Robotics

Forward and Inverse Kinematics of Serial Manipulators

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Direct and inverse kinematics

Robot usually directly measures its inner kinematic parameters - joint coordinates. Those coordinates measure the position of joints. We denote them usually as $\vec{q}$, joint coordinate of the revolute joint is denoted as $\theta$, joint coordinate of the prismatic joint is denoted as $d$.

User is interested in the position of the end effector or the position of the manipulated rigid body. It has 6 DOF and it could be described in number of ways, e.g. by the transformation matrix describing position of the end effector coordinate system in the world coordinate system.

We are interested in the mapping between those two descriptions of the robot position.

Students often confuse position of the end effector (6 DOF) with the position of the center of the gripper (point - has 3 DOF). This is a crucial error as orientation of the gripper is important for both manipulation or operation (e.g. welding).
Direct (forward) kinematics is a mapping from joint coordinate space to space of end-effector positions. That is we know the position of all (or some) individual joints and we are looking for the position of the end effector. Mathematically:

\[ \vec{q} \rightarrow \mathbf{T}(\vec{q}) \]

Direct kinematics could be immediately used in coordinate measurement systems. Sensors in the joints will inform us about the relative position of the links, joint coordinates. The goal is to calculate the position of the reference point of the measuring system.

Let us emphasize, that direct kinematics is mapping, not a mathematical function. It could have none, one, several, or infinitely many solutions for particular \( \vec{q} \).
Inverse kinematics

Inverse kinematics is a mapping from space of end-effector positions to joint coordinate space. That is we know the position of the end effector and we are looking for the coordinates of all individual joints. Mathematically:

\[ T \rightarrow \vec{q}(T) \]

Inverse kinematics is needed in robot control, one knows the required position of the gripper, but for control the joint coordinates are needed.

Let us emphasize, that inverse kinematics is mapping, not a mathematical function. It could have none, one, several, or infinitely many solutions for particular \( \vec{q} \).
Open kinematic chain modelling

Open kinematic chain is formed by the sequence of links connected by joints. If we know the description joints using geometrical transformations we can easily find transformation of point coordinates from end effector coordinate system to base coordinate system and vice versa. This transformation is called kinematic equations.
Open kinematic chain modeling in plane

Homogeneous coordinate system transformation could be described by transformation matrix $A$

$$ x = Ax^b, $$

$$ A = \begin{bmatrix} R & x_o \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos(\phi) & -\sin(\phi) & x_o \\ \sin(\phi) & \cos(\phi) & y_o \end{bmatrix}, $$

where $\phi$ is relative rotation of second coordinate system to first coordinate system. Inverse matrix is immediately:

$$ A^{-1} = \begin{bmatrix} R^T & -R^T x_o \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos(\phi) & \sin(\phi) & -\cos(\phi)x_o - \sin(\phi)y_o \\ -\sin(\phi) & \cos(\phi) & \sin(\phi)x_o - \cos(\phi)y_o \end{bmatrix}, $$

The simple serial planar manipulator is shown on the figure. The manipulator consist of the revolute joint (joint variable $\theta_1$), link of the length $l_1$, then there is a prismatic joint with the joint variable $d_2$, which is tilted by angle $\alpha_2$. Consequent joint is revolute $\theta_3$. At the end of the link with the length $l_3$ is gripper. The angle of the gripper to the base coordinate frame is denoted as $\phi$, the origin of the gripper in the base coordinate system is $G_0 = (x_0, y_0)^T$.

We shall assign coordinate systems to each end of the link, the transformations between coordinate systems will then have the simplest form of either pure translation (prismatic joint) or pure rotation, revolute joint. Starting with the base coordinate system $0$, rotating by $\theta_1$, translation by $l_1$, rotation by $\alpha_2$, translation by $d_2$, rotation by $\theta_3$, and translation by $l_3$. The individual transformation matrices will look like:

$$ A_0^1 = \begin{bmatrix} \cos(\theta_1) & -\sin(\theta_1) & 0 \\ \sin(\theta_1) & \cos(\theta_1) & 0 \end{bmatrix}, $$

$$ A_1^2 = \begin{bmatrix} 1 & 0 & l_1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, $$

$$ A_2^3 = \begin{bmatrix} \cos(\alpha_2) & -\sin(\alpha_2) & 0 \\ \sin(\alpha_2) & \cos(\alpha_2) & 0 \end{bmatrix}, $$

$$ A_3^4 = \begin{bmatrix} 1 & 0 & d_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, $$

$$ A_4^5 = \begin{bmatrix} \cos(\theta_2) & -\sin(\theta_2) & 0 \\ \sin(\theta_2) & \cos(\theta_2) & 0 \end{bmatrix}, $$

$$ A_5^6 = \begin{bmatrix} 1 & 0 & l_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, $$

$$ x_0 = A_0^1 A_1^2 A_2^3 A_3^4 A_4^5 A_5^6 x_6. $$
Transversal of two lines is the shortest line connecting the point on one line with the point on the other line.

Relative position of two straight lines in space could be:

- coincident lines, both end points of degenerate transversal could be placed to any point on the line,
- parallel lines, transversal could be placed anywhere along the parallel lines,
- crossing lines, degenerate transversal is located in the intersection of lines,
- nonparallel and nonintersecting lines, the transversal is perpendicular to both lines.
Unique and efficient description of transformations can be found by the method Denavit-Hartenberg. The description is then called Denavit-Hartenberg notation.

Euler’s theorem about the existence of motion axis says roughly that each motion in 3D space could be represented as a composition of rotation around certain axis and translation around the same axis. This theorem allows to formulate the algorithm for forward kinematics of open kinematic chains. D–H notation is just one of those formalisms. D–H notation is based on the idea of mathematical induction. Therefore **D–H notation could be used only for open kinematic chains** (think about why).

We describe the joint $i$.

1. Find the axes of rotation of joints $i - 1$, $i$, $i + 1$.
2. Find the common normal of joint axes $i - 1$ and $i$ and axes of joints $i$ a $i + 1$.
3. Find points $O_{i-1}$, $H_i$, $O_i$.
4. Axis $z_i$ shall be placed into the axis of the joint $i + 1$.
5. Axis $x_i$ shall be placed into the common normal $H_iO_i$.
6. Axis $y_i$ forms with the other axes right-hand coordinate system.
7. Name the distance of points $|O_{i-1}, H_i| = d_i$.
8. Name the distance of points $|H_i, O_i| = a_i$.
9. Name the angle between common normals $\theta_i$.
10. Name the angle between axes $i$, $i + 1$ $\alpha_i$.
11. The origin of a base coordinate system $O_o$ can be placed anywhere on the joint axis and axis $x_0$ can be oriented arbitrarily. For example to get $d_1 = 0$.
12. The origin $O_n$ of the end effector coordinate system and orientation of the axis $z_n$ can be placed arbitrarily when other rules hold.
13. When the axis of two consecutive joints are parallel, the common normal position can be placed arbitrarily, e.g. to get $d_i = 0$.
14. The position of joint axis can be arbitrarily chosen for prismatic joints.
Adjacent coordinate frames in DH
Position of end effector in base coordinate system
The transformation in the joint is uniquely described by four parameters $\theta_i$, $d_i$, $a_i$, $\alpha_i$. Parameters $a_i$, $\alpha_i$ are constant, one of the parameters $d_i$, $\theta_i$ is changing when the joint moves. The joints are usually:

- **Revolute**, then $d_i$ is constant and $\theta_i$ is changing,
- **Prismatic**, then $\theta_i$ is constant and $d_i$ is changing.

The matrix of transformation $A$ can be calculated as:

$$A^{-1}_i = A_{int}^{-1}A_{int}^1,$$

where

$$A_{int}^{-1} = \begin{bmatrix}
\cos \theta_i & -\sin \theta_i & 0 & 0 \\
\sin \theta_i & \cos \theta_i & 0 & 0 \\
0 & 0 & 1 & d_i \\
0 & 0 & 0 & 1
\end{bmatrix},$$

$$A_{int}^1 = \begin{bmatrix}
1 & 0 & 0 & a_i \\
0 & \cos \alpha_i & -\sin \alpha_i & 0 \\
0 & \sin \alpha_i & \cos \alpha_i & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}.$$

It can be shown, that:

$$A^{-1}_i = \begin{bmatrix}
\cos \theta_i & -\sin \theta_i & \sin \theta_i & \sin \alpha_i & a_i \cos \theta_i \\
\sin \theta_i & \cos \theta_i & \cos \theta_i & \sin \alpha_i & a_i \sin \theta_i \\
0 & \sin \alpha_i & \cos \alpha_i & 0 & 0 \\
0 & 0 & 0 & 1 & d_i
\end{bmatrix}.$$

Denote $q_i$, the parameter $\theta_i$, $d_i$, which is changing. Expression ?? can be redrawn to

$$x^0 = A_1^0(q_1)A_2^1(q_2)A_3^2(q_3)A_4^3(q_4) \ldots A_n^{n-1}(q_n)x^n.$$

For each value of the vector $q = (q_1, q_2, q_3, q_4, \ldots q_n) \in Q = \mathbb{R}^n$ we can calculate coordinates of point $P$ in base coordinate system from given $P$ coordinates in end effector coordinate system and vice versa.

*Kinematic equation are always solvable analytically for open kinematic chain.*
Denavit–Hartenberg

Matrix representing transformation from one link to the successive link

\[ A_{i-1}^i = \begin{bmatrix}
\cos \theta_i & -\sin \theta_i \cos \alpha_i & \sin \theta_i \sin \alpha_i & a_i \cos \theta_i \\
\sin \theta_i & \cos \theta_i \cos \alpha_i & -\cos \theta_i \sin \alpha_i & a_i \sin \theta_i \\
0 & \sin \alpha_i & \cos \alpha_i & d_i \\
0 & 0 & 0 & 1
\end{bmatrix}. \]
Inverse kinematics

We call inverse kinematics the task when there is given a matrix

\[ T(q) = A_0(q_1)A_1(q_2)A_2(q_3)A_3(q_4) \cdots A_n(q_n). \]

and we are looking for values of vector \( q \). The system of nonlinear equations (usually trigonometric) is typically not possible to solve analytically.

Solving inverse kinematics:

- Analytically, if possible. There is not a unique description, how to solve the system.
- Numerically.
- Look up table, precalculated for the working space \( W \subset Q \).

There are manipulator structures, which can be solved analytically. We call them solvable.

The sufficient condition of solvability is e.g. when the 6 DOF robot has three consecutive revolute joints with axes intersecting in one point.

The other property of inverse kinematics is ambiguity of solution in singular points. There is often the subspace \( Q_s \) of the space \( Q \), which gives the same \( T \).

\[ \forall q \in Q_s : T(q) = T \]

To decide which \( q \) solving \( T(q) = T \) to select has to be taken into account mainly:

1. Is the selected value \( q \) applicable, i.e. can the robot be sent to \( q \)?
2. How to reach the singular point. The function \( q \) shall always be a continuous function of time. The preceding values of \( q \) should make with the selected value continuous function of time.
3. How to continue from the singular point? The future values of \( q \) should form with the selected value continuous function of time.
4. Will not the selected value of \( q \) guide us to the situation where we will not be able to satisfy above conditions?
5. Will the required operational space limit us during manipulation? The example is the insertion of the seat into car.

We design sometimes redundant robots (with more degrees of freedom, e.g. 8), to increase the space \( Q_s \), from which we select \( q \) to allow more freedom to fulfill above requirements.

Think about following:

- Is it possible to design the robot with prismatic joints only which can position arbitrarily the rigid body in 3D space? Why?
- Choose some manipulating task and design the structure of redundant robot for it.
Inverse Kinematics - example
Inverse Kinematics - example

View A: Detail of mechanical interface

View D bottom view drawing: Detail of installation dimension

View C: Detail of screw holes for fixing wiring hookup

* Dimensions when installing a solenoid valve (optional)

* Dimensions when installing a solenoid valve (optional)

Machine cable

(For customer use)
Inverse Kinematics - example

Restriction on wide angle in the rear section

Note 1) $J_2 + J_3 \times 2 \geq -200$ degree when $-45$ degree $\leq J_2 < 15$ degree.

Note 2) $J_2 + J_3 \geq 8$ degree when $|J_1| \leq 75$ degree, $J_2 < -45$ degree.

Note 3) $J_2 + J_3 \geq -40$ degree when $|J_1| > 75$ degree, $J_2 < -45$ degree.

Restriction on wide angle in the front section

Note 4) $J_3 \geq -40$ degree when $-105$ degree $\leq J_1 \leq 95$ degree, $J_2 \geq 123$ degree.

Note 5) $J_2 \geq 110$ degree when $J_1 < -105$ degree, $J_1 < -95$ degree.

However, $J_2 - J_3 \leq 150$ degree when 95 degree $J_2 \leq 110$ degree.
Inverse Kinematics - example

Fig. 2-5: Operating range diagram: RV-6S/6SC

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Note 5) $J_2 \geq 110$ degree when $J_1 < -105$ degree, $J_1 < -95$ degree.

However, $J_2 - J_3 \leq 150$ degree when $85$ degree $\leq J_2 \leq 110$ degree.
Multiple configurations in inverse kinematics
The configuration flag indicates the robot posture. For the 6-axis type robot, the robot hand end is saved with the position data configured of X, Y, Z, A, B and C. However, even with the same position data, there are several postures that the robot can change to. The posture is expressed by this configuration flag, and the posture is saved with FL1 in the position constant (X, Y, Z, A, B, C) (FL1, FL2).

The types of configuration flags are shown below.

1. **RIGHT/LEFT**
   - Q is center of J5 axis rotation in comparison with the plane through the J2 axis vertical to the ground.
   - Fig. 6-1: Configuration flag (RIGHT/LEFT)

2. **ABOVE/BELOW**
   - Q is center of J5 axis rotation in comparison with the plane through both the J3 and the J2 axis.
   - Fig. 6-2: Configuration flag (ABOVE/BELOW)

3. **NONFLIP/FLIP** (6-axis robot only.)
   - This means in which side the J6 axis is in comparison with the plane through both the J4 and the J5 axis.
   - Fig. 6-3: Configuration flag (NONFLIP/FLIP)
2DOF planar manipulator with 2 revolute joints have two solutions within the circle, where it can reach. On the border of the circle there is a single solution, where two solutions basically coincide (compare 1 solution of quadratic equation), this border is singular surface, where one configuration can switch to the other configuration. Outside of the circle there is no solution. There is infinitely many solutions in the center of the circle, this is another singular point of the robot.

This planar manipulator has only 2 DOF but it operates in the 3D working space, that is e.g. $x$, $y$, and orientation of the gripper $\phi$. DOF deficiency thus causes that only some points in the working space are reachable, that is only some combinations of $(x, y, \phi)$. The picture of the working space shows the reachable points, green color represent first configuration, red color representing second configuration. The $\phi$ axis is a singular point, where any orientation is reachable, the boundary between green and red surface is also singular, where both solutions meet. Working space is shown here in the interval $< 0, 720^\circ >$, the spiral is actually from $-\infty$ to $\infty$.

It shall be stressed that ideally the working space shall occupy some compact but dense region, where all orientations of the end effector could be reached in all locations. Visualisation of six dimensional working space of spatial manipulator is of course difficult.
Manipulator with links of different length cannot reach near first joint.
Mapping between joint space and working space is for robot with revolute joints quite nonlinear.
**Direct and inverse kinematics - summary**

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<td>Hybrid</td>
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<td></td>
<td>Parallel</td>
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Number of solutions and difficulty to solve the particular kinematics for particular robot is given by the mathematical nature of the problem, the transformation is described by set of nonlinear equations, which has to be solved. The equations are basically polynomial in variables or their sines and cosines, goniometric functions causing the nonlinearity. The equations have in some cases unique solution, e.g. direct kinematics of the open kinematic chain (serial manipulator) and are relatively easily solvable. In other cases the task is not solvable analytically or its solution is not known. Numerical methods are used in such cases or such structures are avoided altogether.
The robot controller usually allows using a pendant interactive control of end-effector position in various coordinate systems:

- joint coordinates (standard),
- cartesian coordinates in world coordinate system (almost standard),
- cylindrical coordinate system in world coordinate system,
- cartesian coordinate system in end-effector coordinates system,...