Robotics

Inverse Kinematics of 6–DOF Serial Manipulator

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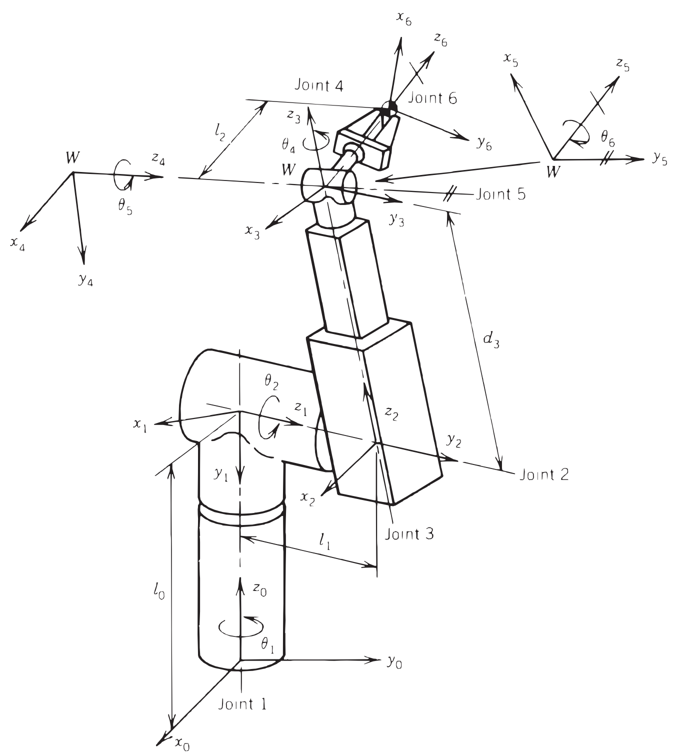
Inverse kinematics of 6-DOF manipulator with three intersecting axes



One has to understand that a single point could have different coordinates in diferent coordinate system. In certain system the coordinates could be trivial, in other it shall be calculated. Let us introduce the notation the point P has the coordinates P^j in j-th coordinate system.

For solving the task, one has to make few observations:

$$\mathbf{O}_j^j = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \tag{1}$$

$$\mathbf{P}^i = \mathbf{A}_j^i \mathbf{P}^j, \tag{2}$$

$$\mathbf{T} = \mathbf{A}_1^0(q_1)\mathbf{A}_2^1(q_2)\mathbf{A}_3^2(q_3)\mathbf{A}_4^3(q_4)\dots\mathbf{A}_n^{n-1}(q_n), \tag{3}$$

$$\mathbf{A}_j^i = (\mathbf{A}_i^0)^{-1}\,\mathbf{T}(\mathbf{A}_n^j)^{-1}, \tag{4}$$

Let

$$\mathbf{A}_j^i = \left[\begin{array}{cccc} & & & t_{jx}^i \\ & \mathbf{R}_j^i & & t_{jy}^i \\ & & & t_{jz}^i \\ 0 & 0 & 0 & 1 \end{array}\right]. \tag{5}$$

Then its inverse:

$$(\mathbf{A}_j^i)^{-1} = \left[\begin{array}{cccc} & \mathbf{R}_j^{iT} & & -\mathbf{R}_j^{iT}\vec{t}_j^i \\ 0 & 0 & 0 & 1 \end{array}\right]. \tag{6}$$

Correspondingly rotation matrices for rotating vectors can be composed and the order could changed by similar formulas. Note that vectors are not in homogeneous coordinates. Vector coordinates can be calculated by well known formula from vector algebra as a difference of euclidean coordinates of two points, defining the vector. The vector is denoted here as $\vec{r}$.

$$\vec{r}^i = \mathbf{P}_{e2}^i - \mathbf{P}_{e1}^i, \tag{7}$$

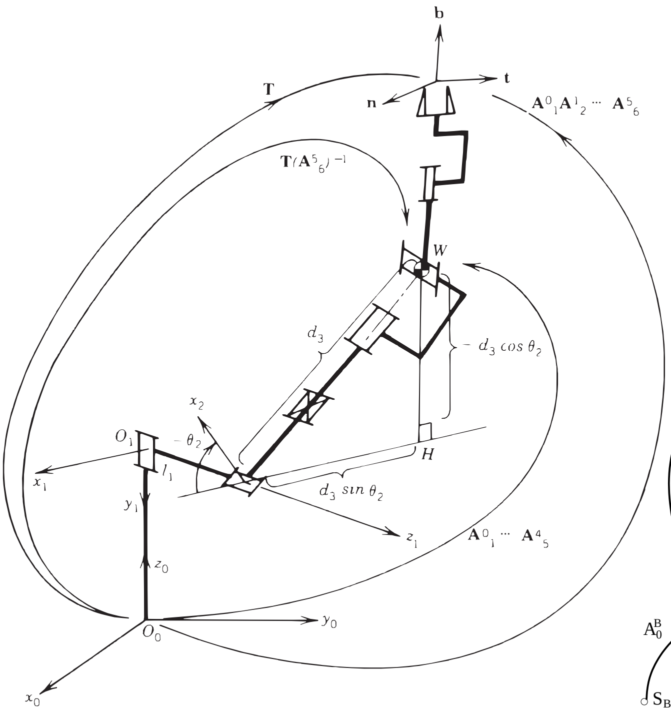
$$\mathbf{P}_{e2}^i = \mathbf{P}_{e1}^i + \vec{r}^i, \tag{8}$$

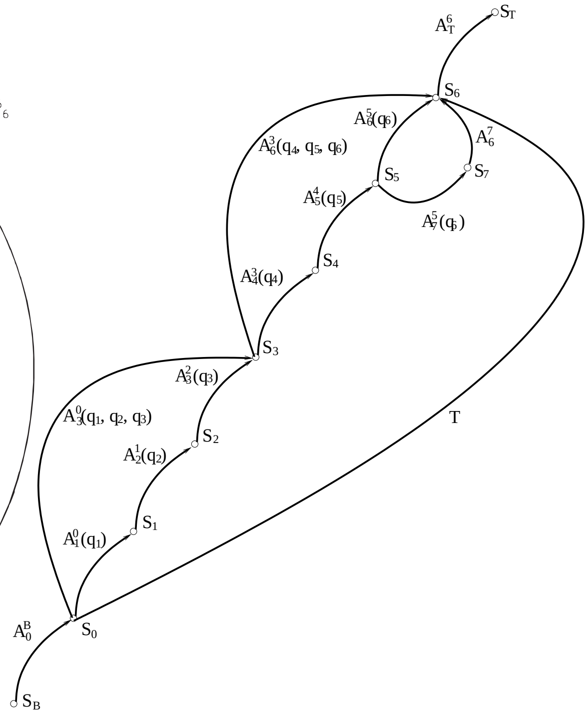
$$\vec{r}^i = \mathbf{R}_j^i \vec{r}^j, \tag{9}$$

$$\mathbf{R} = \mathbf{R}_1^0(q_1)\mathbf{R}_2^1(q_2)\mathbf{R}_3^2(q_3)\mathbf{R}_4^3(q_4)\dots\mathbf{R}_n^{n-1}(q_n), \tag{10}$$

$$\mathbf{R}_j^i = (\mathbf{R}_i^0)^{-1}\,\mathbf{R}(\mathbf{R}_n^j)^{-1}. \tag{11}$$

Inverse kinematics of 6-DOF manipulator with three intersecting axes





Inverse kinematics

Inverse kinematics means, we are given position of the end efector $\mathbf{T}$ and we have to find joint coordinate vector $\vec{q}$. We will use in our reasoning following assumptions:

1. The manipulator is serial (open kinematic chain). This transforms into the equation (3), where each transformation matrix is a function of one joint variable.
2. The manipulator has 6 DOF. With less DOF we will not be able to reach arbitrary position and orientation even in some limited 6D working space. With more DOF one will get infinitely many solutions within some 6D working space.
3. The manipulator has three consecutive intersecting revolute joints near the end effector of the manipulator. This is a common situation, the robot has a wrist allowing any orientation of manipulated object.

To solve inverse kinematics of the 6-DOF serial manipulator with three consecutive intersecting axes we could use following reasoning. Let us have the three intersecting axes 4, 5, 6. The intersection of axes z_4, z_5, z_6 is marked as point $\mathbf{W}$. Using above simple formulas we can compute inverse kinematics this way (next page). It is given $\mathbf{T}$ or equivalent information e.g. coordinates of a center of gripper and its orientation by quaternion. Let us calculate matrices $\mathbf{A}^i_{i+1}$ for $i = 0..5$ symbolically using D-H notation. For coordinates of the point W then holds:

$$\mathbf{W}^3 = \mathbf{O}^3_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \tag{12}$$

$$\mathbf{W}^4 = \mathbf{O}^4_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \tag{13}$$

$$\mathbf{W}^5 = \mathbf{O}^5_5 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{14}$$

Let us define auxiliary coordinate system O_7 by following conditions:

$$\mathbf{W}^7 = \mathbf{O}^7_7 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \tag{15}$$

$$\mathbf{R}^7_6 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{E} \tag{16}$$

that is $\mathbf{A}^5_7$ represents only rotation and $\mathbf{A}^7_6$ represents only translation:

$$\mathbf{A}^5_6(q_6) = \mathbf{A}^5_7(q_6)\mathbf{A}^7_6, \tag{17}$$

$$\mathbf{A}^5_7(q_6) = \begin{bmatrix} & \mathbf{R}^5_6(q_6) & & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{18}$$

$$\mathbf{A}^7_5(q_6) = (\mathbf{A}^5_7(q_6))^{-1} = \begin{bmatrix} & \mathbf{R}^5_6(q_6)^T & & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{19}$$

$$\mathbf{A}_6^7 = \begin{bmatrix} 1 & 0 & 0 & t_{6x}^7 \\ 0 & 1 & 0 & t_{6y}^7 \\ 0 & 0 & 1 & t_{6z}^7 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{20}$$

$$\mathbf{A}_7^6 = (\mathbf{A}_6^7)^{-1} = \begin{bmatrix} 1 & 0 & 0 & -t_{6x}^7 \\ 0 & 1 & 0 & -t_{6y}^7 \\ 0 & 0 & 1 & -t_{6z}^7 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{21}$$

$$\mathbf{A}_6^5 = \mathbf{A}_7^5(q_6)\mathbf{A}_6^7 = \begin{bmatrix} & \mathbf{R}_6^5(q_6) & & \mathbf{R}_6^5(q_6)\mathbf{t}_6^7 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{22}$$

$\mathbf{A}_7^6$ is not a function of q_6. $\mathbf{A}_6^5$ is known symbolically from D-H notation or the decomposition could be done from geometrical analysis.

Inversion using Eq. 6

$$\mathbf{A}_5^6(q_6) = (\mathbf{A}_6^5(q_6))^{-1} = \tag{23}$$

$$= \begin{bmatrix} & \mathbf{R}_6^5(q_6)^T & & -\mathbf{R}_6^5(q_6)^T\mathbf{R}_6^5(q_6)\mathbf{t}_6^7 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \tag{24}$$

$$= \begin{bmatrix} & \mathbf{R}_6^5(q_6)^T & & -\mathbf{t}_6^7 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \tag{25}$$

$$= \begin{bmatrix} 1 & 0 & 0 & -t_{6x}^7 \\ 0 & 1 & 0 & -t_{6y}^7 \\ 0 & 0 & 1 & -t_{6z}^7 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} & & & 0 \\ & \mathbf{R}_6^5(q_6)^T & & 0 \\ & & & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \tag{26}$$

$$= \mathbf{A}_7^6\mathbf{A}_5^7(q_6). \tag{27}$$

Note that $\mathbf{A}_5^6(q_6)$ could be easily factorized into two matrices, one independent of q_6.

Thus $\mathbf{A}_7^6$ could be found by comparison in the above formula. Transformations of the point $\mathbf{W}$:

$$\mathbf{W}^6 = \mathbf{A}_7^6\,\mathbf{W}^7, \tag{28}$$

$$\mathbf{W}^0 = \mathbf{T}\,\mathbf{W}^6. \tag{29}$$

Solving following equation for q_1, q_2, q_3:

$$\mathbf{W}^0 = \mathbf{A}_1^0(q_1)\mathbf{A}_2^1(q_2)\mathbf{A}_3^2(q_3)\mathbf{W}^3, \tag{30}$$

$$\mathbf{A}_3^0 = \mathbf{A}_1^0(q_1)\mathbf{A}_2^1(q_2)\mathbf{A}_3^2(q_3) \tag{31}$$

is then known, as this is equivalent to 3-DOF manipulator IKT solved in lab previously.

Then our task is

$$(\mathbf{A}_3^0)^{-1}\mathbf{T}(\mathbf{A}_6^7)^{-1} = \mathbf{A}_4^3(q_4)\mathbf{A}_5^4(q_5)\mathbf{A}_7^5(q_6). \tag{32}$$

Note that q_4, q_5, q_6 are revolute joints. As $\mathbf{O}_4 = \mathbf{O}_5 = \mathbf{O}_7$, we can forget about translation part of the transformations matrices and simplify (32) into

$$(\mathbf{R}_3^0)^{-1}\mathbf{R}_T(\mathbf{R}_6^7)^{-1} = \mathbf{R}_4^3(q_4)\mathbf{R}_5^4(q_5)\mathbf{R}_7^5(q_6). \tag{33}$$

After simplification:

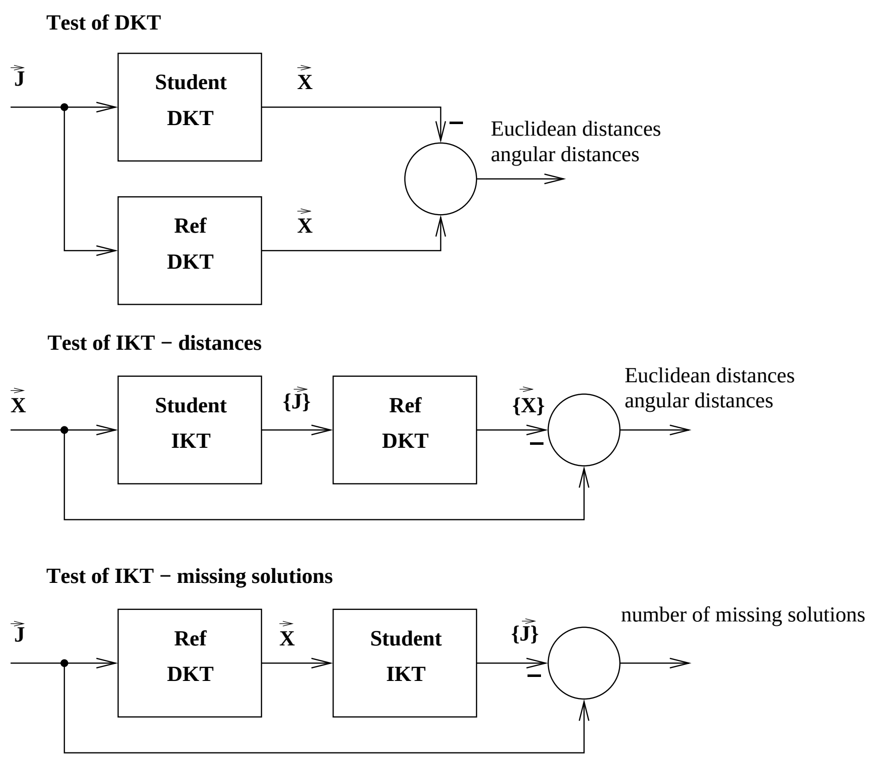
$$\mathbf{R}_3^{0T}\mathbf{R}_T = \mathbf{R}_4^3(q_4)\mathbf{R}_5^4(q_5)\mathbf{R}_6^5(q_6). \tag{34}$$

Solving (34) will give us q_4, q_5, q_6. This can be achieved by comparison with the rotation matrix of Euler angles. Then the IKT is solved.

Cookbook

1. Factorize $\mathbf{A}_5^6(q_6)$ into $\mathbf{A}_7^6\mathbf{A}_5^7(q_6)$.
2. Identify $\mathbf{t}_6^7$ and $\mathbf{R}_6^5(q_6)$.
3. Calculate $\mathbf{W}^0 = \mathbf{T}\,\mathbf{A}_7^6 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$.
4. Using lower 3 DOF inverse kinematics, solve (30) for q_1, q_2, q_3.
5. Solve equation (34) for q_4, q_5, q_6.

How to check direct and inverse kinematic task



To check DKT and IKT, we need to have some kind of ground truth. The ground truth could be:

1. **IKT in robot's control unit** When we have a working robot and we need to make its mathematical model we could compare results from mathematical model with real robot
2. **Measured positions on the real robot** When we are designing control unit for new robot, we could measure certain positions in space and then check, if tested IKT will move the robot to the required positions.
3. **DKT from mathematical model** Mathematical model of DKT is for open kinematic chain much easier to design as we have tools like Denavit–Hartenberg notation. Then we could compare IKT and DKT in closed loop. Similarly testing of DKT of parallel manipulators could be done using IKT which could be easier to model.

In all cases we should have in mind, that we are not mathematicaly proving the IKT is correct but testing it on the set of positions. The fundamental problem is selection of these test positions. The robot with 6-DOF have upto 16 configurations (solutions) for each position. Further the robot is working in 6-D space with 2^6 "quadrants". It is recommended to represent all those "quadrants" in testing set to get reasonable certainty that IKT is not working just for e.g. positive values in joint 3.

Another problem is to check, if all configurations are properly generated.

We recommend to use the following methodology, which is also applied for check of your home assignments (see Fig.):

1. If there is ground truth or reliable DKT available, one can compare tested IKT and ground truth DKT.

2. We insert positions into IKT which could or need not be reachable. All generated solutions (different configurations) are inserted into DKT and the results compared with original positions.

3. We insert reachable joint coordinates into ground truth DKT. The reachable joint coordinates usually could be found in robot's manual or from design parameters, they often form n-dimensional interval. The output positions is inserted into tested IKT and the output shall contain orginal joint coordinates among other configurations.

One has to mention the method for evaluation of the differences between reference and calculated solution. If we calculate the difference between positions we actually have two coordinate transformation between base and end effector coordinate systems. The translation part of the transformation could be easily compared using standard euclidean distance. The rotation part of the transformation is easy to compare in 2-D case, where difference between angles calculated modulo 2π is the rotation from one end-efector coordinate system to the other. In spatial case we have two rotational matrices R_1, R_2. We will calculate the rotation from reference end-efector coordinate system to calculated end-efector coordinate system $R = R_1 R_2^{-1}$, and then we convert it into the axis-angle description. The angle is then evaluated and is expected to be close to zero.