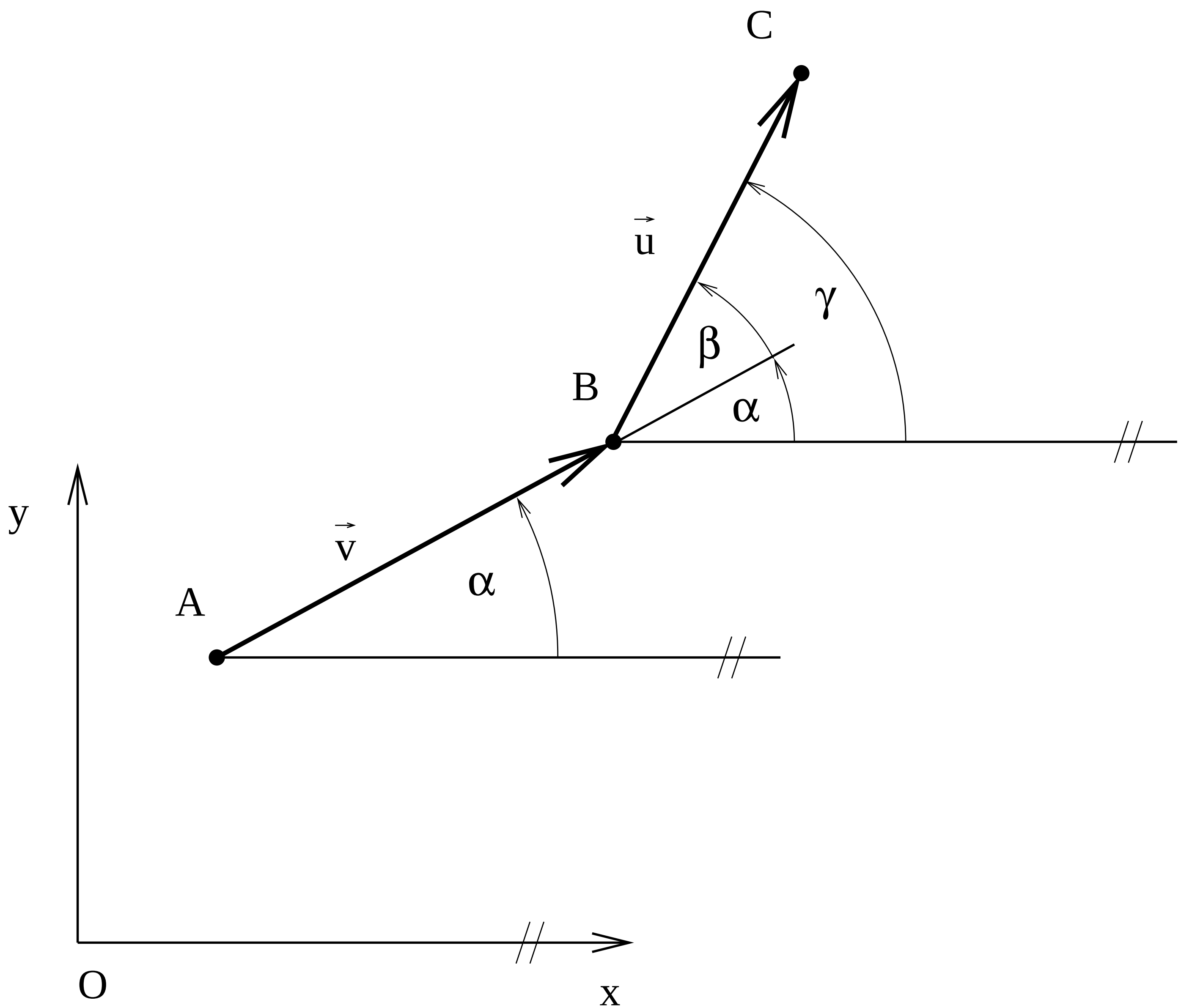

A
$\vec{v}$
B


A
$\vec{v}$
B
$\vec{a}$
$\vec{b}$
O


A
z
O
y
x

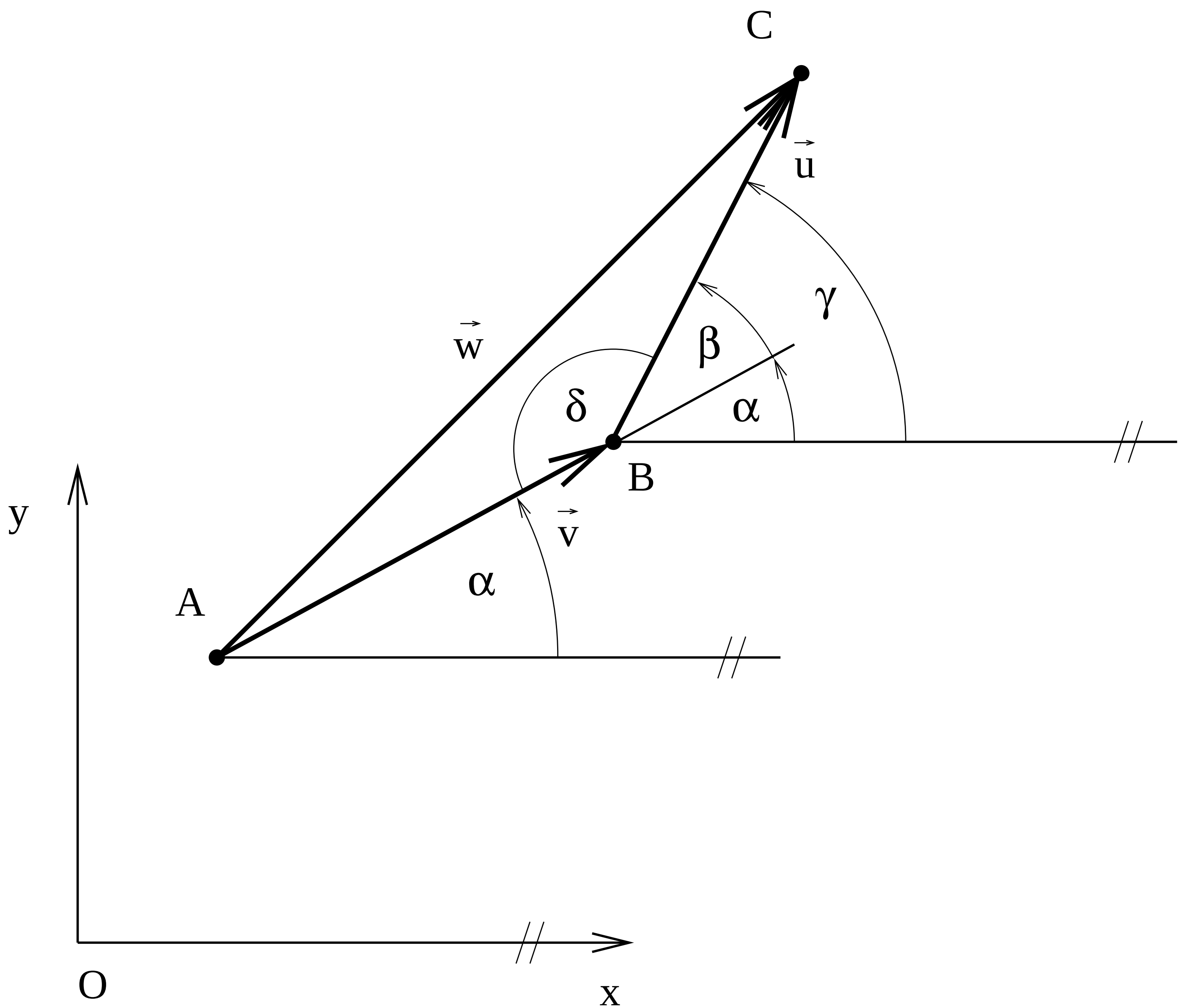

A
z
$\vec{a}$
O
y
x

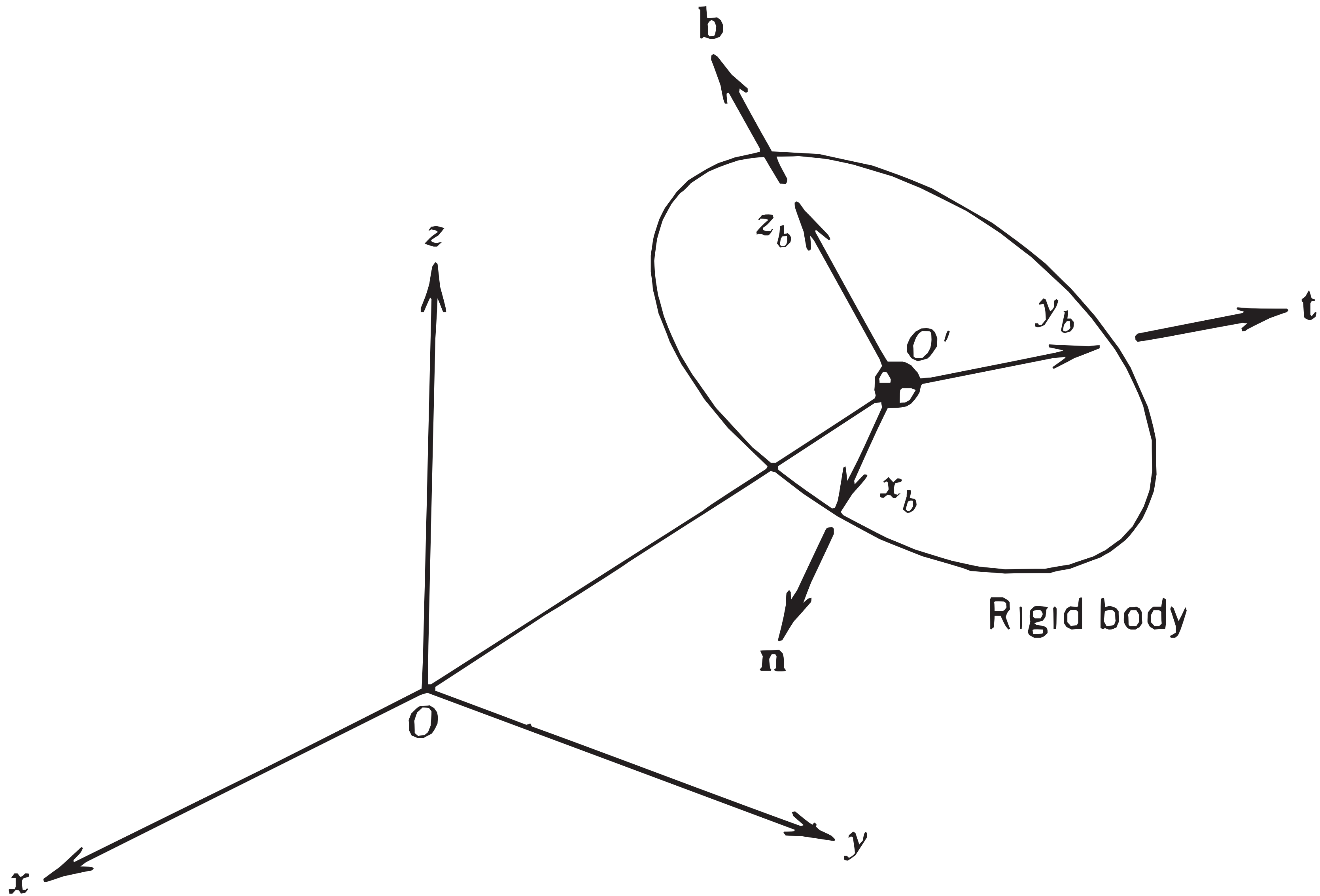

B
A
$\vec{v}$
z
$\vec{a}$
$\vec{b}$
O
y
x



1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31

Technical intermezzo II




b
z_b
z
O'
y_b
t
x_b
n
Rigid body
O
x
y

Description of Body Position

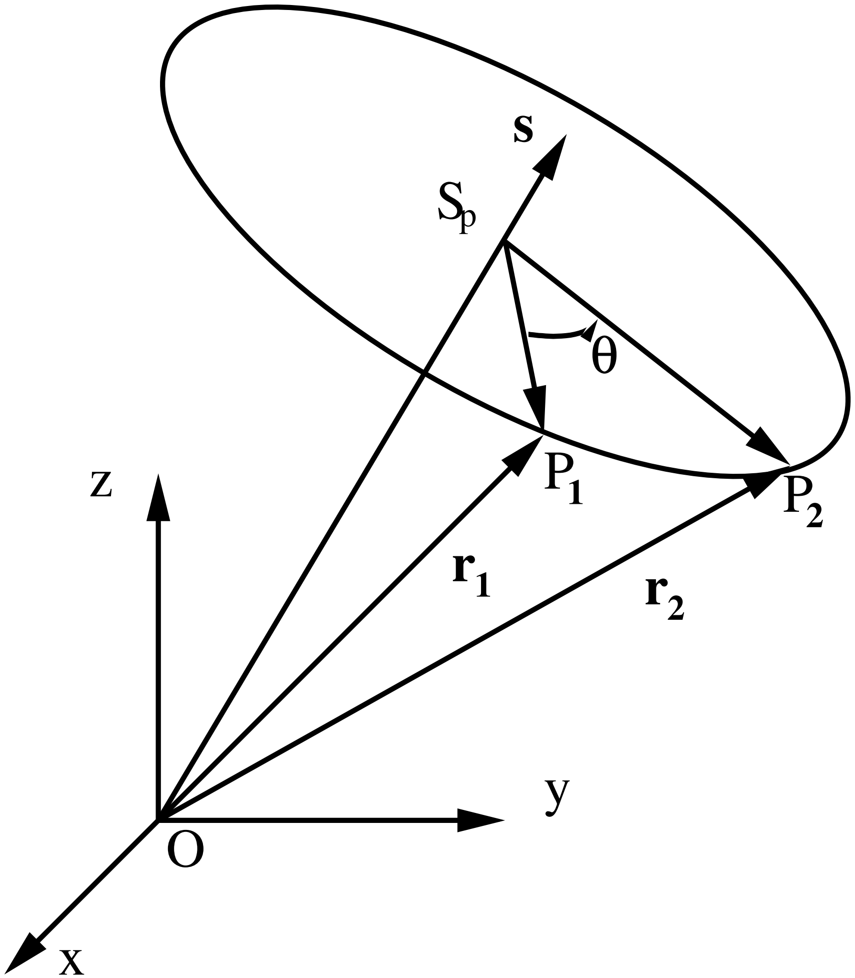
Point in 3D - described by three coordinates.

Rigid body in 3D - described by 6 coordinates:

- 3 coordinates of reference point $\mathbf{t}_0^0 = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$,
- orientation could be described e.g. by:
  - coordinates of vectors attached to the body $(\mathbf{n}, \mathbf{t}, \mathbf{b})$,
  - Euler angles (ϕ, θ, ψ),
  - rotational matrix $\mathbf{R}$,
  - axis – angle,
  - quaternions,
  - rotation vector.

Coordinates of reference point and rotation matrix could be combined into transformation matrix.

Axis–Angle, Quaternions, Rotation Vector



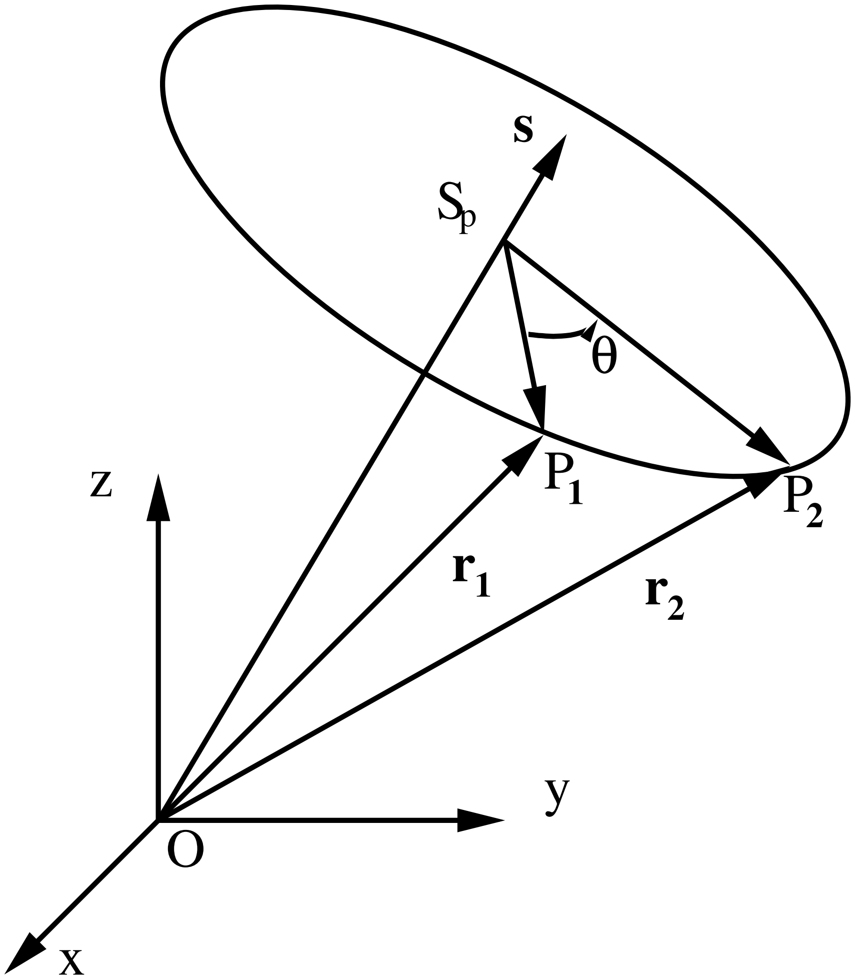
Euler's rotation theorem states than any rotation in 3–D could be represented as a single rotation around a certain axis. We can describe this axis as $\mathbf{s}$ and angle of the rotation as θ. This pair $(\mathbf{s}, \theta)$ could represent rotation and is called axis–angle. Quaternions can be expressed as:

$$\mathbf{q} = (\cos(\theta/2), \sin(\theta/2)\mathbf{s}^T) =$$

$$(\cos(\theta/2), \sin(\theta/2)s_x, \sin(\theta/2)s_y, \sin(\theta/2)s_z)$$

Rotation vector uses the fact, that vector $\mathbf{s}$ is normalized and has only 2 DOF, so one can express rotation using three numbers $\mathbf{v} = (\theta\mathbf{s})$.

Rodrigues Rotation Formula



Rodrigues' rotation formula:

$$\mathbf{r}_2 = \mathbf{r}_1 \cos\theta + (\mathbf{s} \times \mathbf{r}_1)\sin\theta + \mathbf{s}(\mathbf{s}\cdot\mathbf{r}_1)(1-\cos\theta)$$

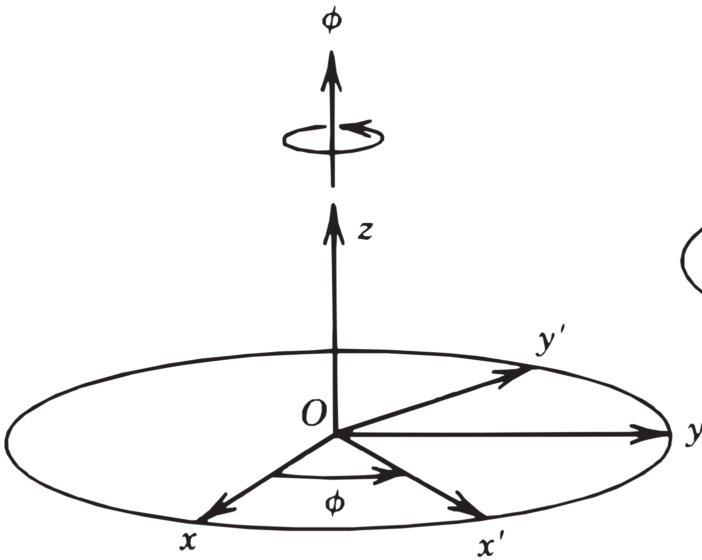
The rotation matrix can be calculated from axis angle representation as:

$$R = I\cos\theta + [\mathbf{s}]_x \sin\theta + \mathbf{s}\mathbf{s}^T(1-\cos\theta)$$

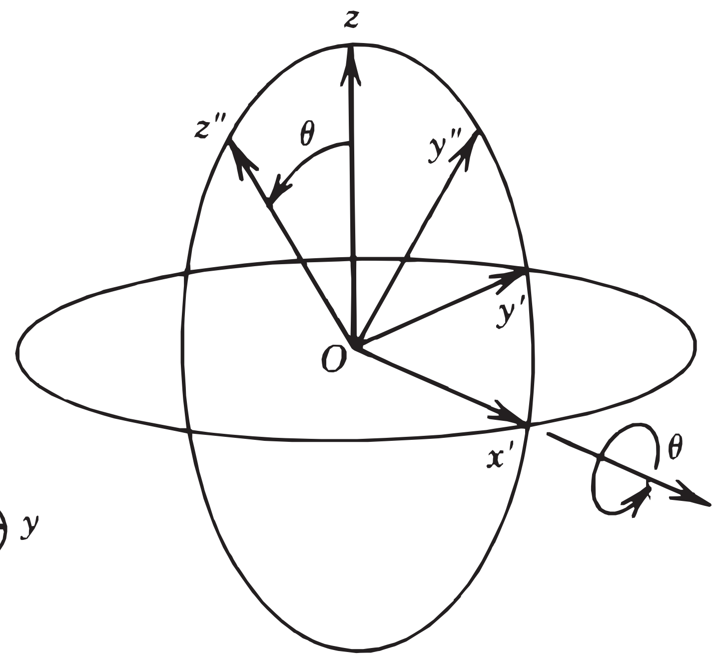
where $[\mathbf{s}]_x$ stands for skew symmetric or antisymetric matrix:

$$[\mathbf{s}]_x = \begin{pmatrix} 0 & -s_z & s_y \\ s_z & 0 & -s_x \\ -s_y & s_x & 0 \end{pmatrix}$$

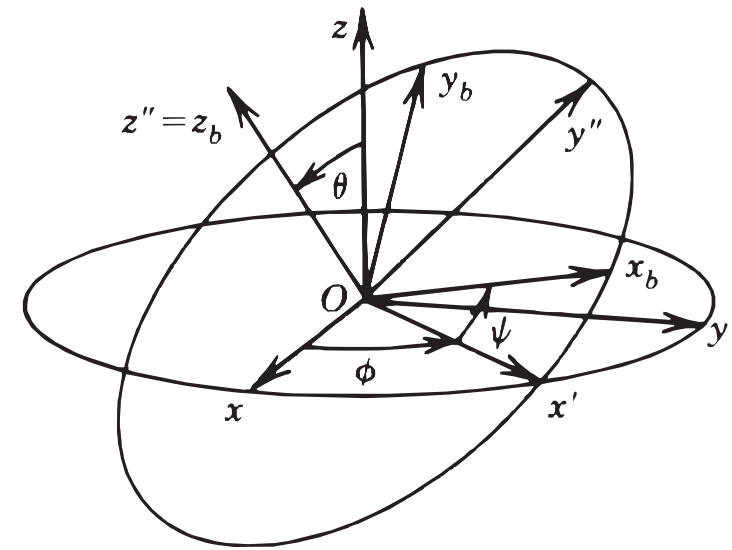
Definition of Euler angles



1 – precession



2 – nutation



3 – rotation

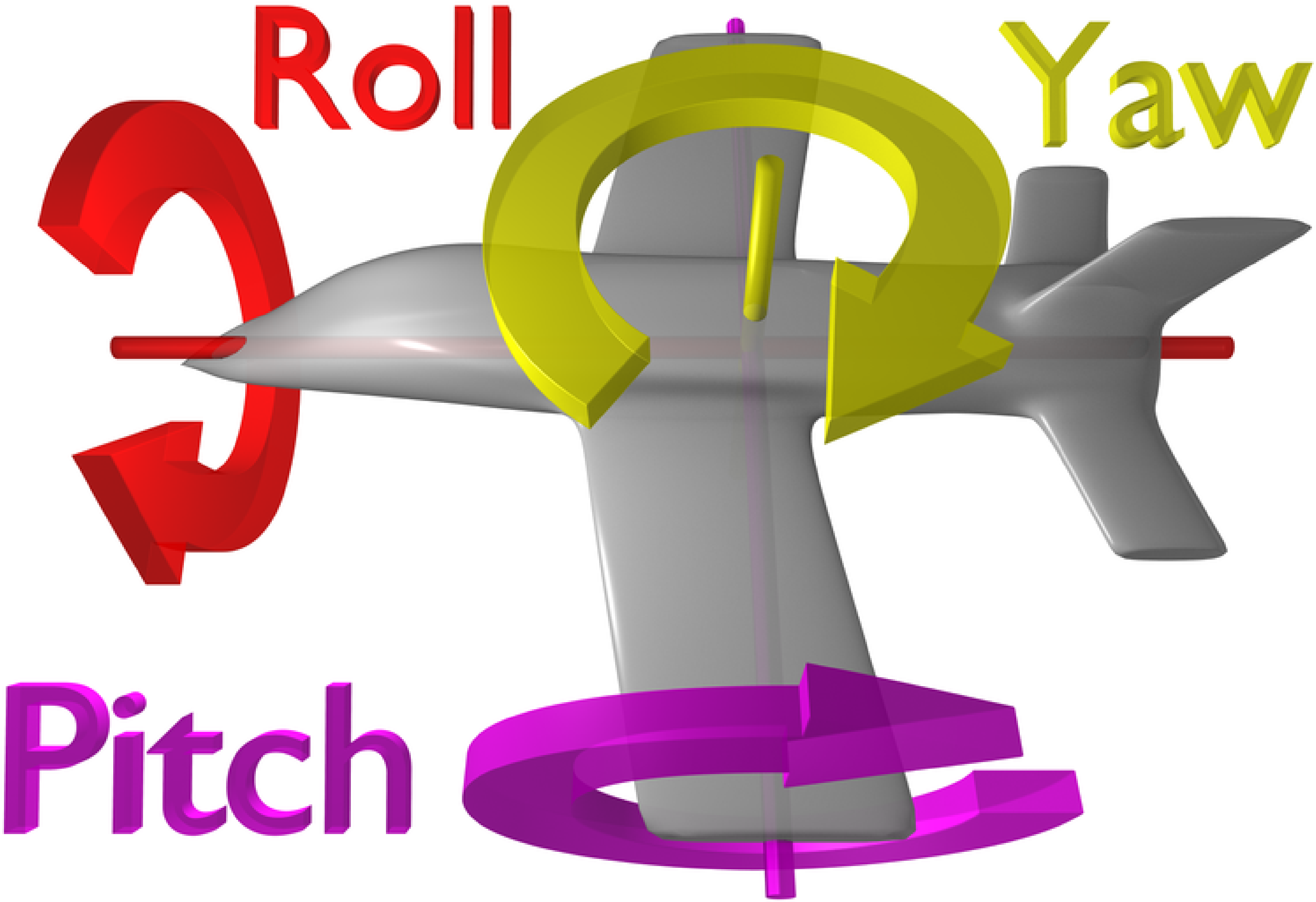
Rotation Matrix Resulting from Euler Angles

Euler angles according definition used here (Asada, Slotine):

$$\begin{pmatrix} \cos\varphi\cos\psi - \cos\vartheta\sin\varphi\sin\psi & -\cos\vartheta\cos\psi\sin\varphi - \cos\varphi\sin\psi & \sin\vartheta\sin\varphi \\ \cos\psi\sin\varphi + \cos\vartheta\cos\varphi\sin\psi & \cos\vartheta\cos\varphi\cos\psi - \sin\varphi\sin\psi & -\cos\varphi\sin\vartheta \\ \sin\vartheta\sin\psi & \cos\psi\sin\vartheta & \cos\vartheta \end{pmatrix}$$

Rotation matrix based on Yaw, Pitch, Roll used in CRS robot, that is rotation around z, then y, then x:

$$\begin{pmatrix} \cos\alpha\cos\beta & \cos\alpha\sin\beta\sin\gamma - \cos\gamma\sin\alpha & \cos\alpha\cos\gamma\sin\beta + \sin\alpha\sin\gamma \\ \cos\beta\sin\alpha & \cos\alpha\cos\gamma + \sin\alpha\sin\beta\sin\gamma & \cos\gamma\sin\alpha\sin\beta - \cos\alpha\sin\gamma \\ -\sin\beta & \cos\beta\sin\gamma & \cos\beta\cos\gamma \end{pmatrix}$$

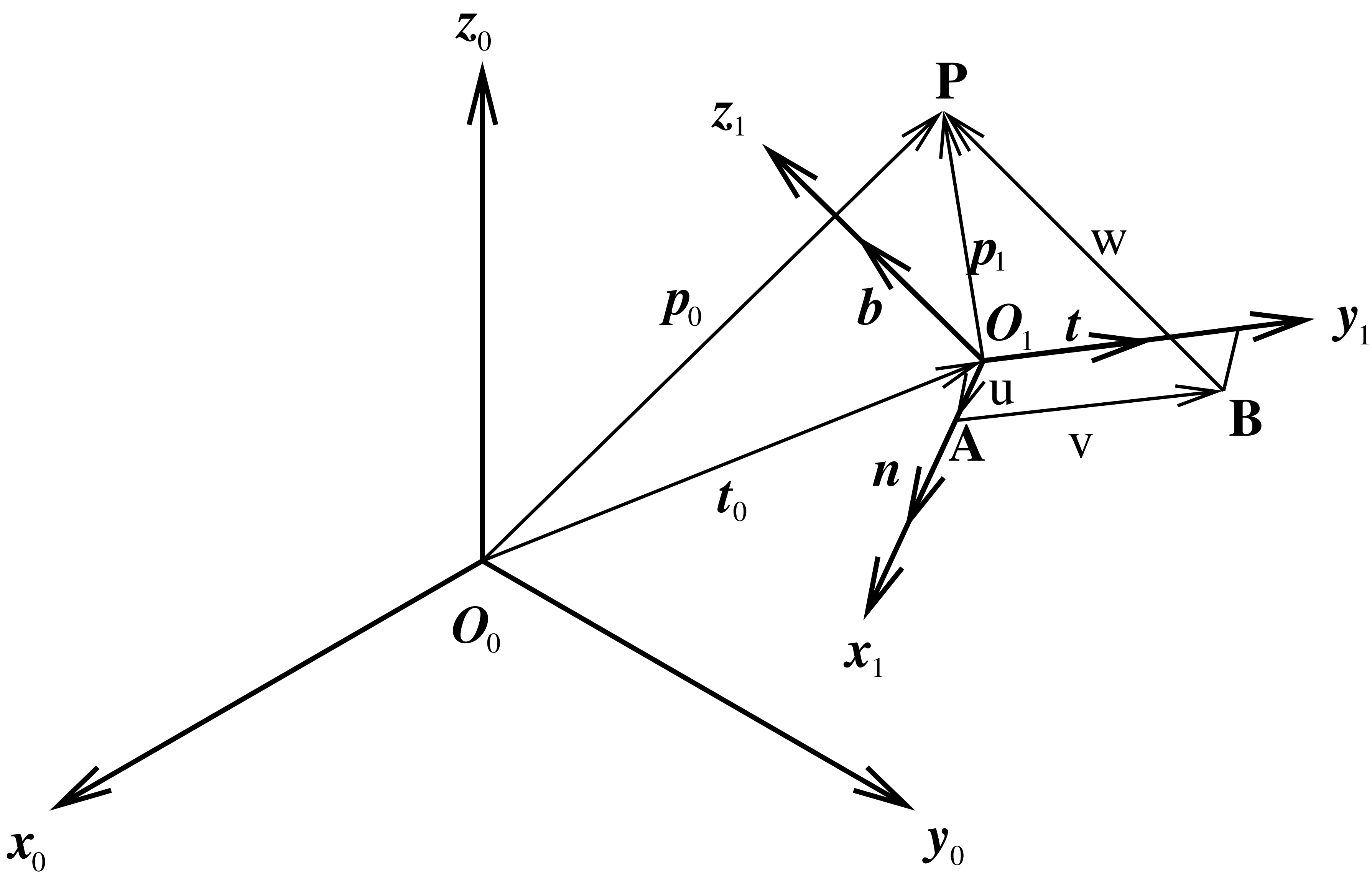

Roll
Yaw
Pitch

Comparison of Rotation Descriptions

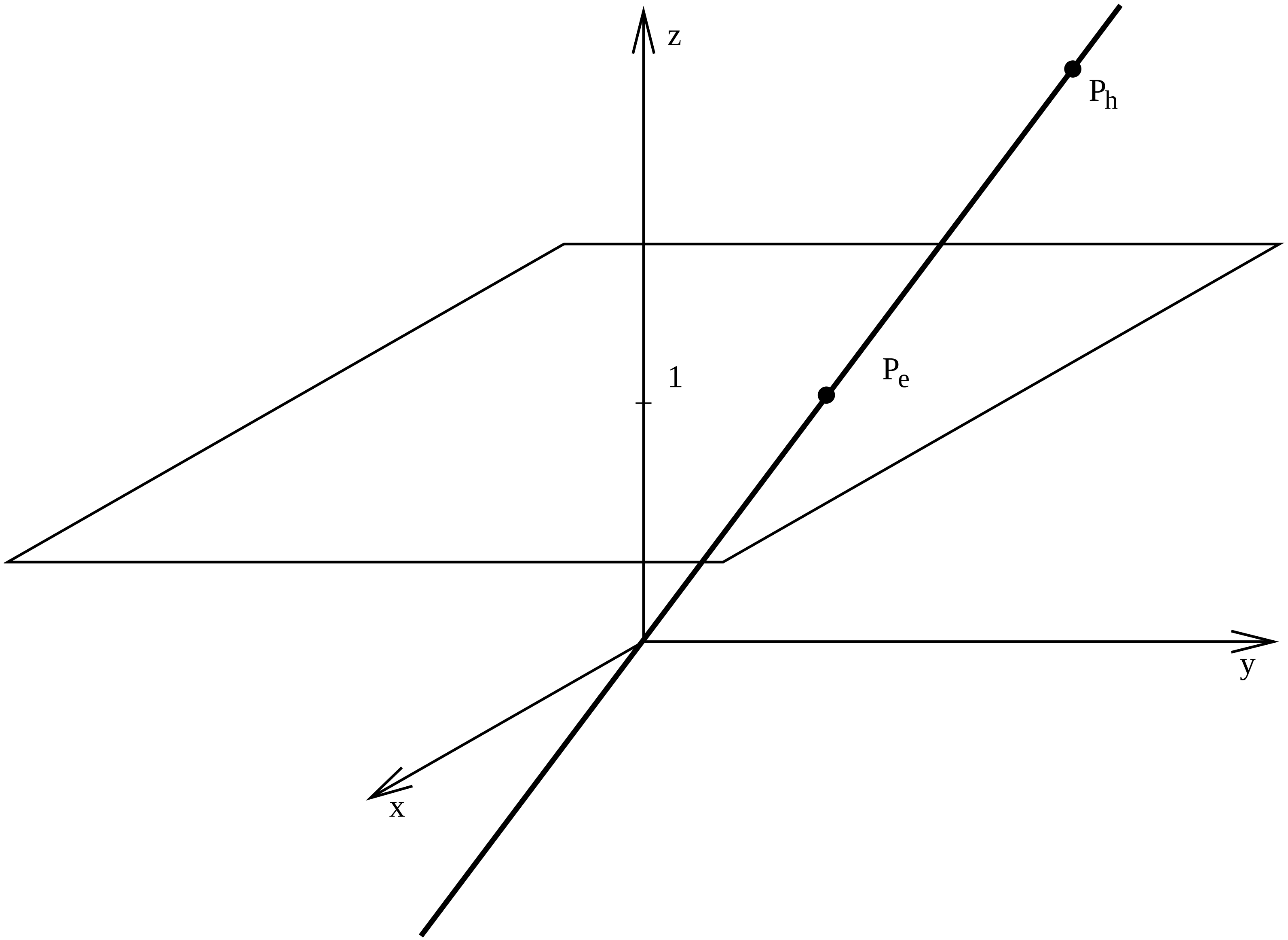
System	Symbol	Equivalent	Pars	Conditions
Rotation matrix	$\mathbf{R}$		9	orthonormal
Vectors of axes	$\mathbf{n}, \mathbf{t}, \mathbf{b}$	$\mathbf{R}$	9	unit vectors, orthogonal
Euler angles	ϕ, θ, ψ	yaw, pitch, roll,...	3	
Axis, angle	$\mathbf{s}, \theta$		4	unit vector
Quaternion	$\mathbf{q}$	axis, angle	4	unit vector
Rotation vector	$\mathbf{v}$	axis, angle	3	

System	Advantages	Disadvantages	Used by
$\mathbf{R}$	good for calculations	redundant	Matlab toolbox
$\mathbf{n}, \mathbf{t}, \mathbf{b}$	human understandable	redundant	
ϕ, θ, ψ	nonredundant	complicated topology	Mitsubishi Staubli, CRS
$\mathbf{s}, \theta$	human understandable	redundant	
	easy interpolation		
$\mathbf{q}$	easy interpolation	redundant	ABB
$\mathbf{v}$	good topology		rotation estimation
	nonredundant		

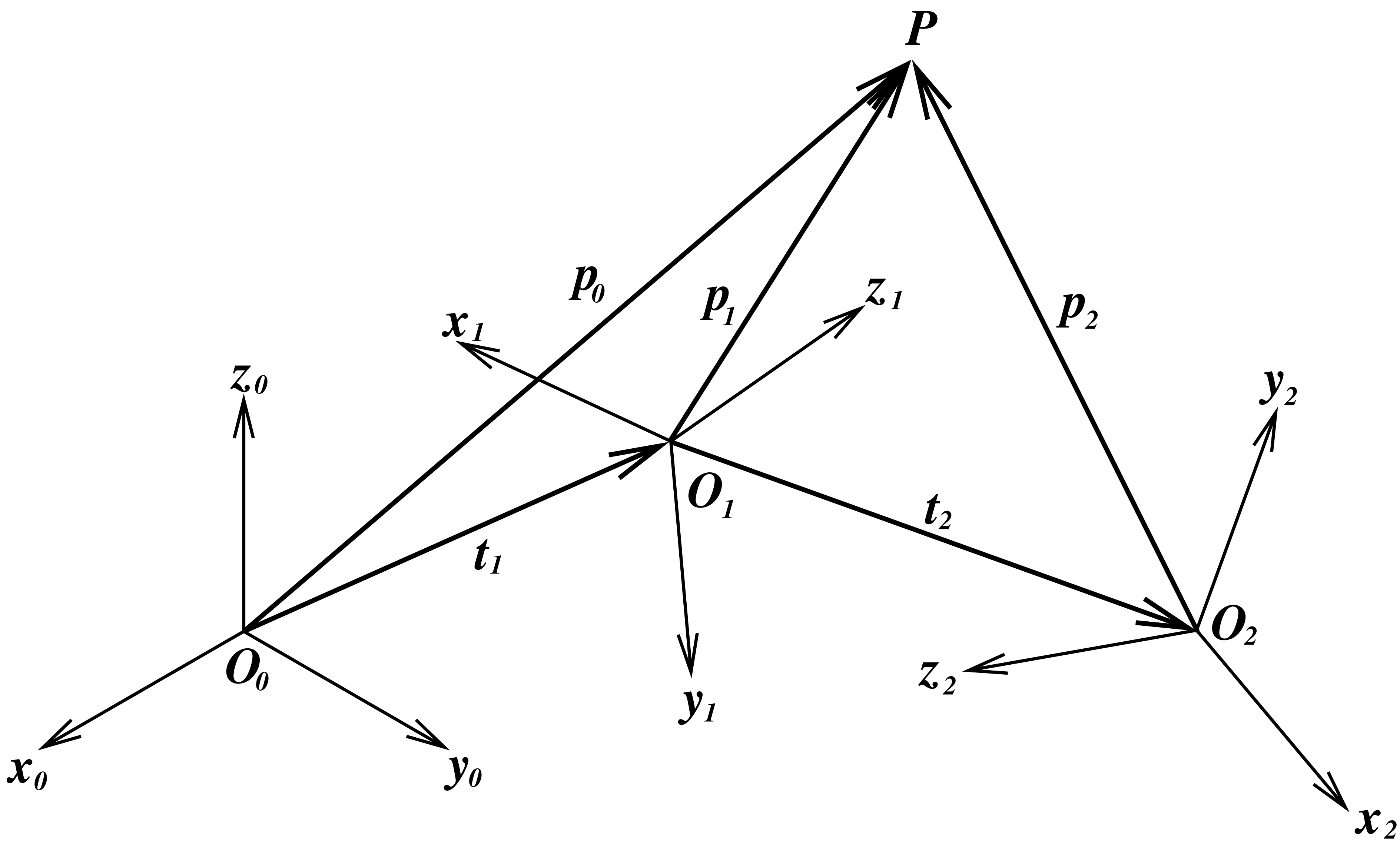
Coordinate transformation



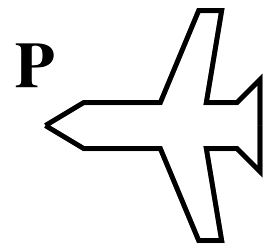
Homogeneous Coordinates

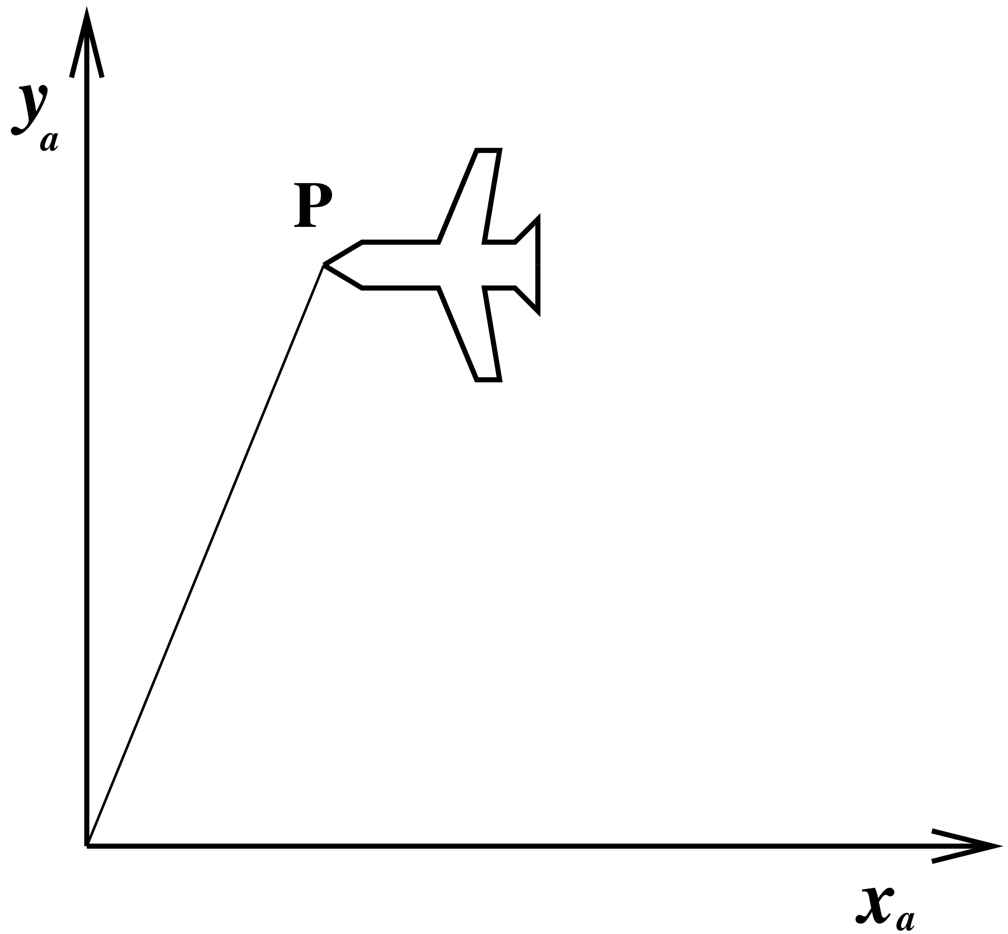


m p
1 2
3 4
5 6
7 8
9 10
11 12
13 14
15 16
17 18
19 20
21 22
23 24
25 26
27 28
29 30
31

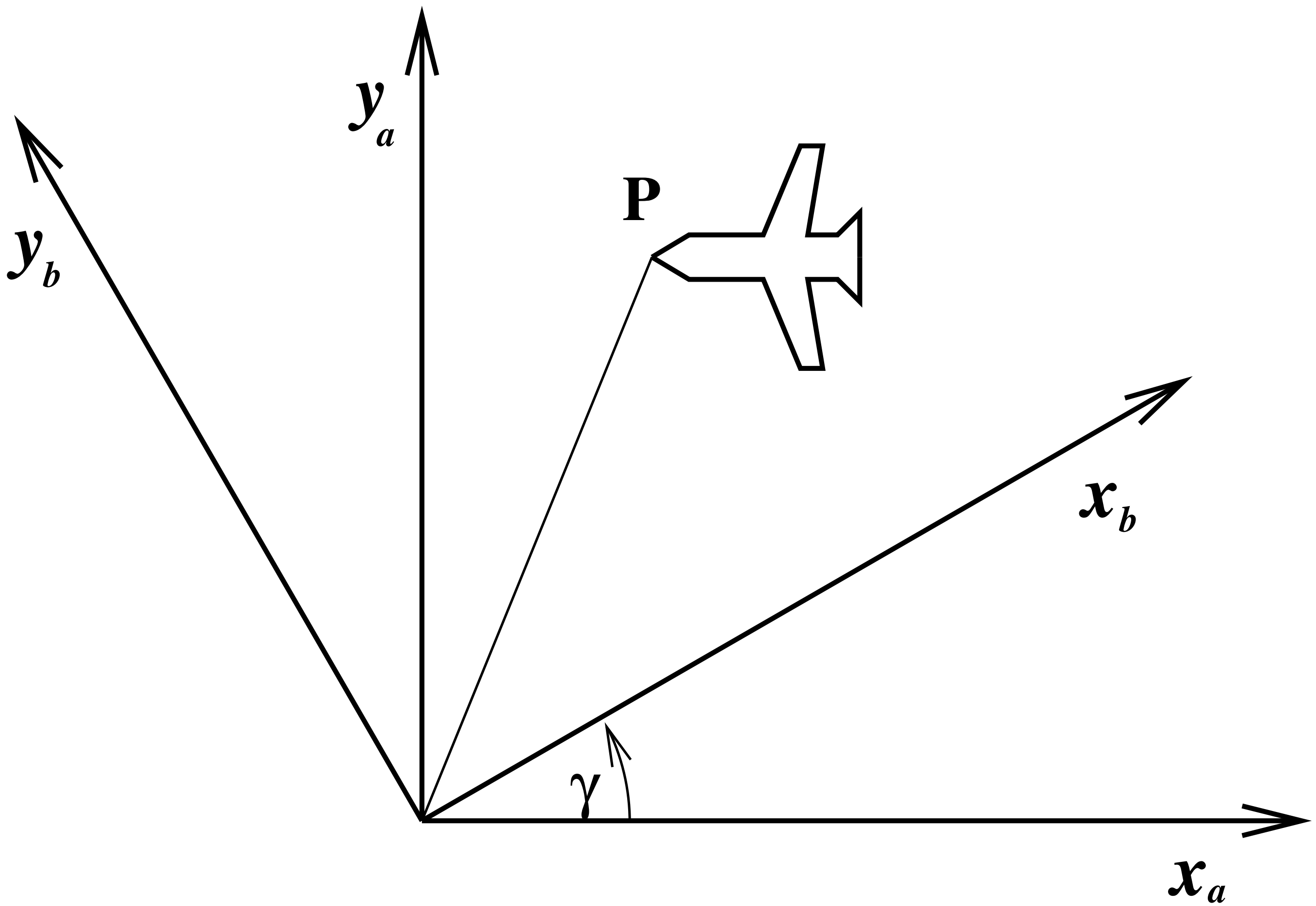

P
p_0
p_1
z_1
p_2
x_1
z_0
y_2
O_1
t_2
t_1
O_2
z_2
O_0
y_1
x_0
y_0
x_2

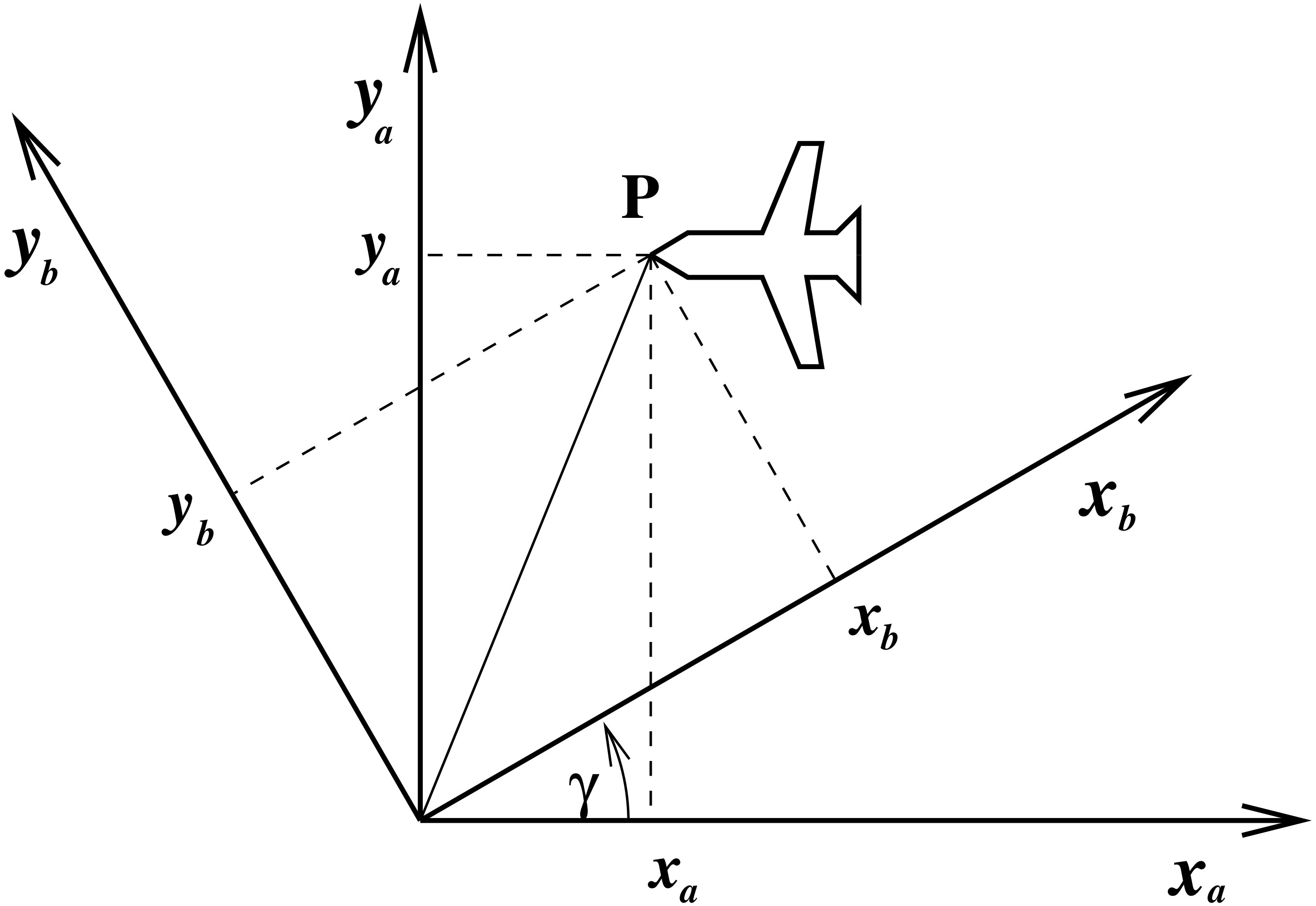
Passive versus Active Transformation – Passive




y_a
P
x_a

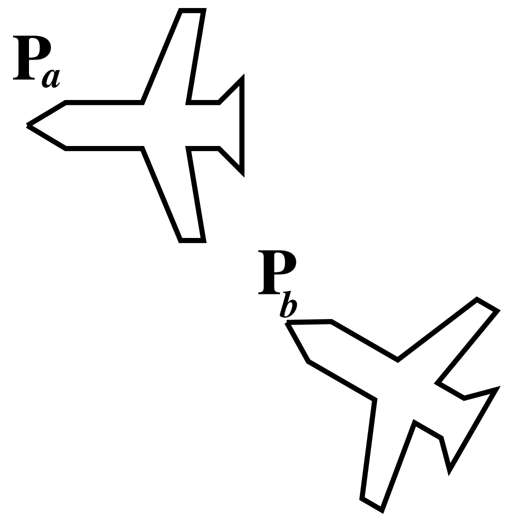
Passive versus Active Transformation – Passive

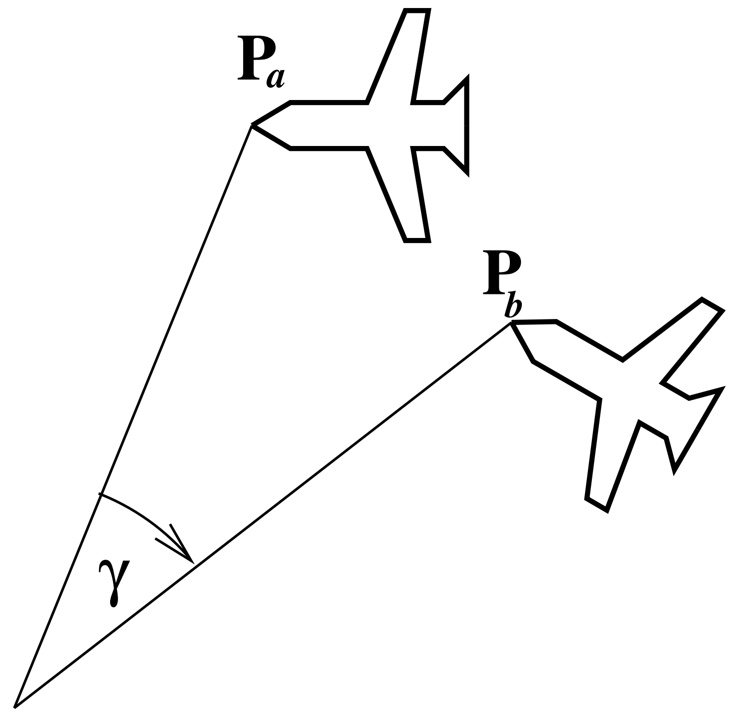




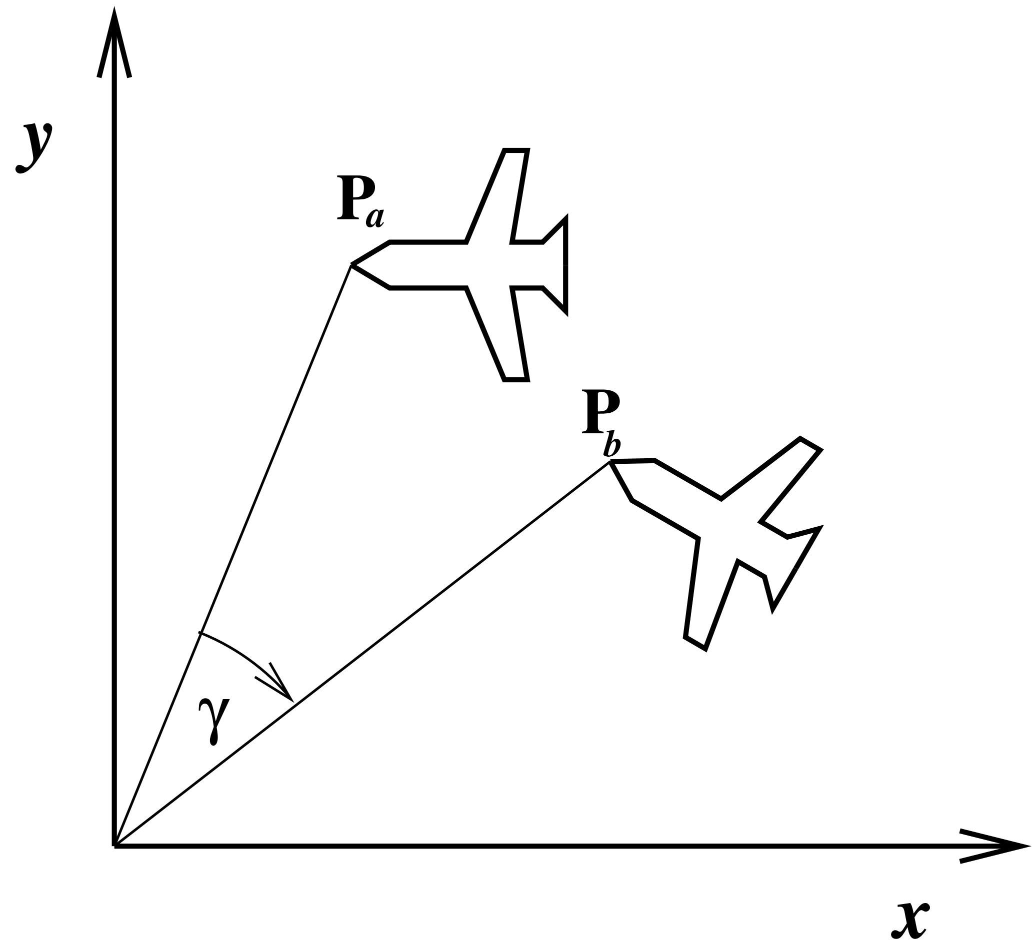
$$\vec{x}_a = \mathbf{R}^a_b(\gamma)\vec{x}_b$$

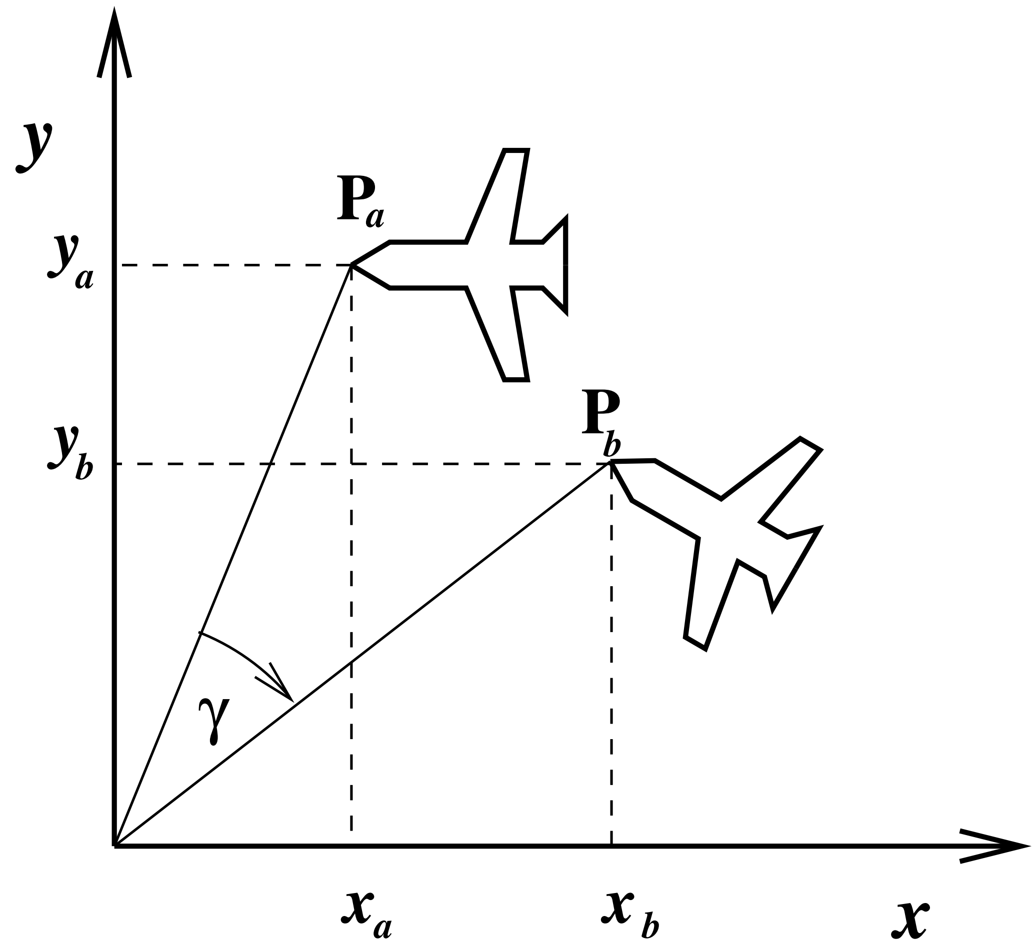
Passive versus Active Transformation - Active





P_a
P_b
γ

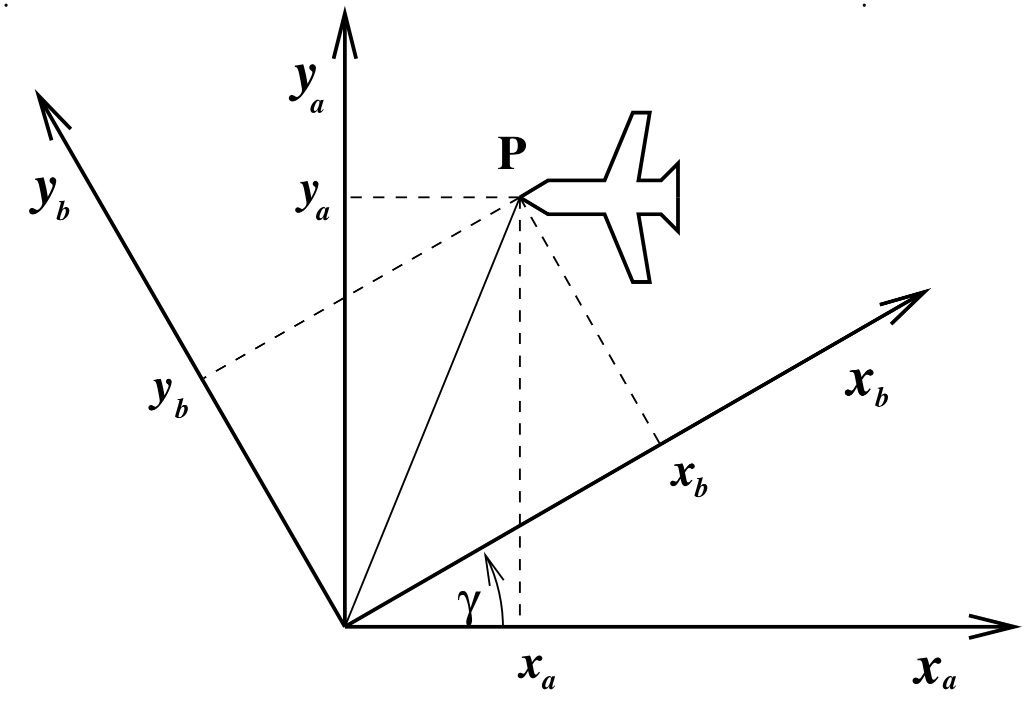

y
P_a
P_b
γ
x



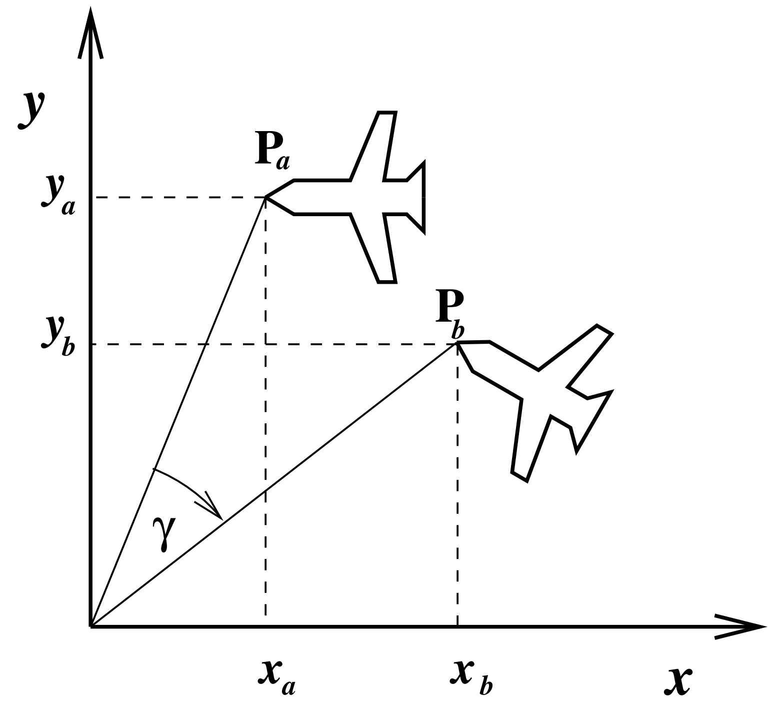
$$\vec{x}_b = \mathbf{R}_b^a(-\gamma)\vec{x}_a = \mathbf{R}_b^a(\gamma)^{-1}\vec{x}_a = \mathbf{R}_a^b(\gamma)\vec{x}_a$$

Passive versus Active Transformation - Comparison

Passive

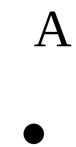


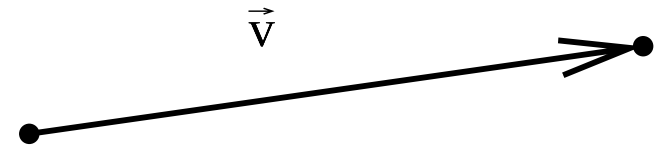
Active

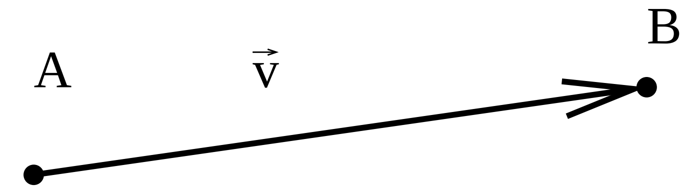


$$\vec{x}_a = \mathbf{R}_b^a(\gamma)\vec{x}_b$$

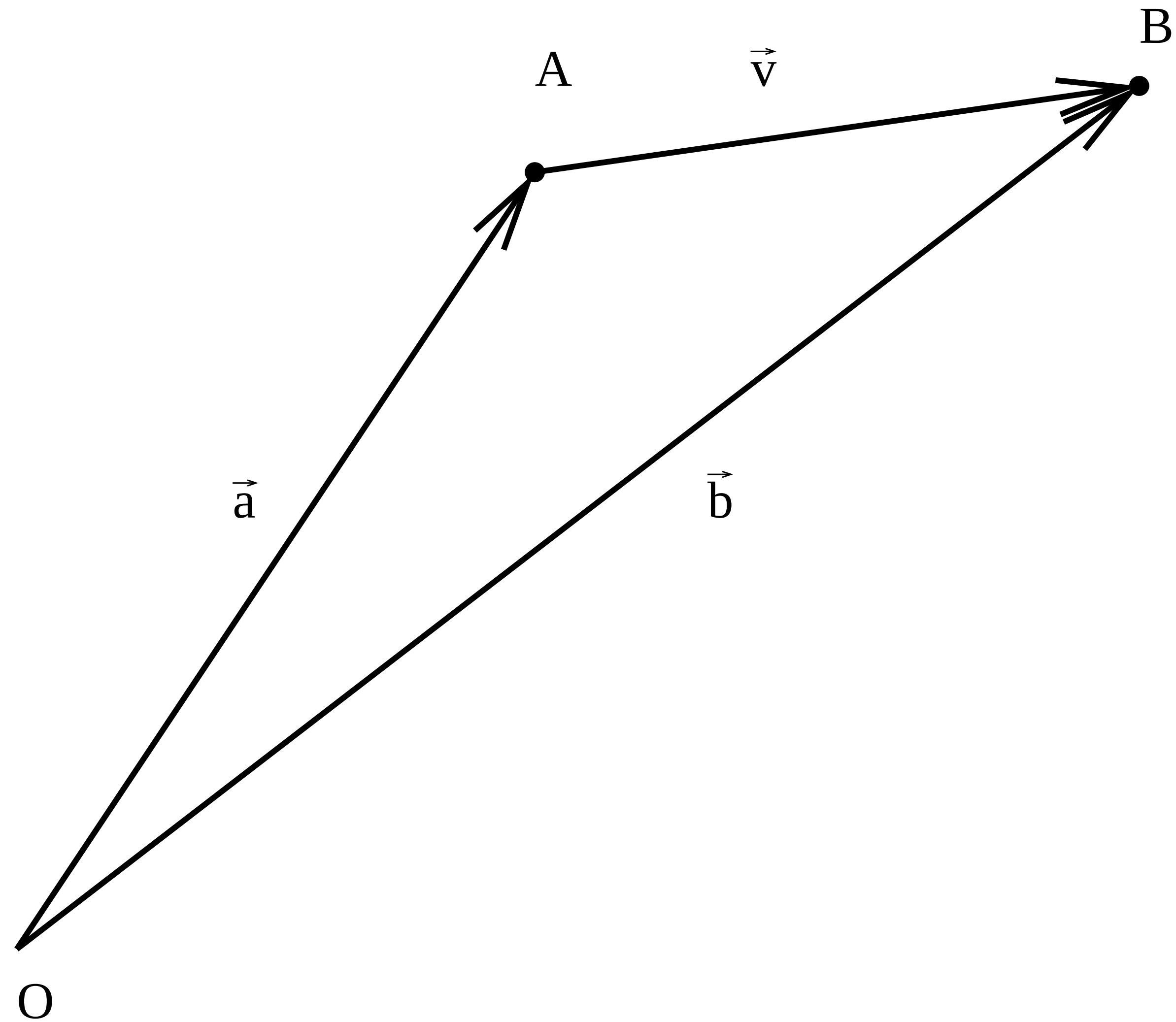
$$\vec{x}_b = \mathbf{R}_b^a(-\gamma)\vec{x}_a = \mathbf{R}_b^a(\gamma)^{-1}\vec{x}_a = \mathbf{R}_a^b(\gamma)\vec{x}_a$$


A

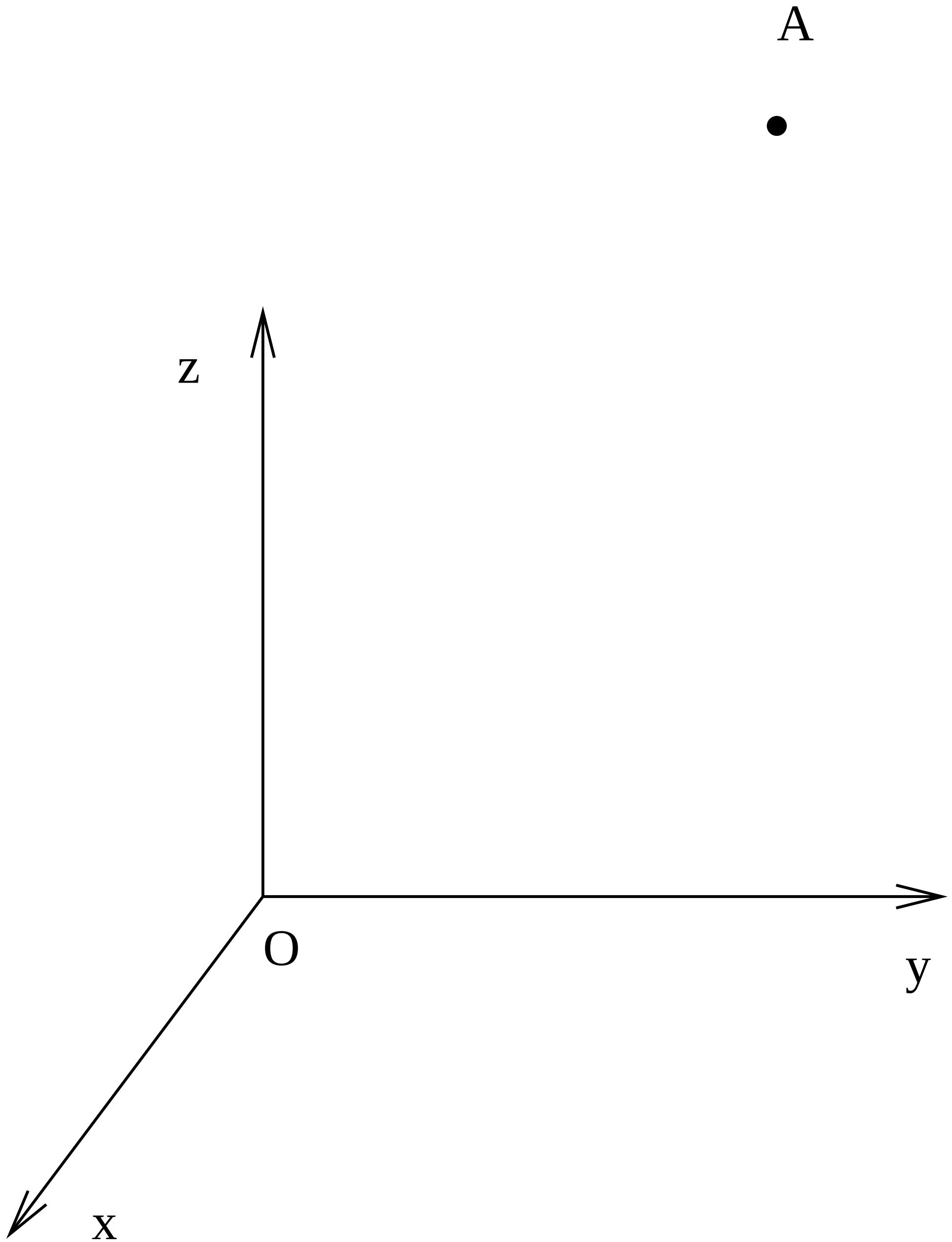

$\vec{v}$

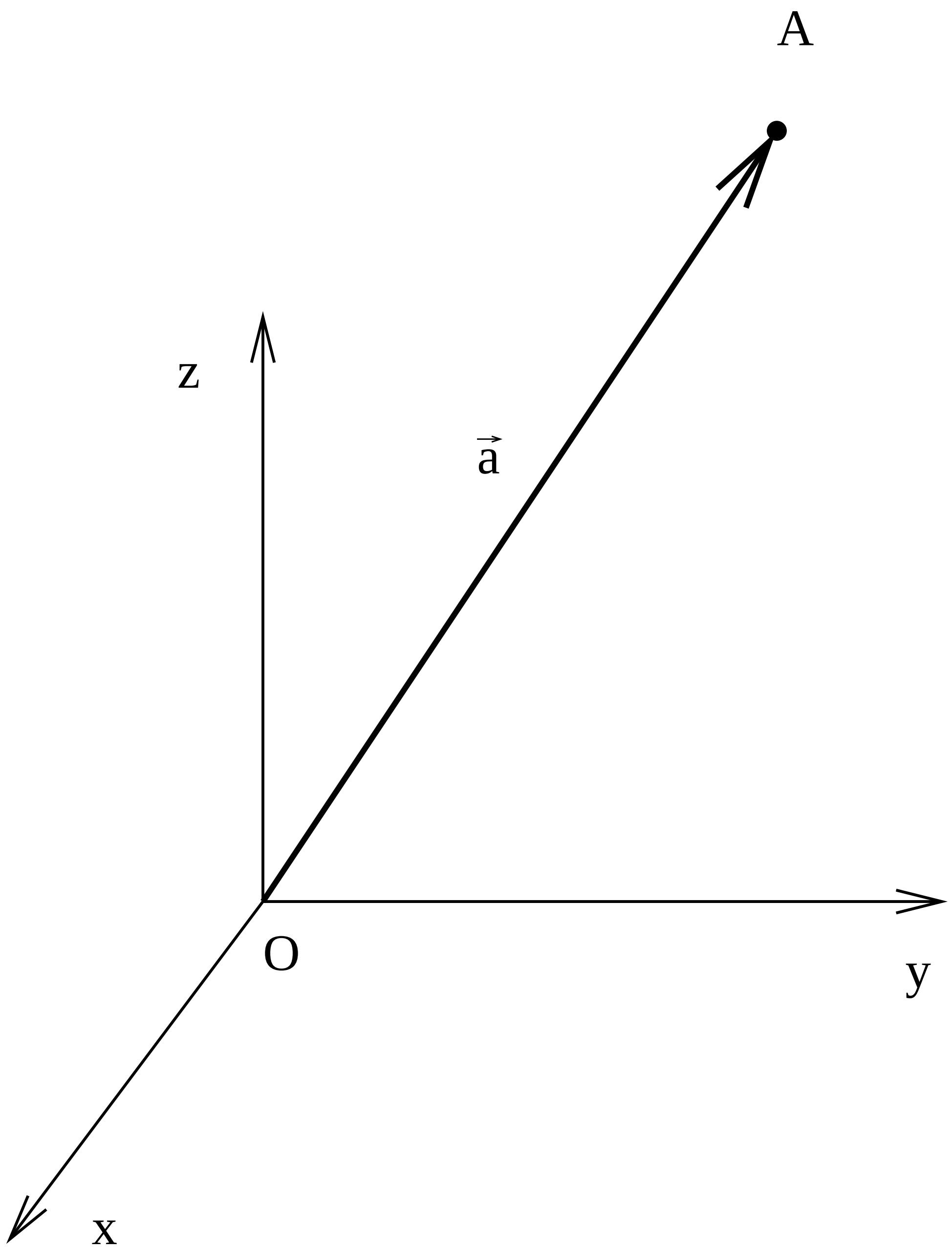


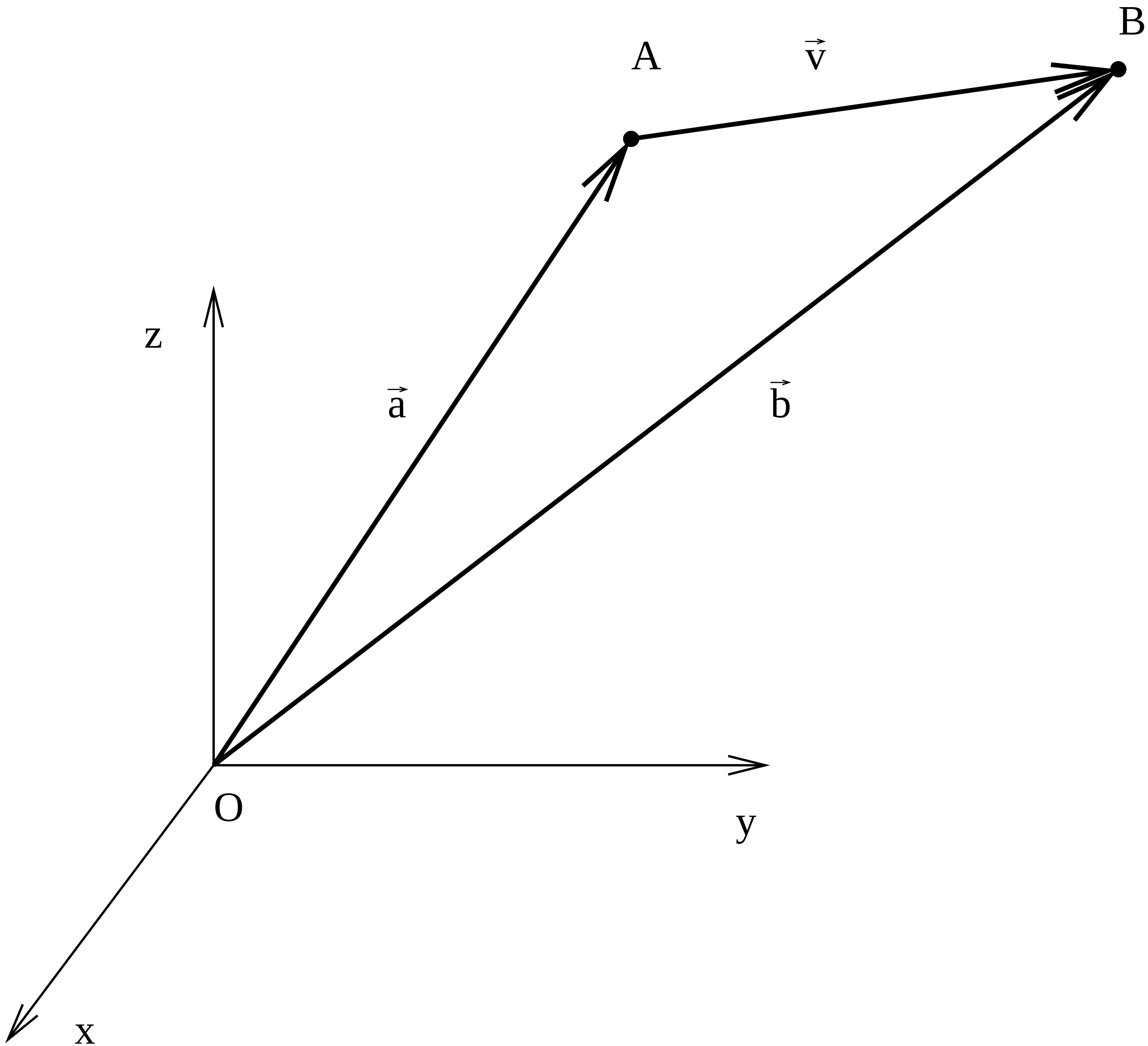
A
$\vec{v}$
B



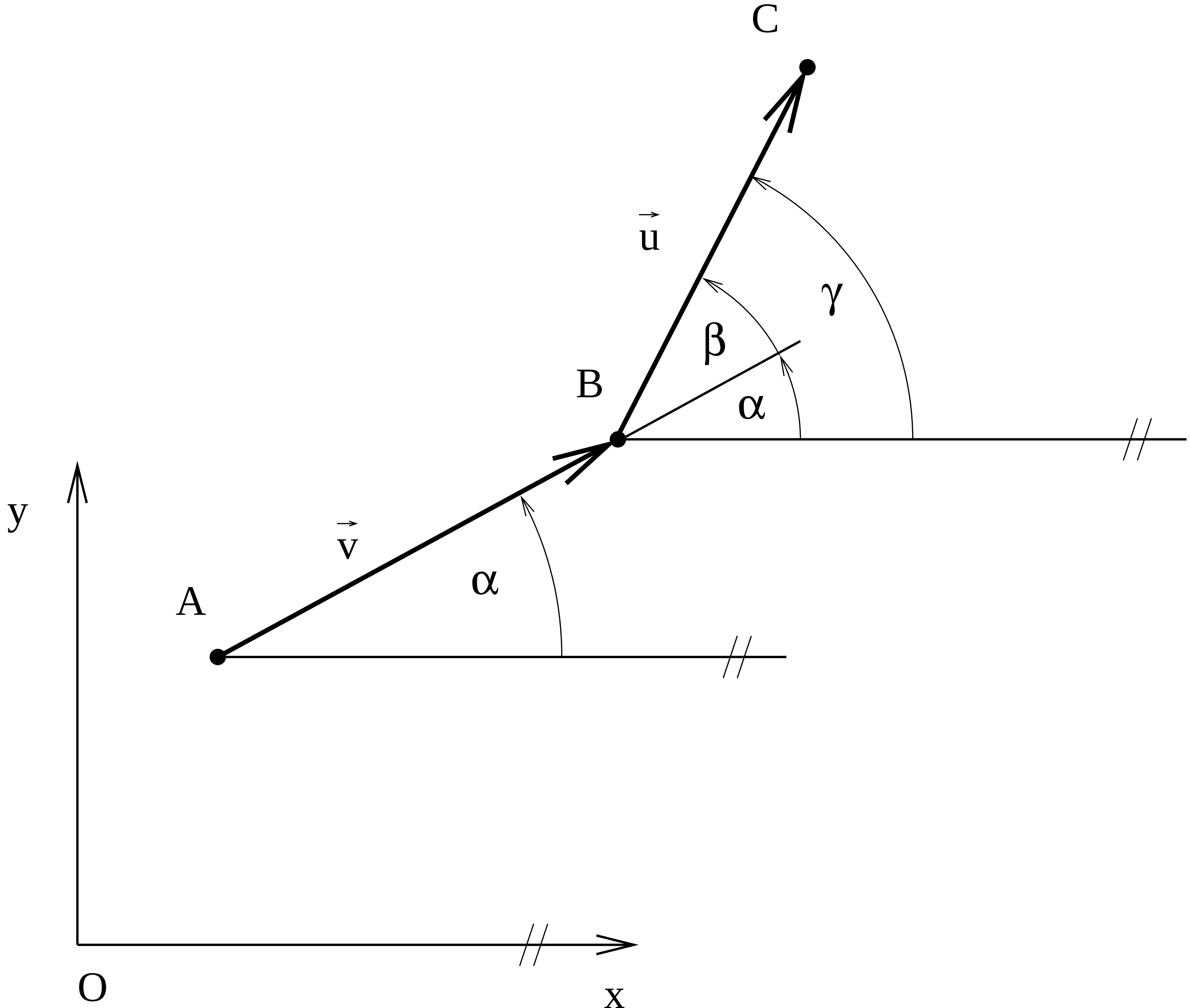
B
A
$\vec{v}$
$\vec{a}$
$\vec{b}$
O

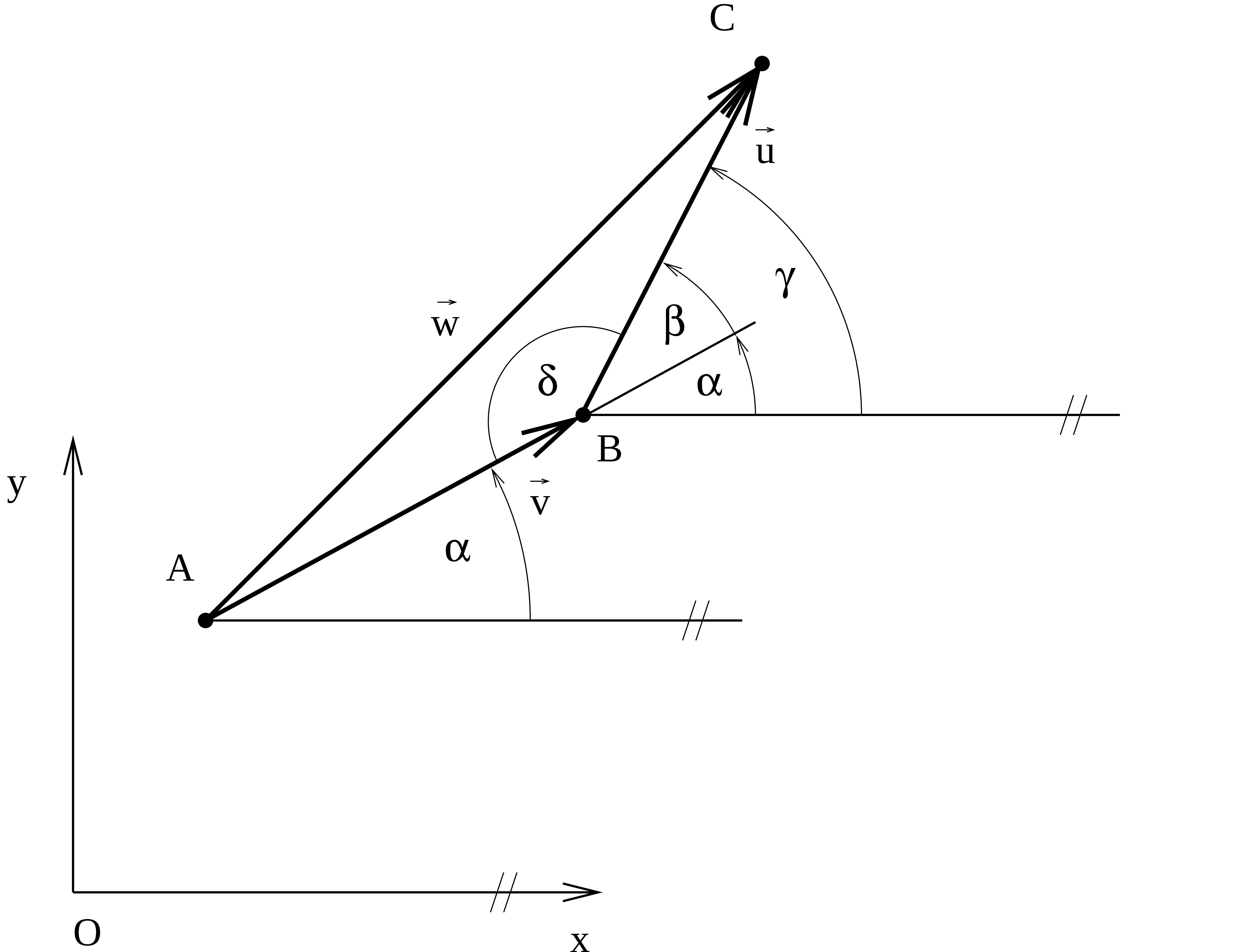

A
z
O
y
x


A
z
$\vec{a}$
O
y
x

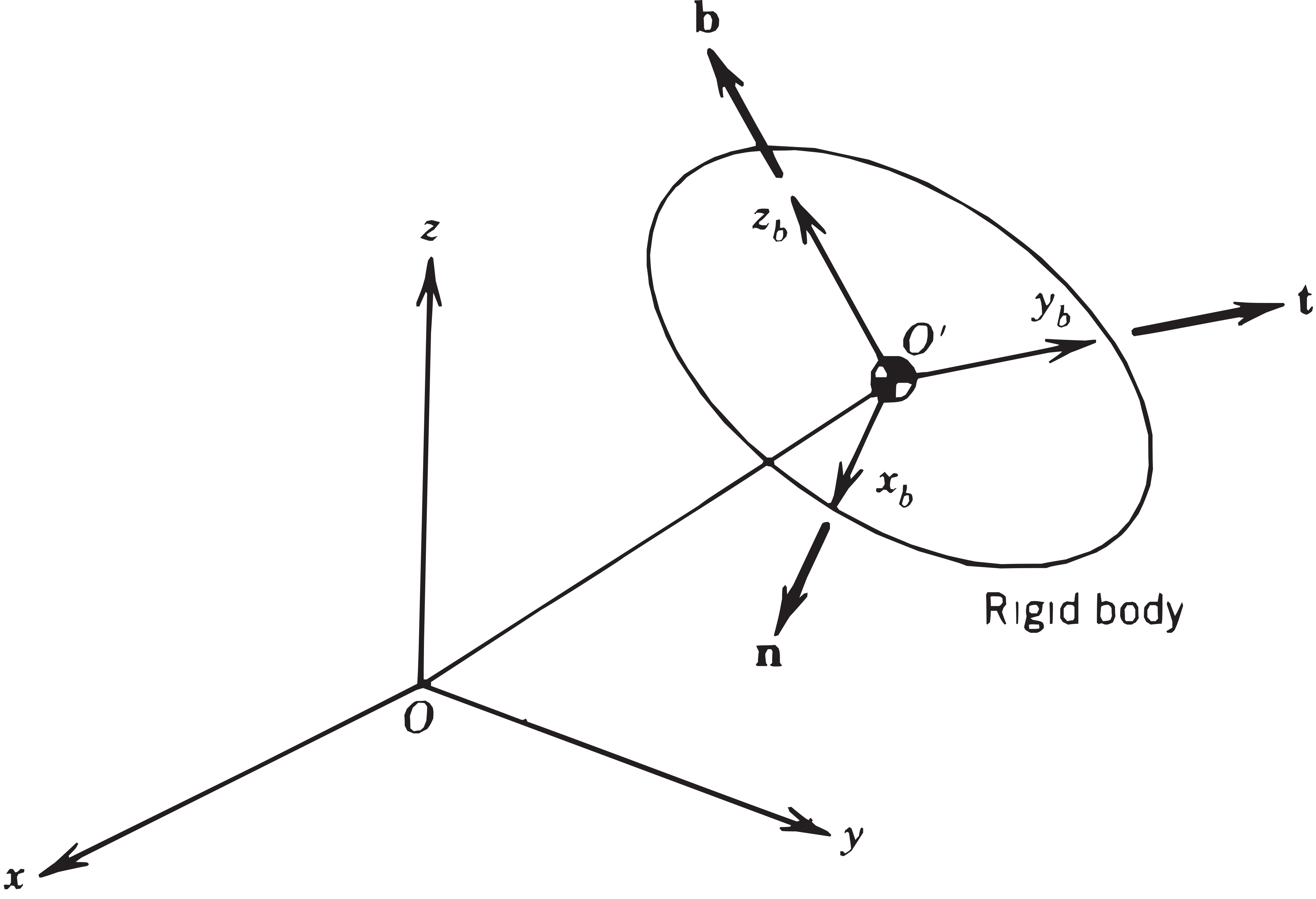


B
A
$\vec{v}$
z
$\vec{a}$
$\vec{b}$
O
y
x

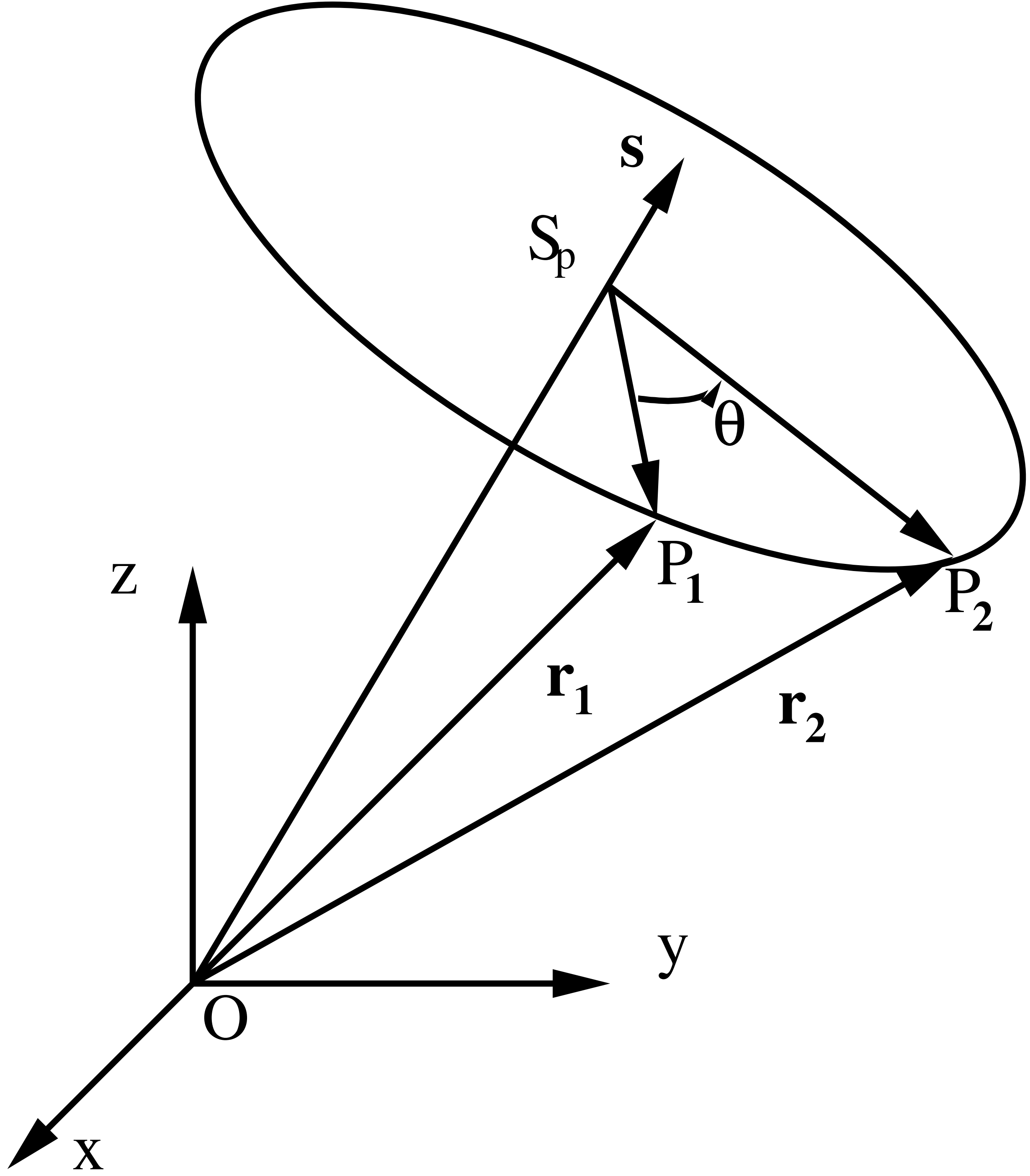

C
u
γ
β
B
α
y
v
α
A
O
x

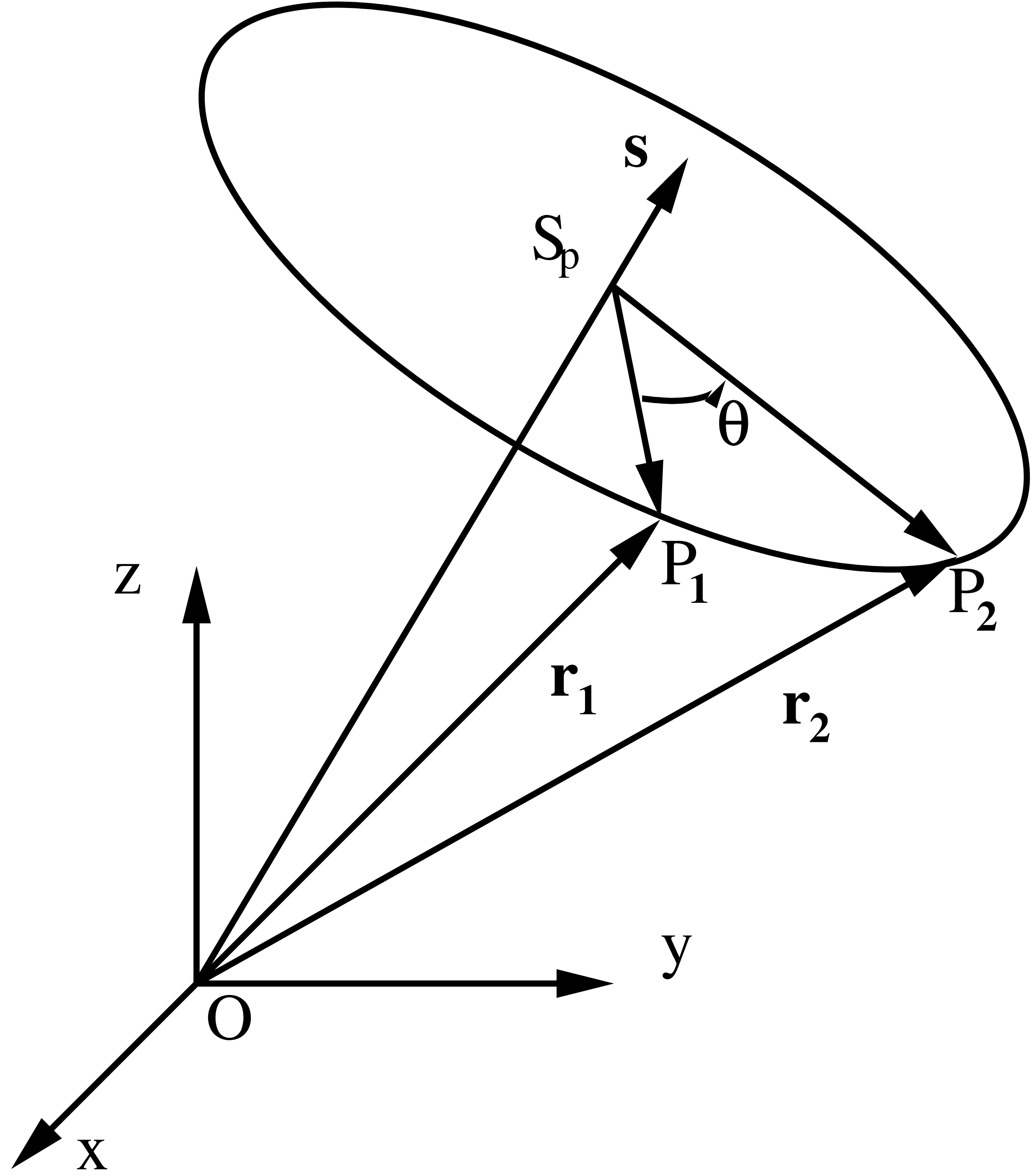


C
$\vec{u}$
γ
$\vec{w}$
β
δ
α
B
y
$\vec{v}$
α
A
O
x

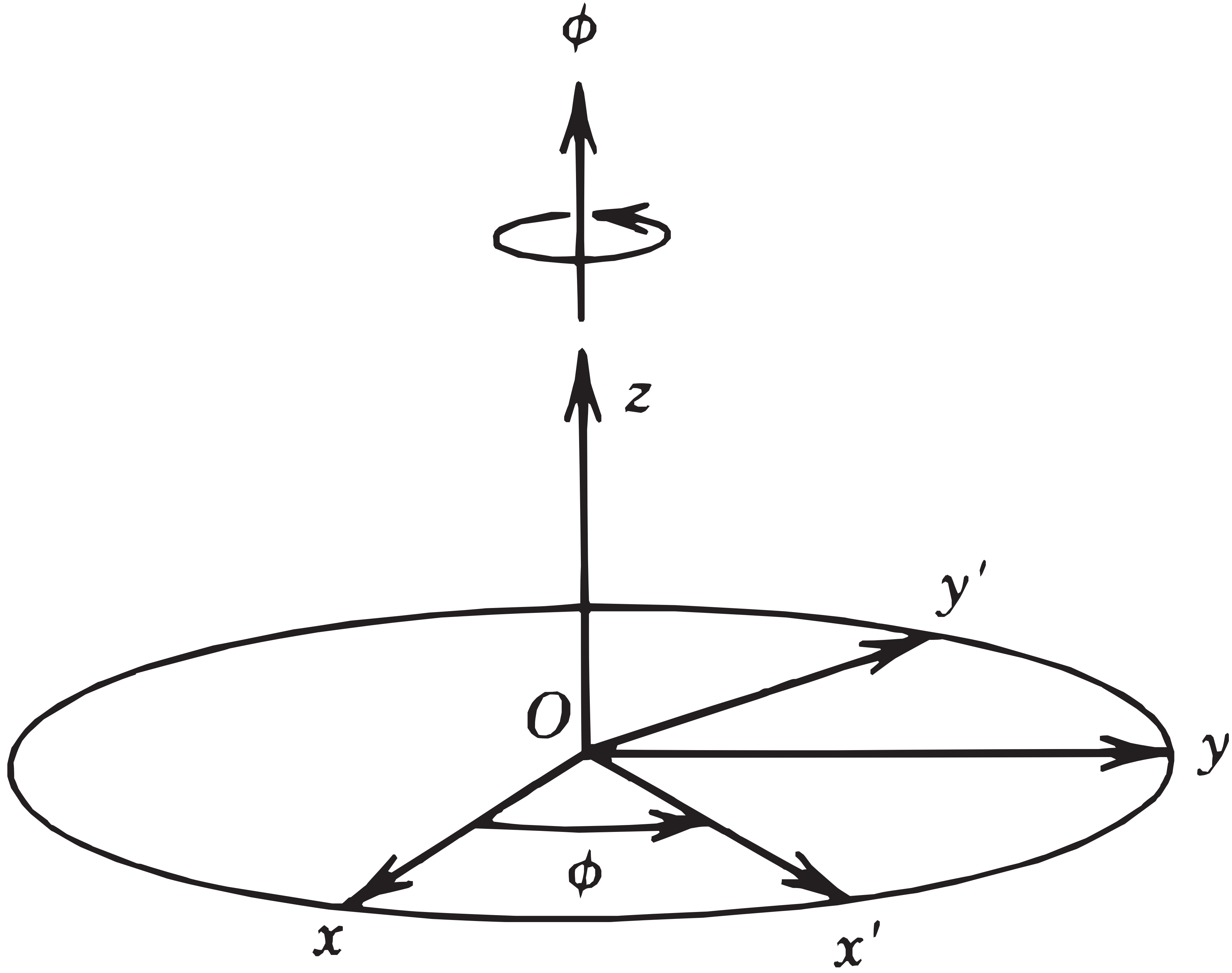


b
z_b
z
y_b
t
O'
x_b
n
Rigid body
O
x
y

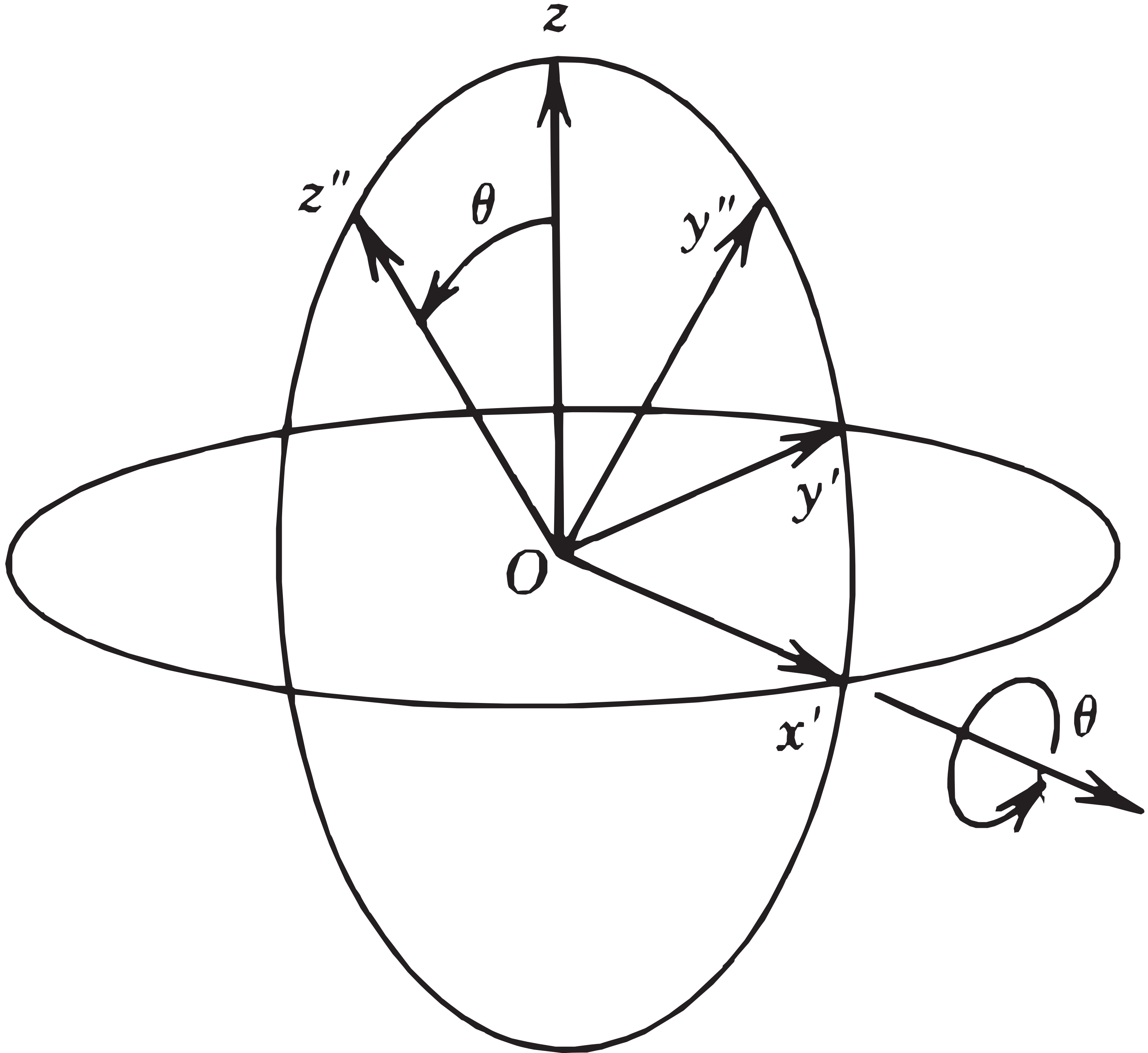

s
Sp
θ
z
P1
P2
r1
r2
y
O
x



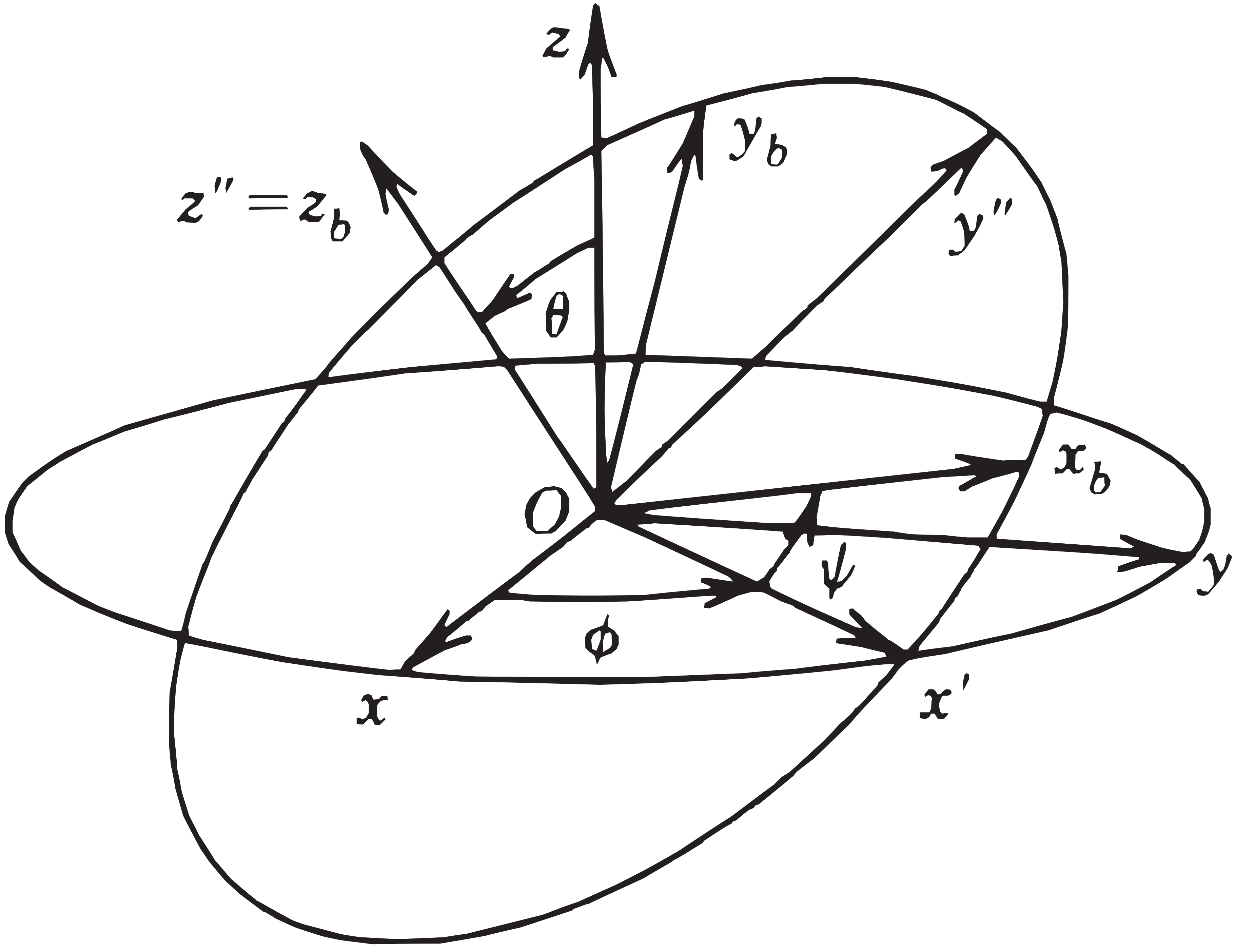
s
S_p
θ
P_1
P_2
z
$\mathbf{r}_1$
$\mathbf{r}_2$
y
O
x

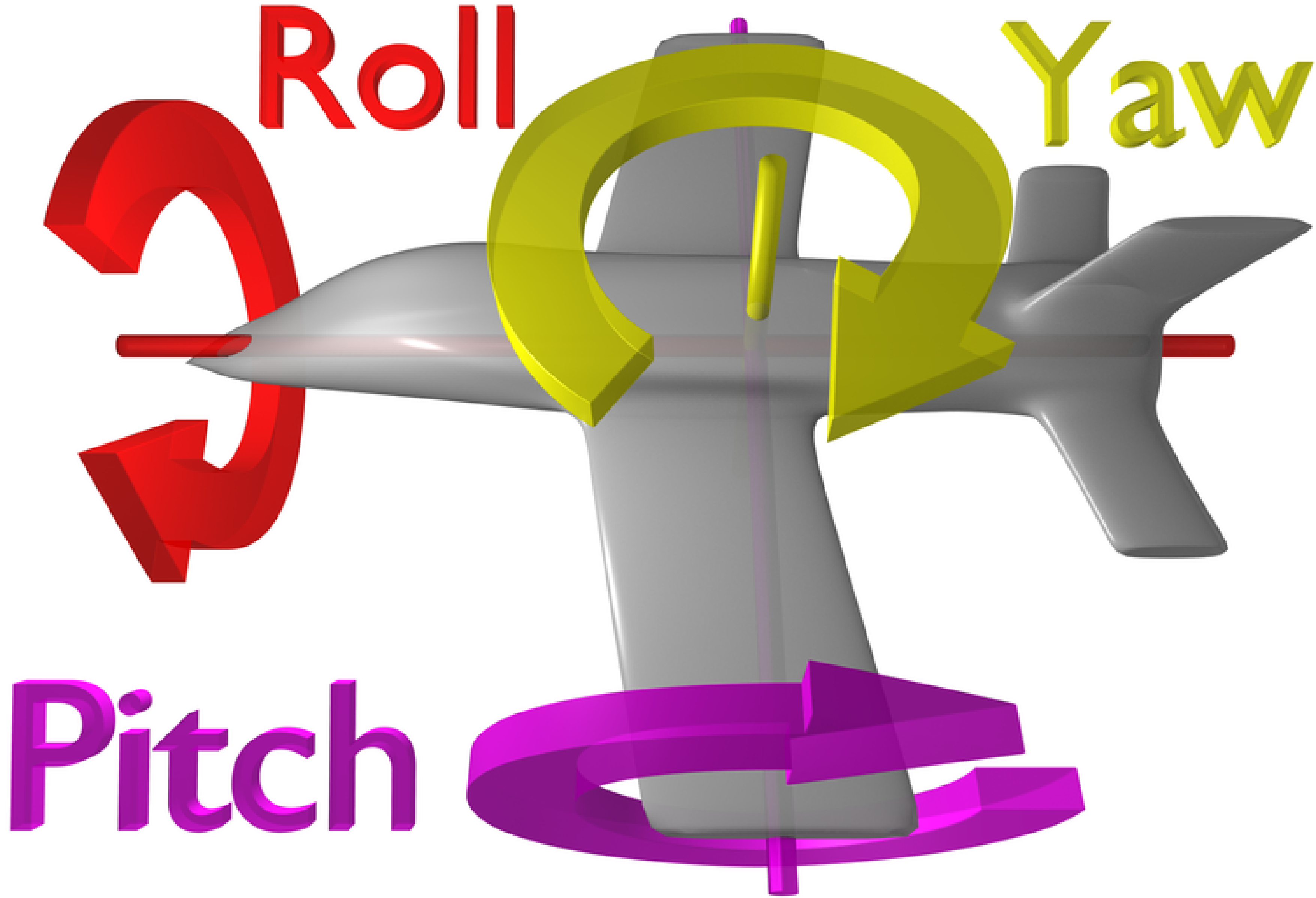


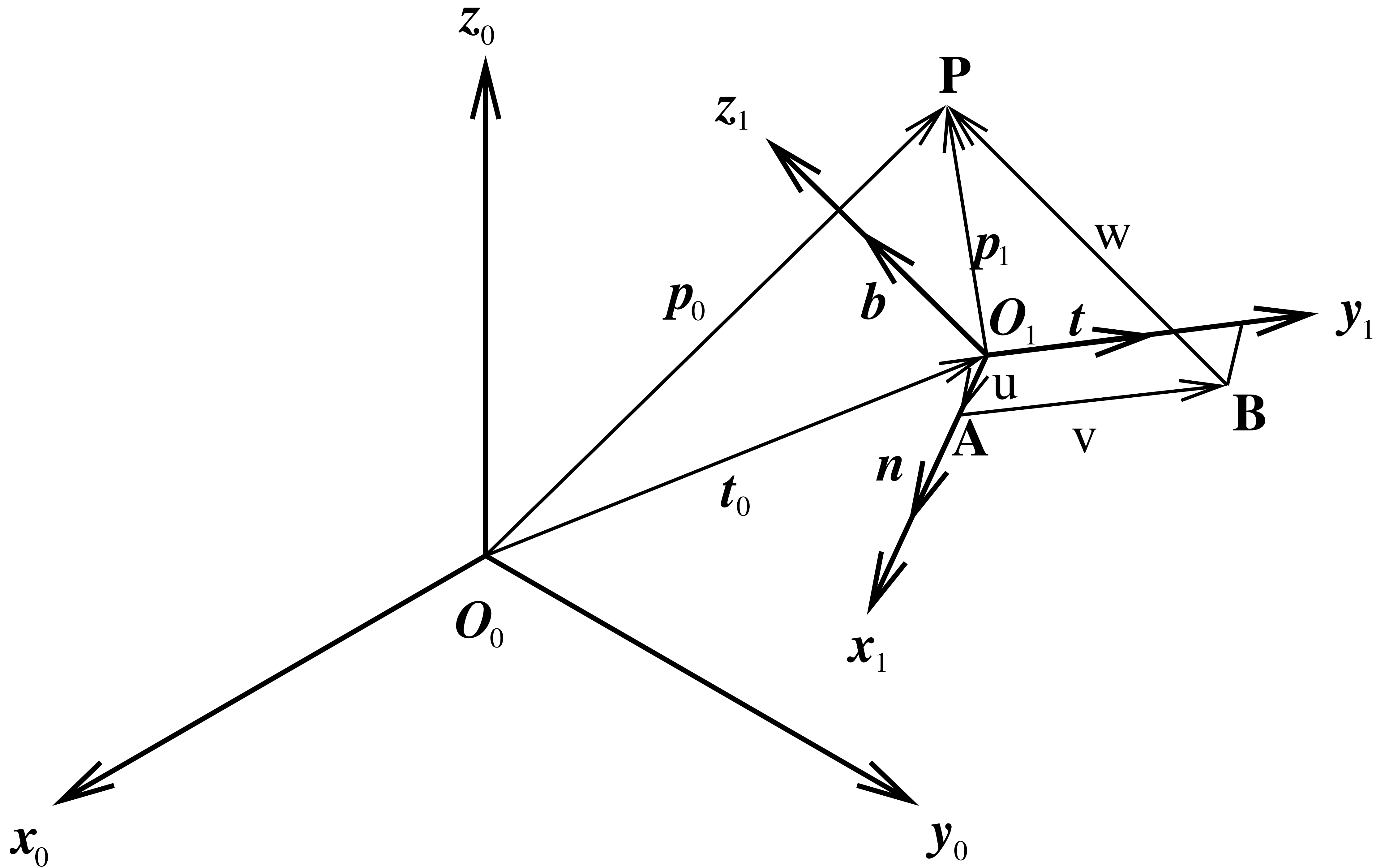
φ
z
y′
O
y
φ
x
x′



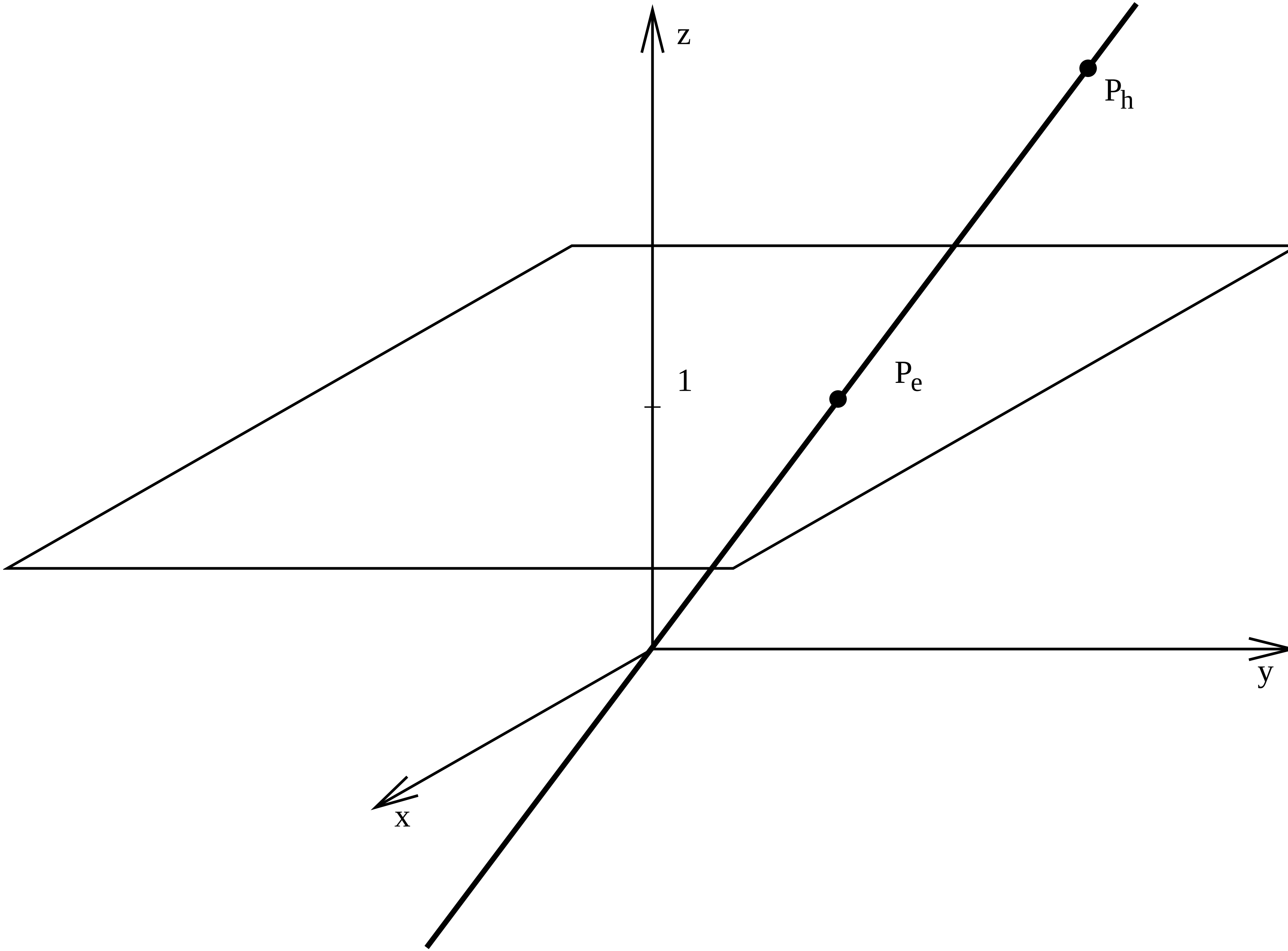
z
z''
θ
y''
y'
O
x'
θ

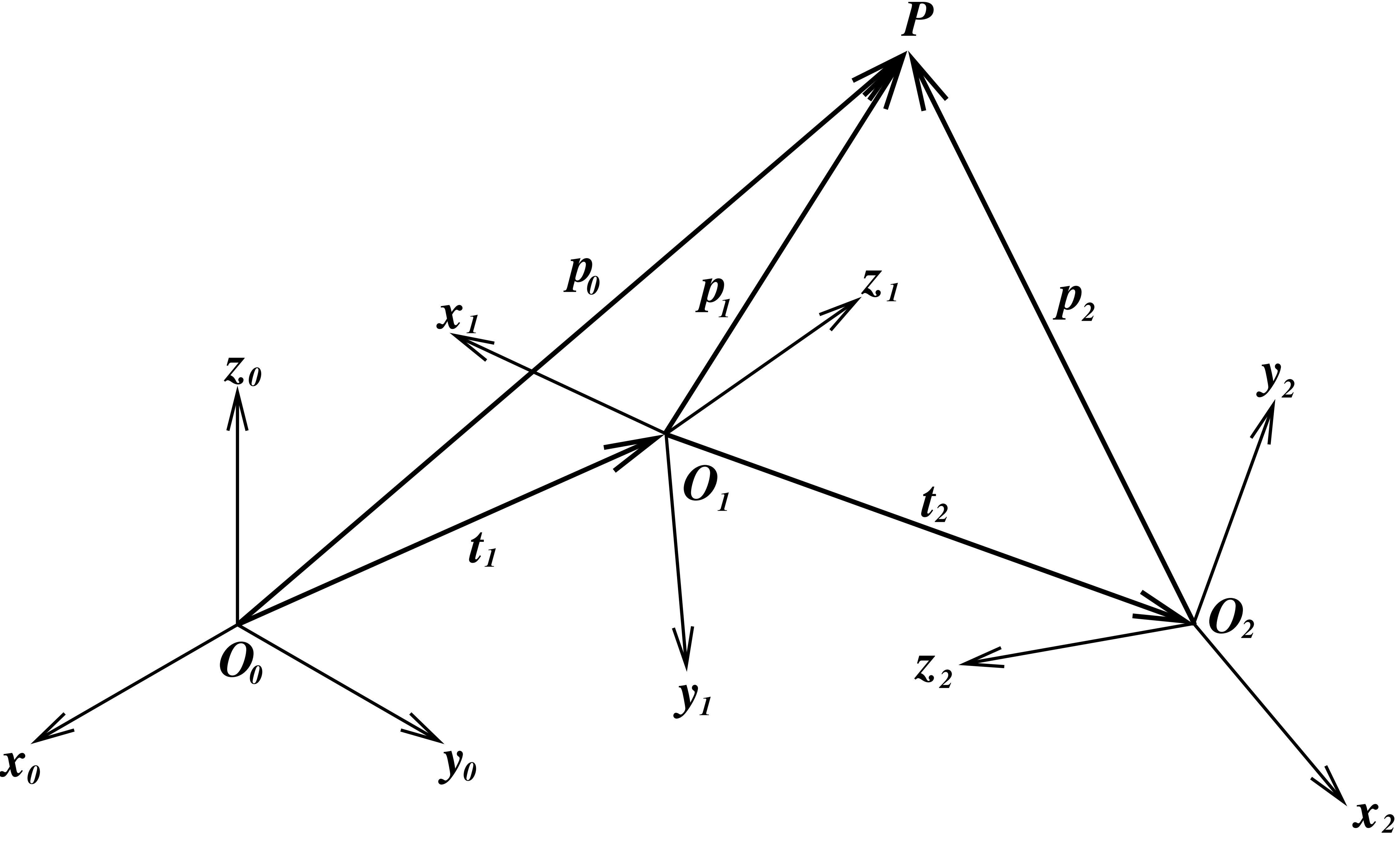

z
y_b
$z'' = z_b$
y''
θ
x_b
O
y
ψ
ϕ
x
x'

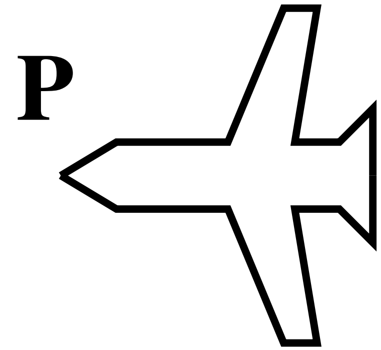

Roll
Yaw
Pitch

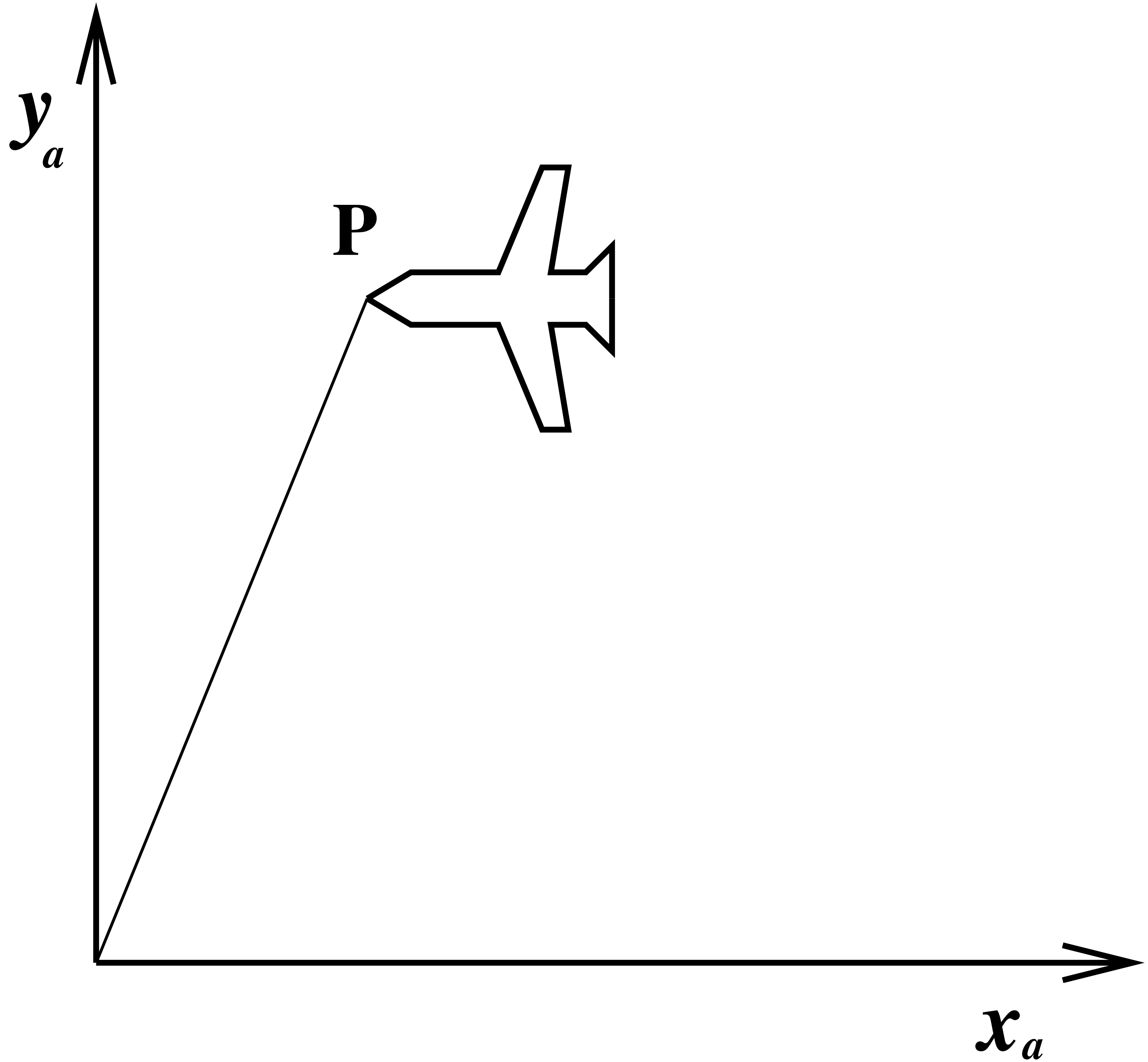


z_0
P
z_1
w
p_1
p_0
b
O_1
t
y_1
u
B
A
v
n
t_0
x_1
O_0
x_0
y_0

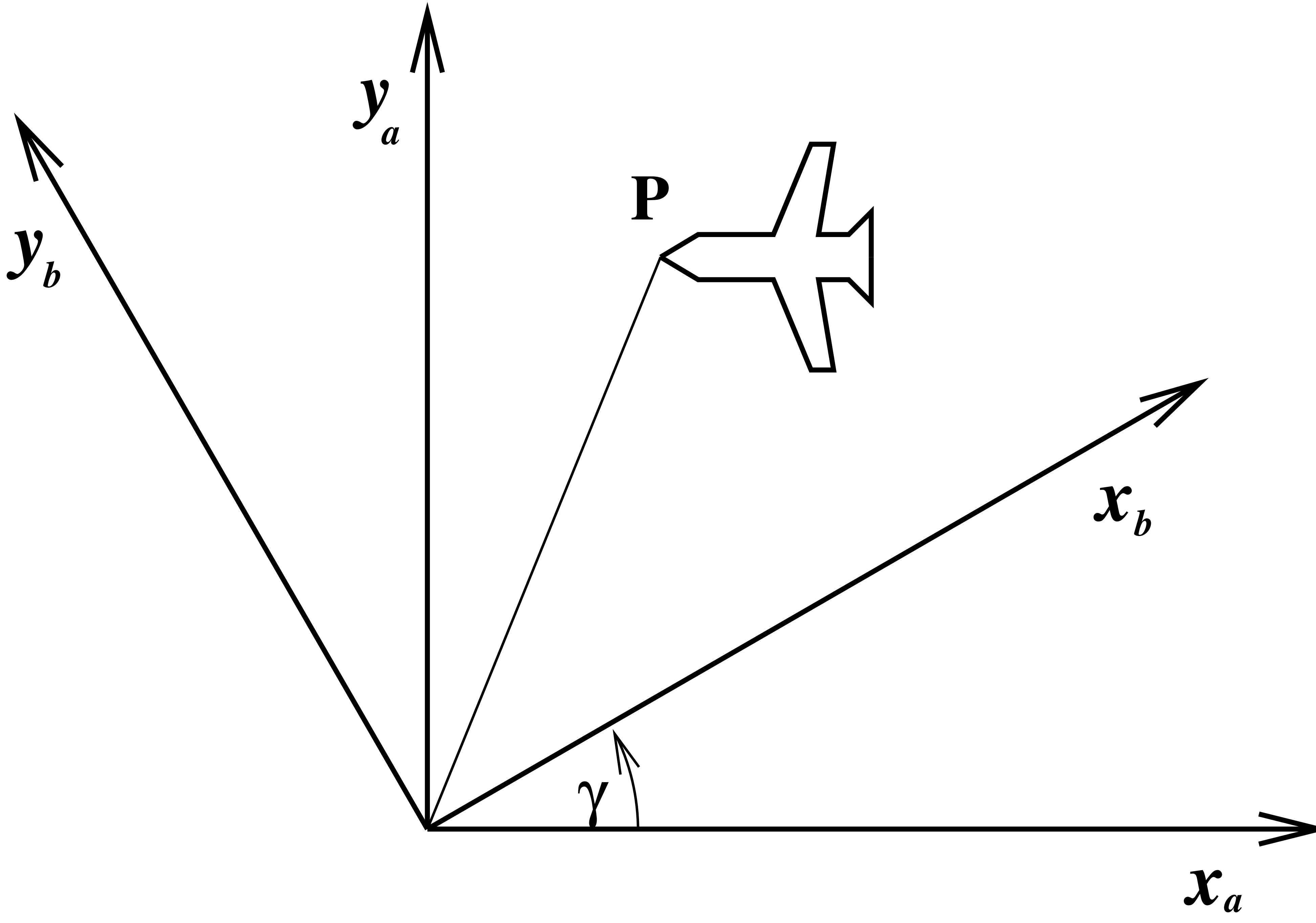

z
P_h
1
P_e
y
x


P
p0
p1
p2
z1
x1
z0
y2
O1
t2
t1
O2
z2
O0
y1
x0
y0
x2

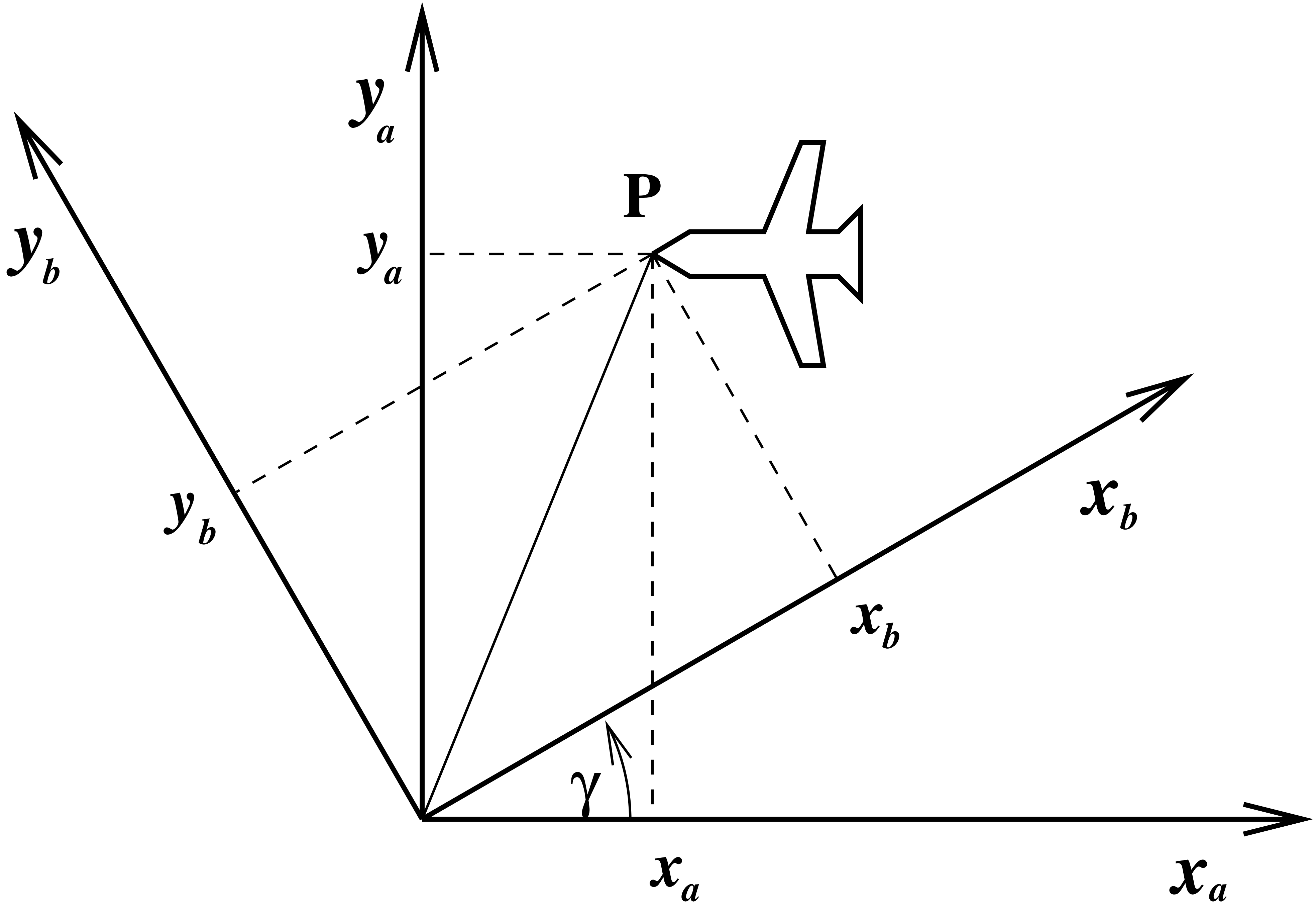

P



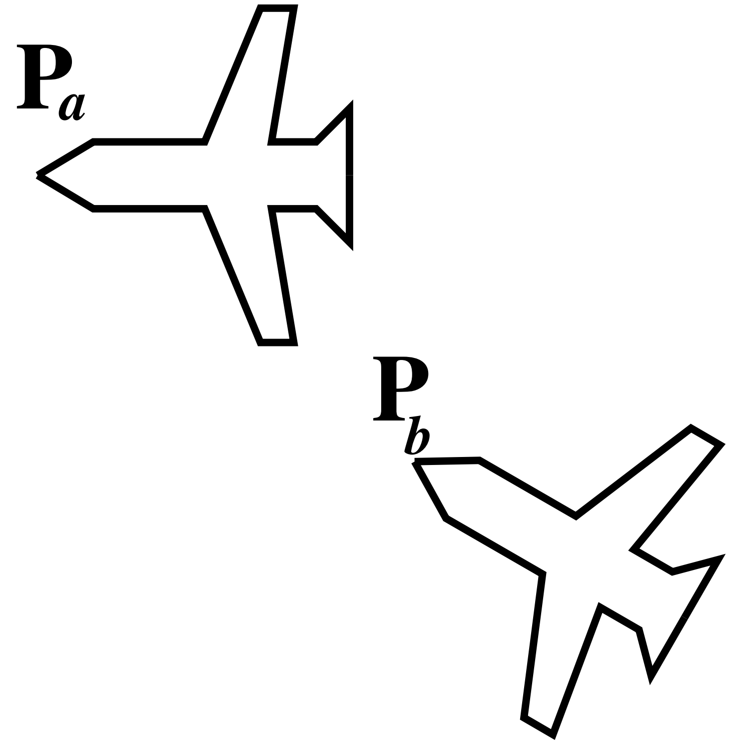
y_a
P
x_a



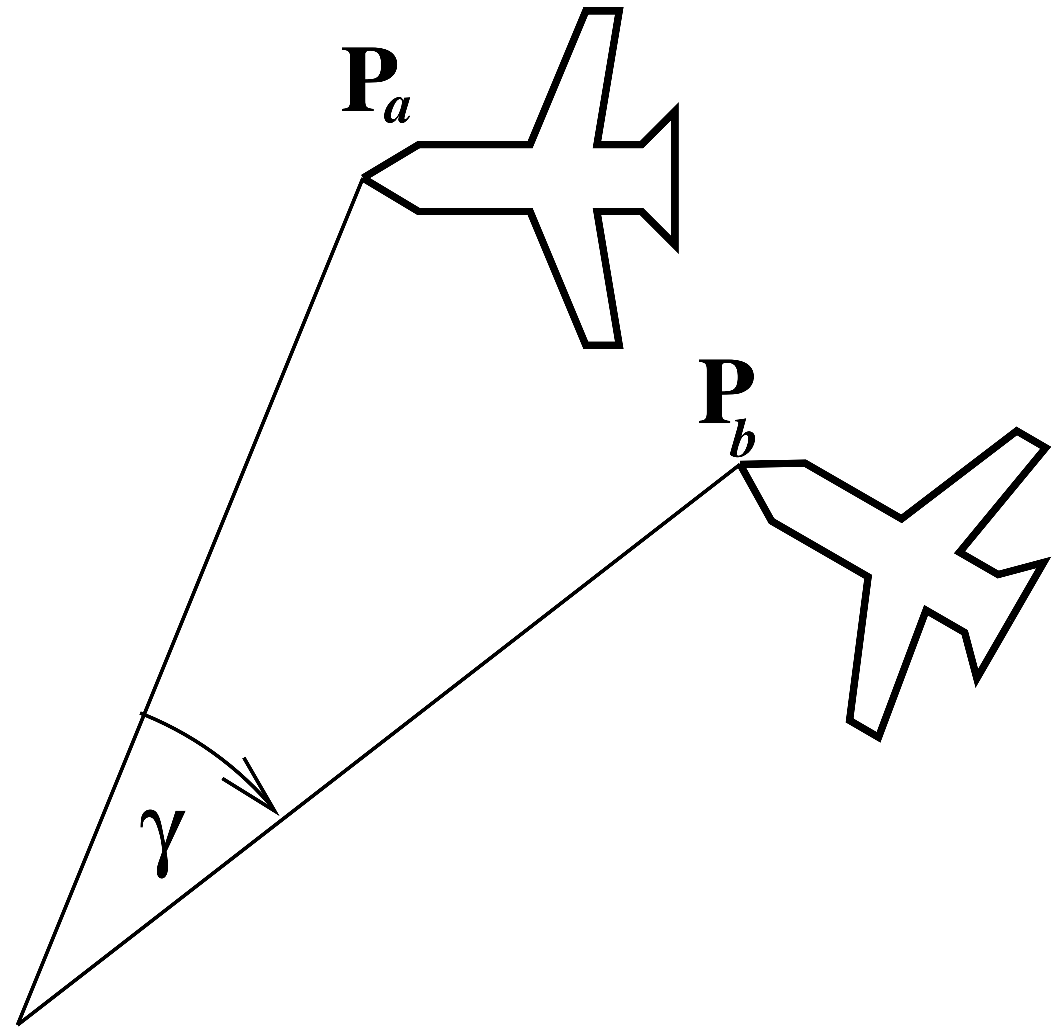
y_a
y_b
P
x_b
γ
x_a



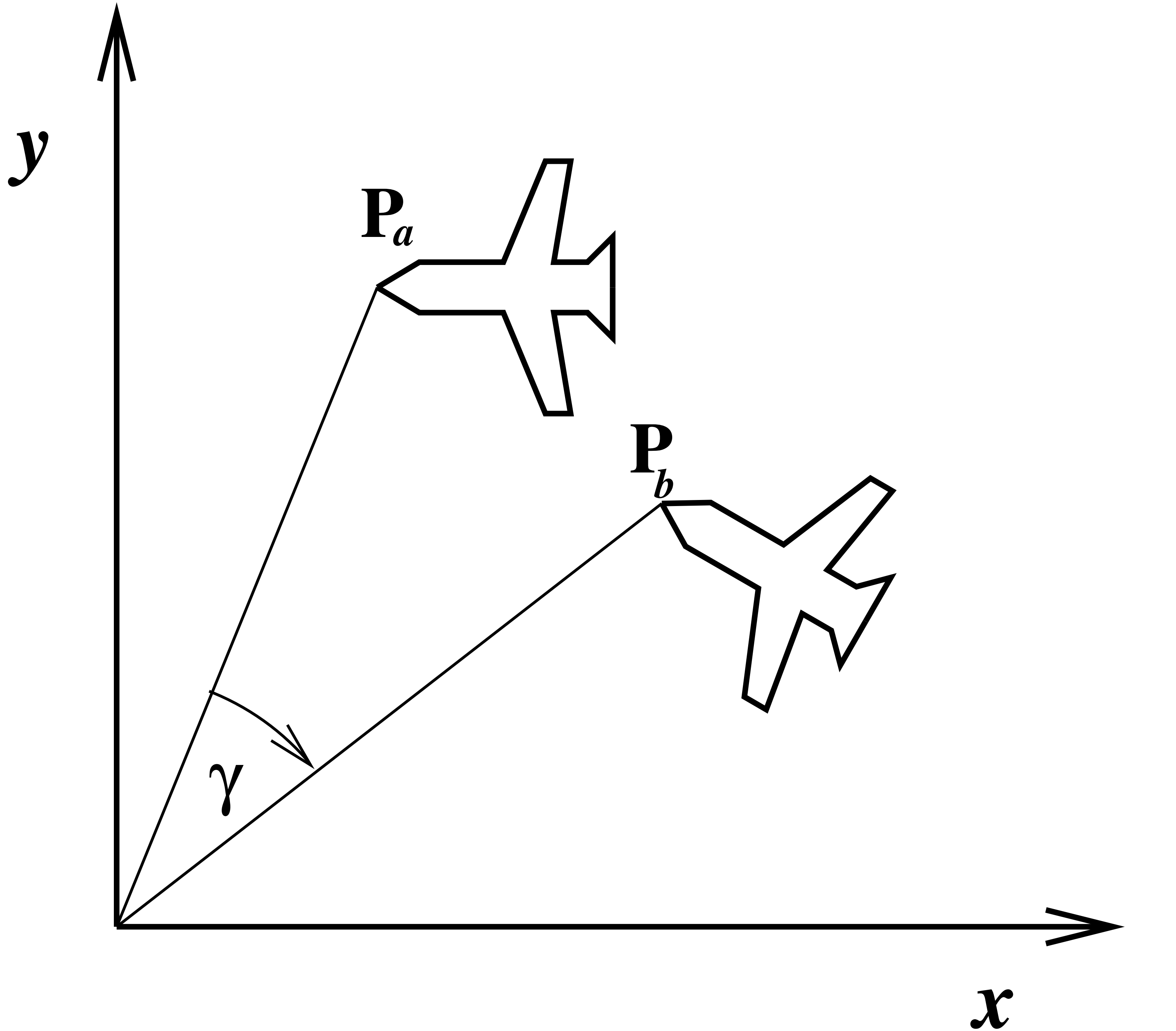
y_a
y_b
P
y_a
y_b
x_b
x_b
γ
x_a
x_a

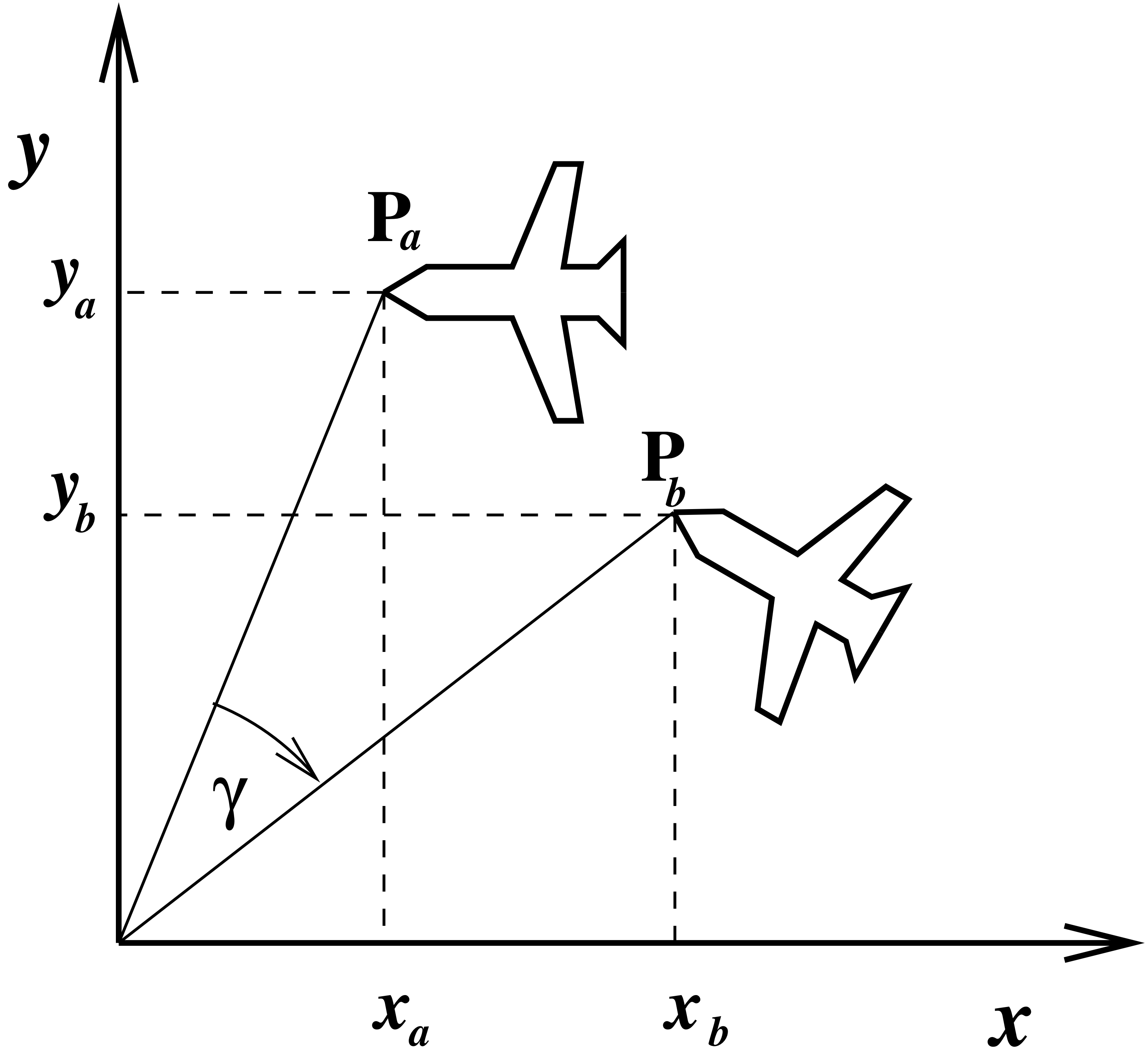


$\mathbf{P}_a$
$\mathbf{P}_b$

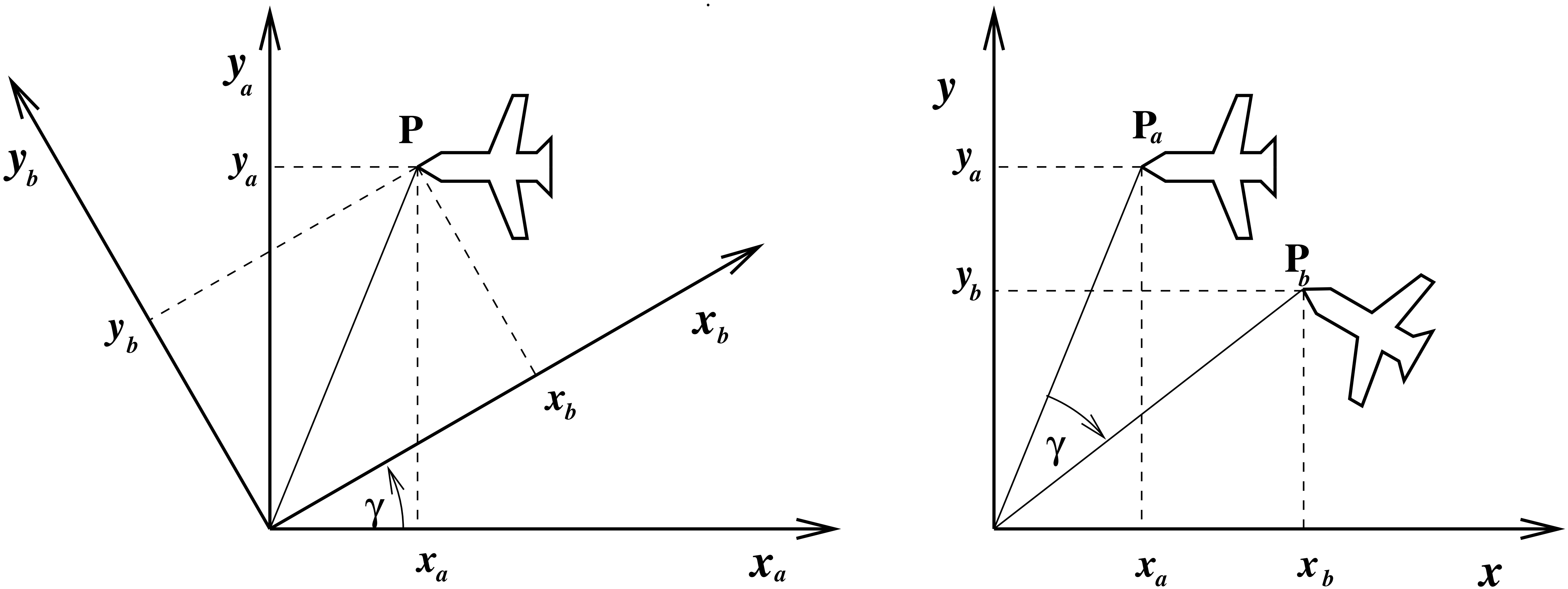


$\mathbf{P}_a$
$\mathbf{P}_b$
γ


y
$\mathbf{P}_a$
$\mathbf{P}_b$
γ
x



y
P_a
y_a
P_b
y_b
γ
x_a
x_b
x



y_a
y_b
P
y_a
y_b
x_b
x_b
γ
x_a
x_a
y
P_a
y_a
P_b
y_b
γ
x_a
x_b
x