Robotics

Forward and Inverse Kinematics of Serial Manipulators

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Forward and inverse kinematics

Robot usually directly measures its inner kinematic parameters - joint coordinates. Those coordinates measure the position of joints. We denote them usually as $\vec{q}$, joint coordinate of the revolute joint is denoted as θ, joint coordinate of the prismatic joint is denoted as d.

User is interested in the position of the end effector or the position of the manipulated rigid body. It has 6 DOF and it could be described in number of ways, e.g. by the transformation matrix describing position of the end effector coordinate system in the world coordinate system.

We are interested in the mapping between those two descriptions of the robot position.

Students often confuse position of the end effector (6 DOF) with the position of the center of the gripper (point - has 3 DOF). This is a crucial error as orientation of the gripper is important for both manipulation or operation (e.g. welding).

Forward kinematics

Forward kinematics is a mapping from joint coordinate space to space of end-effector positions. That is we know the position of all (or some) individual joints and we are looking for the position of the end effector. Mathematically:

$$\vec{q} \to \mathbf{T}(\vec{q})$$

Forward kinematics could be immediately used in coordinate measurement systems. Sensors in the joints will inform us about the relative position of the links, joint coordinates. The goal is to calculate the position of the reference point of the measuring system.

Let us emphasize, that forward kinematics is mapping, not a mathematical function. It could have none, one, several, or infinitely many solutions for particular $\vec{q}$.

Inverse kinematics

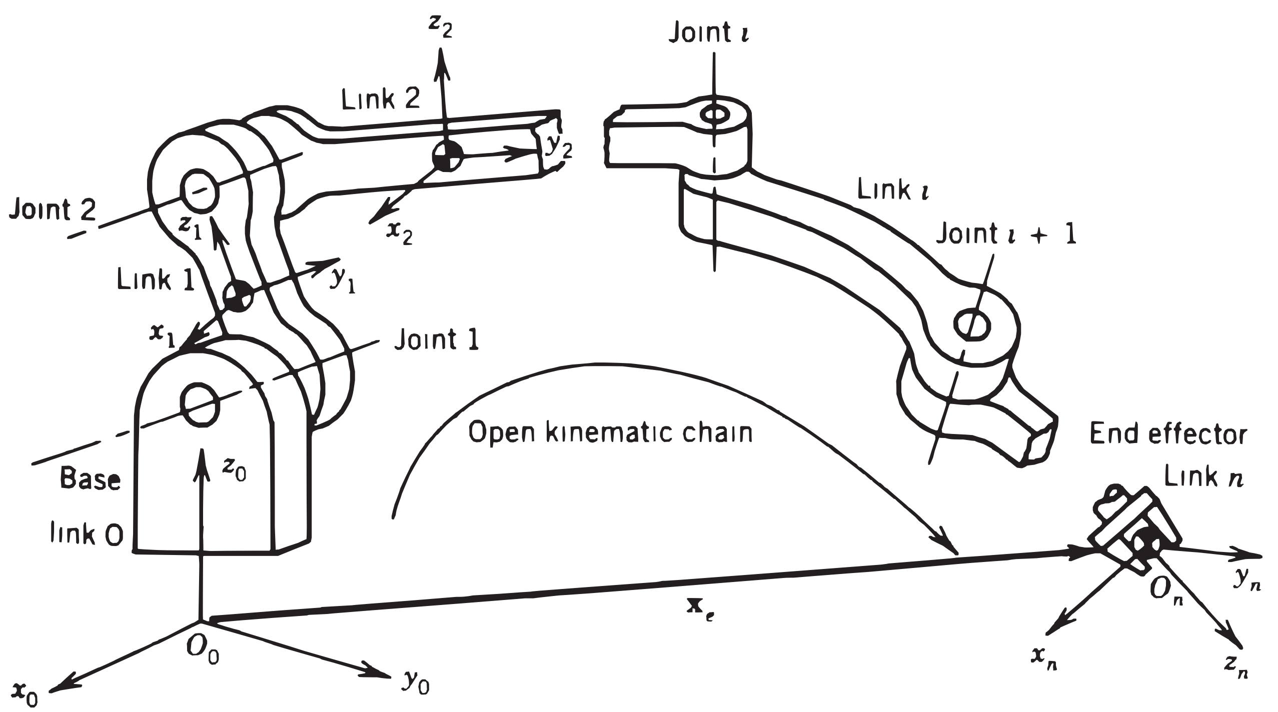
Inverse kinematics is a mapping from space of end-effector positions to joint coordinate space . That is we know the position of the end effector and we are looking for the coordinates of all individual joints. Mathematically:

$$\mathbf{T} \to \vec{q}(\mathbf{T})$$

Inverse kinematics is needed in robot control, one knows the required position of the gripper, but for control the joint coordinates are needed.

Let us emphasize, that inverse kinematics is mapping, not a mathematical function. It could have none, one, several, or infinitely many solutions for particular $\vec{q}$.

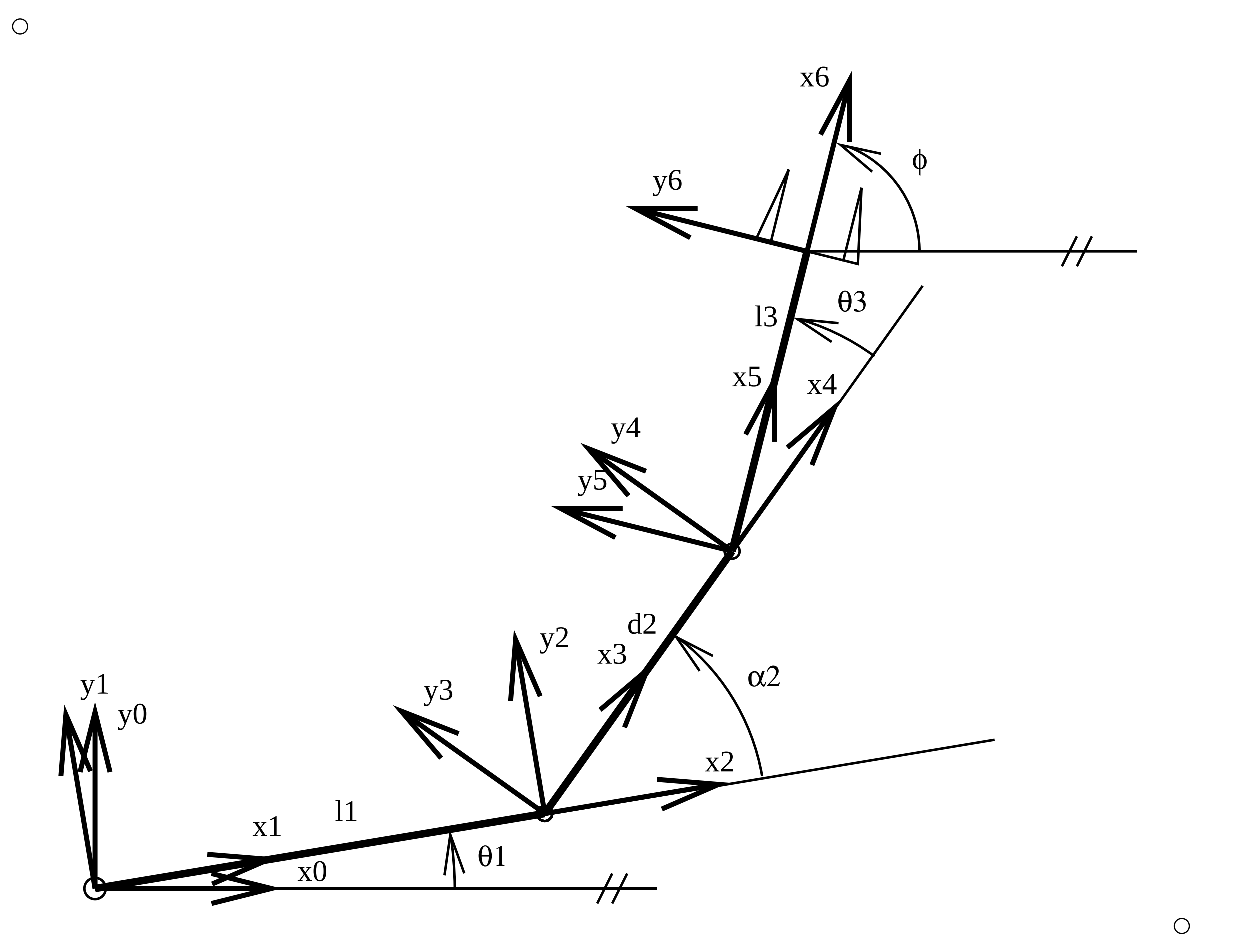
Open kinematic chain



Open kinematic chain modelling

Open kinematic chain is formed by the sequence of links connected by joints. If we know the description joints using geometrical tranformations we can easily find transformation of point coordinates from end effector coordinate system to base coordinate system and vice versa. This transformation is called kinematic equations.

Open kinematic chain modeling in plane



Homogeneous coordinate system transformation could be described by transformation matrix A

$$\mathbf{x} = \mathbf{A}\mathbf{x}^b,$$

$$\mathbf{A} = \begin{bmatrix} \mathbf{R} & \mathbf{x}_o \\ 0\,0 & 1 \end{bmatrix} = \begin{bmatrix} \cos(\phi) & -\sin(\phi) & x_o \\ \sin(\phi) & \cos(\phi) & y_o \\ 0 & 0 & 1 \end{bmatrix},$$

where ϕ is relative rotation of second coordinate system to first coordinate system. Inverse matrix is immediately:

$$\mathbf{A^{-1}} = \begin{bmatrix} \mathbf{R}^T & -\mathbf{R}^T\mathbf{x}_o \\ 0\,0 & 1 \end{bmatrix} = \tag{1}$$

$$= \begin{bmatrix} \cos(\phi) & \sin(\phi) & -\cos(\phi)x_o - \sin(\phi)y_o \\ -\sin(\phi) & \cos(\phi) & \sin(\phi)x_o - \cos(\phi)y_o \\ 0 & 0 & 1 \end{bmatrix}, \tag{2}$$

The simple serial planar manipulator is shown on the figure. The manipulator consist of the revolute joint (joint variable θ_1), link of the length l_1, then there is a prismatic joint with the joint variable d_2, which is tilted by angle α_2. Consequent joint is revolute θ_3. At the end of the link with the length l_3 is gripper. The angle of the gripper to the base coordinate frame is denoted as ϕ, the origin of the gripper in the base coordinate system is $G_0 = (x_0, y_0)^T$.

We shall assign coordinate systems to each end of the link, the transformations between coordinate systems will then have the simplest form of either pure translation (prismatic joint) or pure rotation, revolute joint. Starting with the base coordinate system 0, rotating by θ_1, translation by l_1, rotation by α_2, translation by d_2, rotation by θ_3, and translation by l_3. The individual transformation matrices will look like:

$$\mathbf{A}_1^0 = \begin{bmatrix} \cos(\theta_1) & -\sin(\theta_1) & 0 \\ \sin(\theta_1) & \cos(\theta_1) & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{3}$$

$$\mathbf{A}_2^1 = \begin{bmatrix} 1 & 0 & l_1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{4}$$

$$\mathbf{A}_3^2 = \begin{bmatrix} \cos(\alpha_2) & -\sin(\alpha_2) & 0 \\ \sin(\alpha_2) & \cos(\alpha_2) & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{5}$$

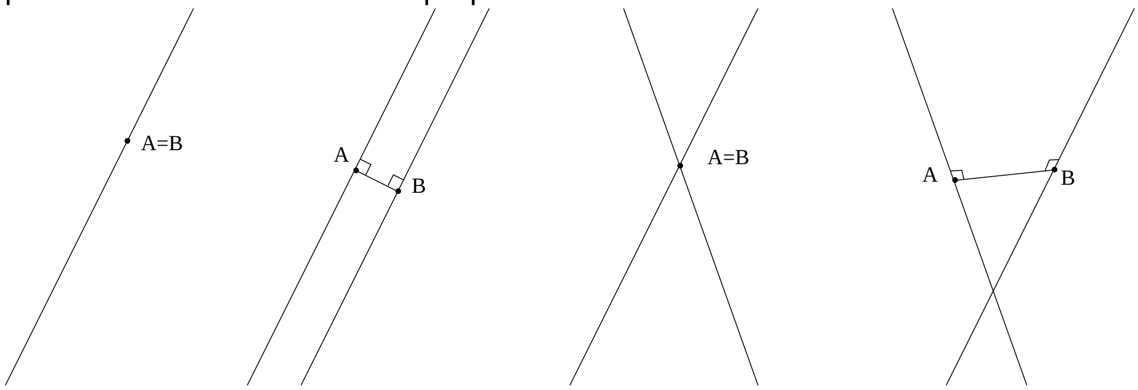
$$\mathbf{A}_4^3 = \begin{bmatrix} 1 & 0 & d_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{6}$$

$$\mathbf{A}_5^4 = \begin{bmatrix} \cos(\theta_2) & -\sin(\theta_2) & 0 \\ \sin(\theta_2) & \cos(\theta_2) & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{7}$$

$$\mathbf{A}_6^5 = \begin{bmatrix} 1 & 0 & l_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{8}$$

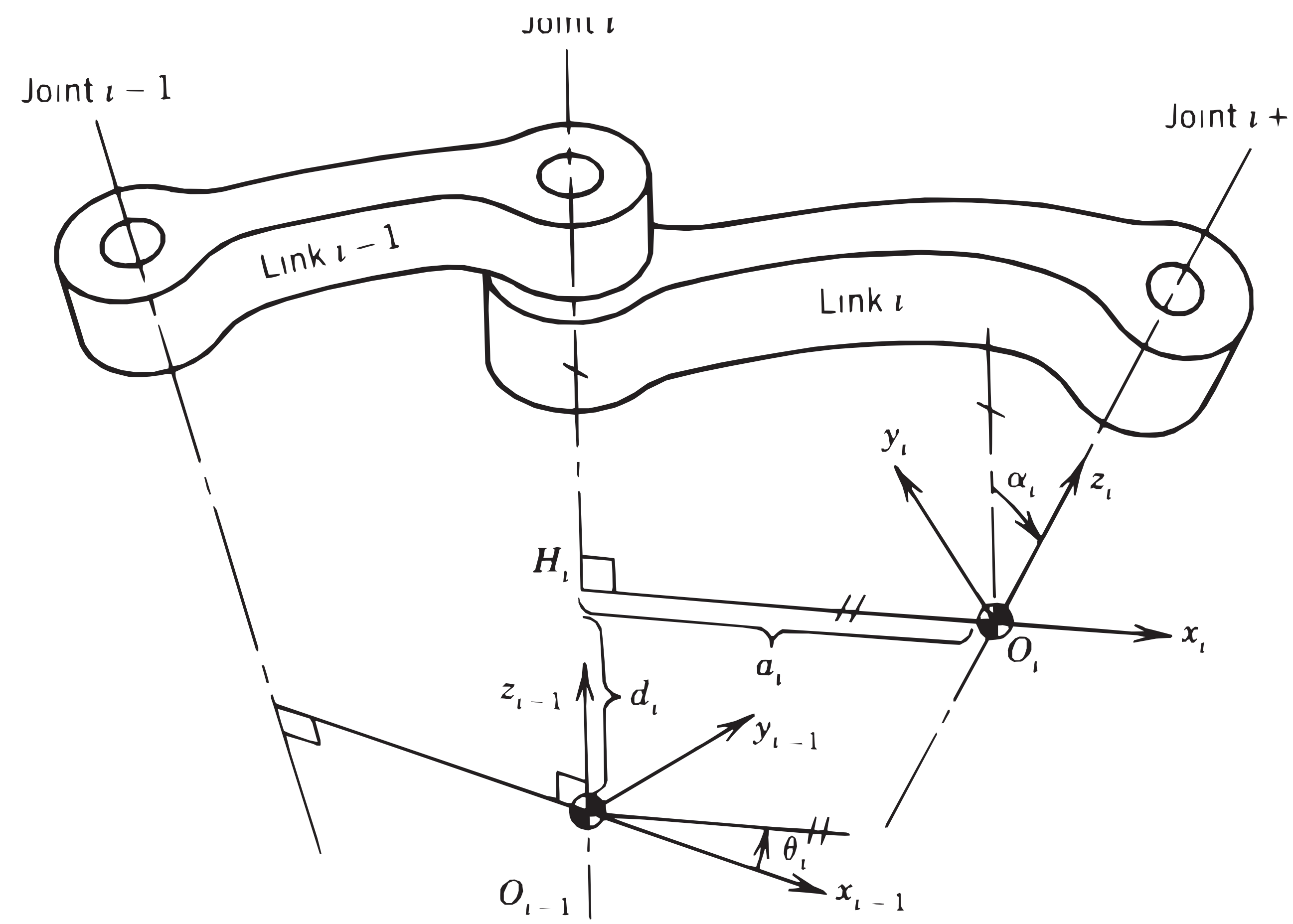
$$\mathbf{x}_0 = \mathbf{A}_1^0\mathbf{A}_2^1\mathbf{A}_3^2\mathbf{A}_4^3\mathbf{A}_5^4\mathbf{A}_6^5 x_6. \tag{9}$$

Transversal of two lines is the shortest line connecting the point on one line with the point on the other line. It is perpendicular to both.



Relative position of two straight lines in space could be:

- coincident lines, both end points of degenerate transversal can be placed to any point on the line,
- parallel lines, transversal can be placed anywhere along the parallel lines,
- crossing lines, degenerate transversal is located in the intersection of lines,
- nonparallel and nonintersecting lines.



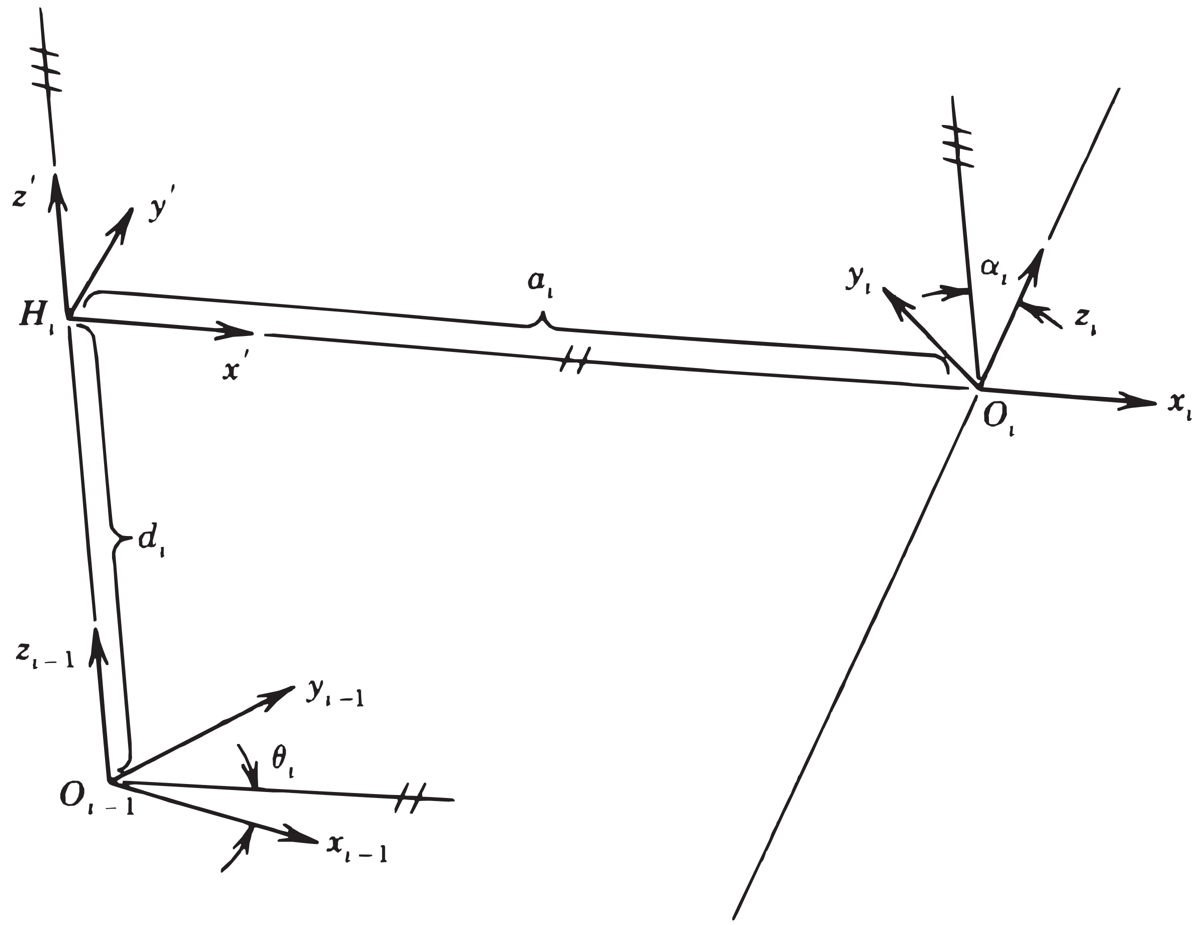
Unique and efficient description of transformations can be found by the method Denavit-Hartenberg. The description is then called Denavit-Hartenberg notation.

Euler's theorem about the existence of motion axis says roughly that each motion in 3D space could be represented as composition of rotation around certain axis and translation around the same axis. This theorem allows to formulate the algorithm for forward kinematics of open kinematic chains. D–H notation is just one of those fomalisms. D–H notation is based on the idea of mathematical induction. Therefore **D–H notation could be used only for open kinematic chains** (think about why).

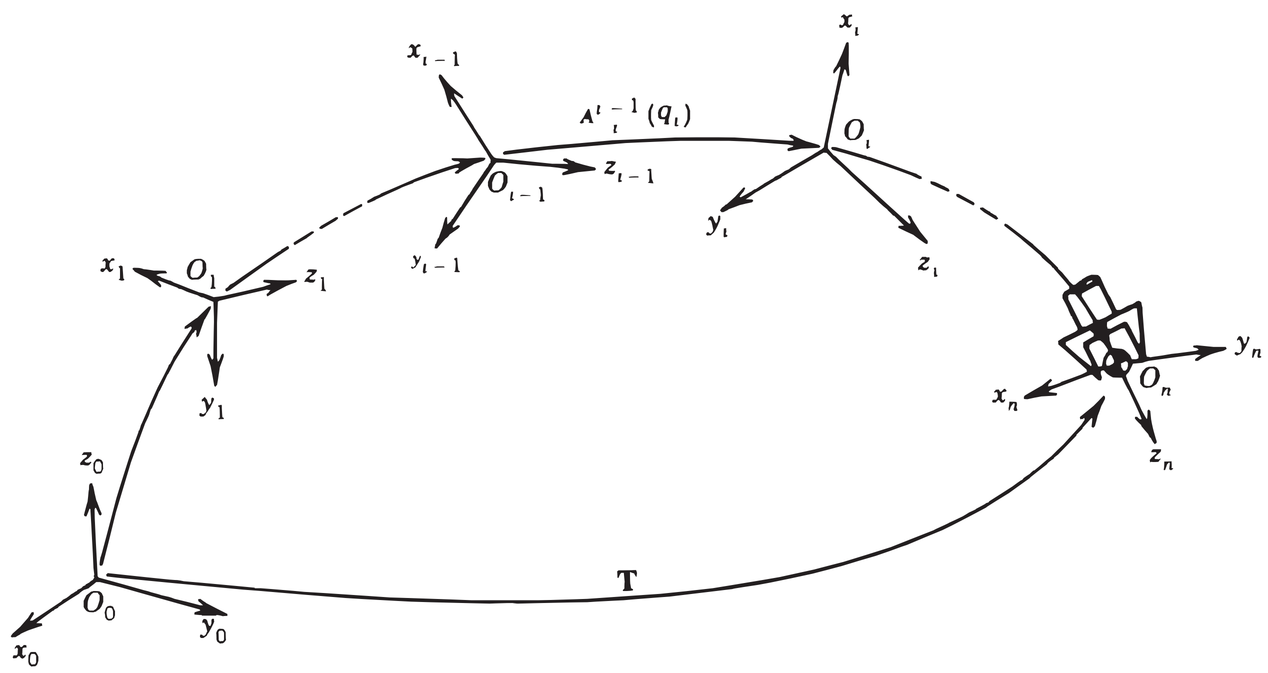
We describe the joint i.

1. Find the axes of rotation of joints $i-1$, i, $i+1$.
2. Find the common normal of joint axes $i-1$ and i and axes of joints i a $i+1$.
3. Find points O_{i-1}, H_i, O_i.
4. Axis z_i shall be placed into the axis of the joint $i+1$.
5. Axis x_i shall be placed into the common normal H_iO_i.
6. Axis y_i forms with the other axes right-hand coordinate system.
7. Name the distance of points $|O_{i-1}, H_i| = d_i$.
8. Name the distance of points $|H_i, O_i| = a_i$.
9. Name the angle between common normals θ_i.
10. Name the angle between axes i, $i+1$ α_i.
11. The origin of a base coordinate system O_o can be placed anywhere on the joint axis and axis x_0 can be oriented arbitrarily. For example to get $d_1 = 0$.
12. The origin O_n of the end effector coordinate system and orientation of the axis z_n can be placed arbitrarily when other rules hold.
13. When the axis of two consecutive joints are parallel, the common normal position can be placed arbitrarily, e.g. to get $d_i = 0$.
14. The position of joint axis can be arbitrarily chosen for prismatic joints.

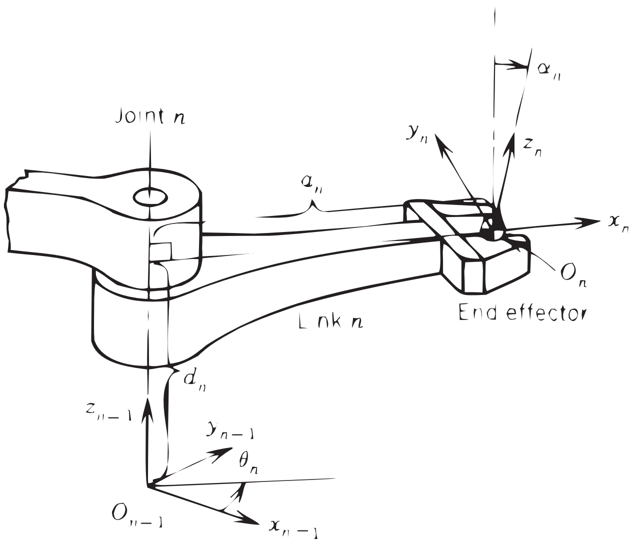
Adjacent coordinate frames in DH

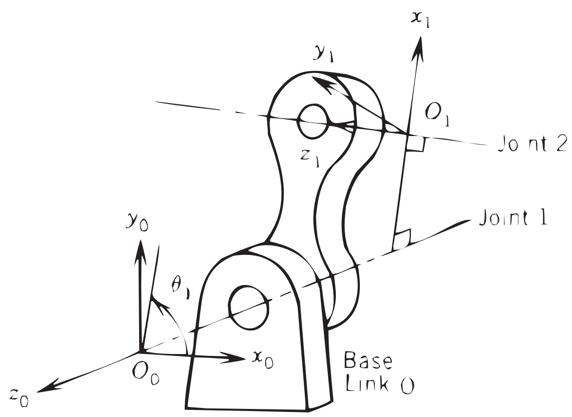


Position of end effector in base coordinate system

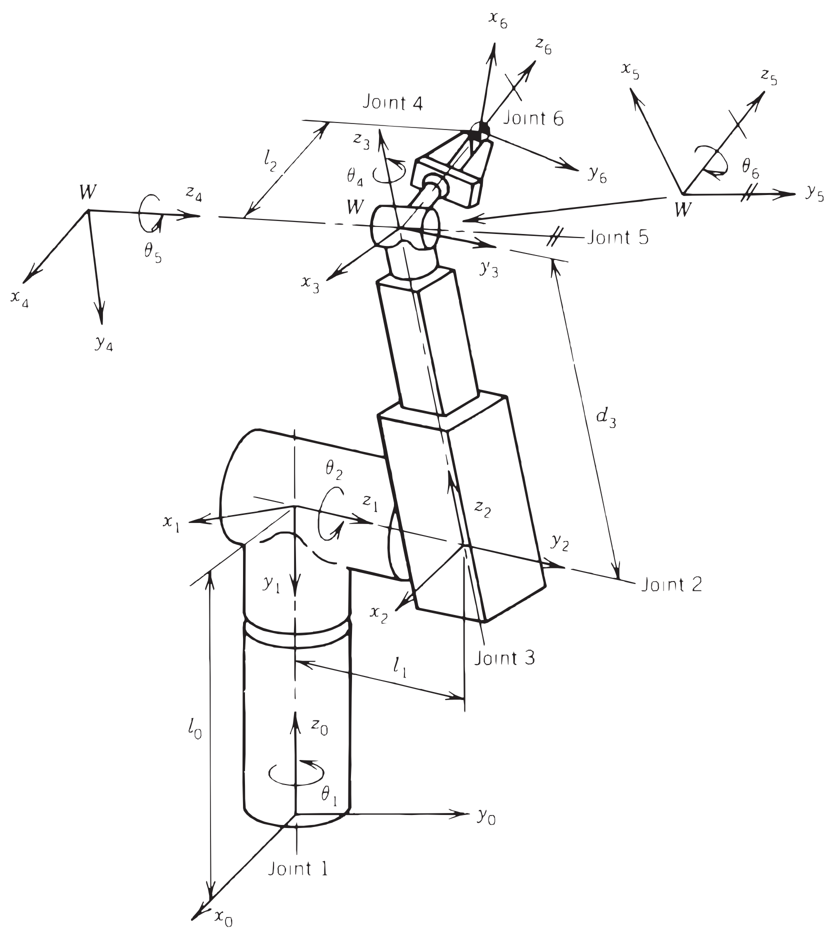


Base and end effector coordinate frames in DH





5-R-1-P manipulator



The tranformation in the joint is uniquelly described by four parameters θ_i, d_i, a_i, α_i. Parameters a_i, α_i are constant, one of the parameters d_i, θ_i is changing when the joint moves.

The joints are usually:

- **Revolute**, then d_i is constant and θ_i is changing,
- **Prismatic**, then θ_i is constant and d_i is changing.

The matrix of transformation **A** can be calculated

$$\mathbf{A}_i^{i-1} = \mathbf{A}_{int}^{i-1}\mathbf{A}_i^{int},$$

where

$$\mathbf{A}_{int}^{i-1} = \begin{bmatrix} \cos\theta_i & -\sin\theta_i & 0 & 0 \\ \sin\theta_i & \cos\theta_i & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\mathbf{A}_i^{int} = \begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & \cos\alpha_i & -\sin\alpha_i & 0 \\ 0 & \sin\alpha_i & \cos\alpha_i & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

It can be shown, that:

$$\mathbf{A}_i^{i-1} = \begin{bmatrix} \cos\theta_i & -\sin\theta_i\cos\alpha_i & \sin\theta_i\sin\alpha_i & a_i\cos\theta_i \\ \sin\theta_i & \cos\theta_i\cos\alpha_i & -\cos\theta_i\sin\alpha_i & a_i\sin\theta_i \\ 0 & \sin\alpha_i & \cos\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Denote q_i the parameter θ_i, d_i, which is changing. Expression 9 can be redrawn to

$$\mathbf{x}^0 = \mathbf{A}_1^0(q_1)\mathbf{A}_2^1(q_2)\mathbf{A}_3^2(q_3)\mathbf{A}_4^3(q_4)\dots\mathbf{A}_n^{n-1}(q_n)\mathbf{x}^n.$$

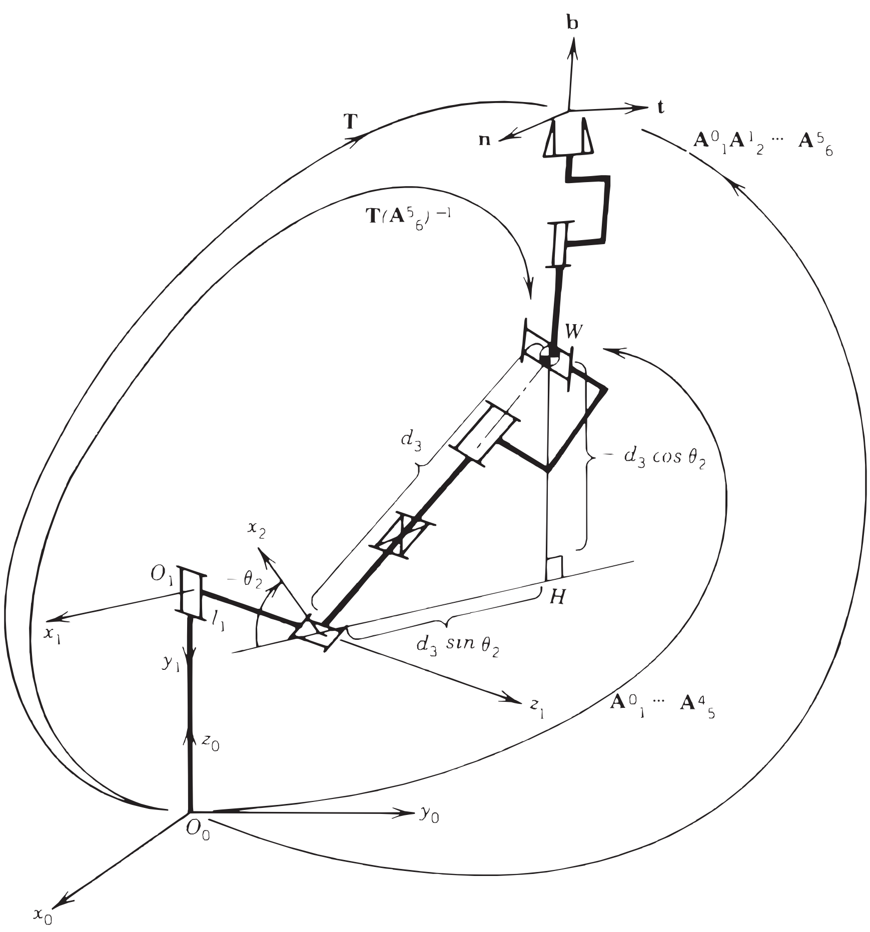
For each value of the vector $\mathbf{q} = (q_1, q_2, q_3, q_4, \dots q_n) \in \mathcal{Q} = \mathcal{R}^n$ we can calculate coordinates of point P in base coordinate system from given P coordinates in end effector coordinate system and vice versa.

Kinematic equation are always solvable analytically for open kinematic chain.

Matrix representing transformation from one link to the succesive link

$$\mathbf{A}_i^{i-1} = \begin{bmatrix} \cos\theta_i & -\sin\theta_i\cos\alpha_i & \sin\theta_i\sin\alpha_i & a_i\cos\theta_i \\ \sin\theta_i & \cos\theta_i\cos\alpha_i & -\cos\theta_i\sin\alpha_i & a_i\sin\theta_i \\ 0 & \sin\alpha_i & \cos\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Inverse kinematics of 5-R-1-P manipulator



Inverse kinematics

We call inverse kinematics the task when there is give a matrix

$$\mathbf{T}(\mathbf{q}) = \mathbf{A}_1^0(q_1)\mathbf{A}_2^1(q_2)\mathbf{A}_3^2(q_3)\mathbf{A}_4^3(q_4)\ldots\mathbf{A}_n^{n-1}(q_n). \quad (10)$$

and we are looking for values of vector $\mathbf{q}$. The system of nonlinear equations (usually trigonometric) is typically not possible to solve analytically.

Solving inverse kinematics:

- Analytically, if possible. There is not a unique description, how to solve the system.
- Numerically.
- Look up table, precalculated for the working space $\mathcal{W} \subset \mathcal{Q}$.

There are a manipulator structures, which can be solved analytically. We call them solvable.

The sufficient condition of solvability is e.g. when the 6 DOF robot has three consecutive revolute joint with axes intersecting in one point.

The other property of inverse kinematics is ambiquity of solution in singular points. There is often the subspace $\mathcal{Q}_s$ of the space $\mathcal{Q}$, which gives the same $\mathbf{T}$.

$$\forall \mathbf{q} \in \mathcal{Q}_s : \mathbf{T}(\mathbf{q}) = \mathbf{T}$$

To decide which $\mathbf{q}$ solving 10 to select has to be taken into account mainly:

1. Is the selected value $\mathbf{q}$ **applicable**, i.e. can the robot be sent to $\mathbf{q}$?
2. How to reach the singular point. The function $\mathbf{q}$ shall allways be a continuous function of time. The preceeding values of $\mathbf{q}$ should make with the selected value continous function of time.
3. How to continue from the singular point? The future values of $\mathbf{q}$ should form with the selected value continous function of time.
4. Will not the selected value of $\mathbf{q}$ guide us to the situation where we will not be able to satisfy above conditions?
5. Will the required **operational space** limit us during manipulation? The example is the insertion of the seat into car.

We design sometimes redundant robots (with more degrees of freedom, e.g. 8), to increase the space $\mathcal{Q}_s$, from which we select $\mathbf{q}$ to allow more freedom to fulfill above requirements.

Think about following:

- Is it possible to design the robot with prismatic joints only which can position arbitrarily the rigid body in 3D space? Why?
- Choose some manipulating task and design the structure of redundant robot for it.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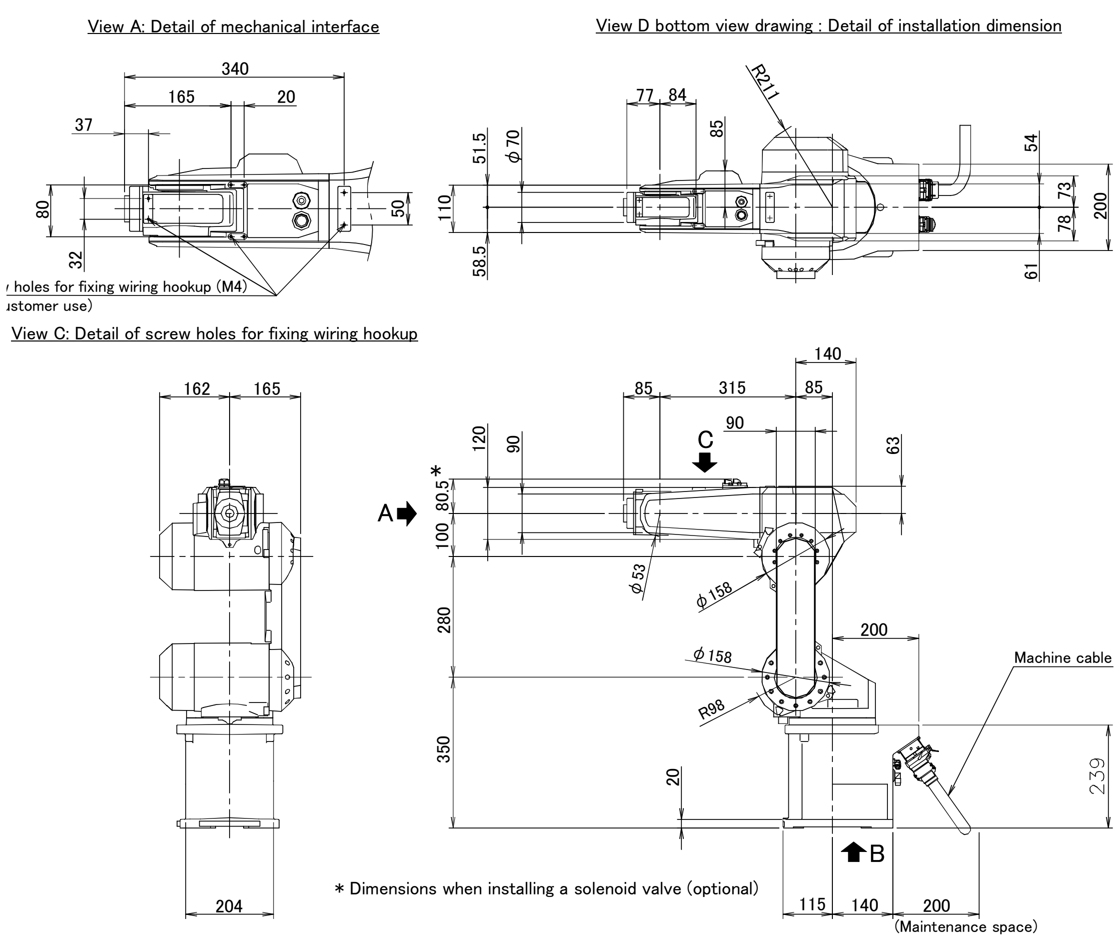
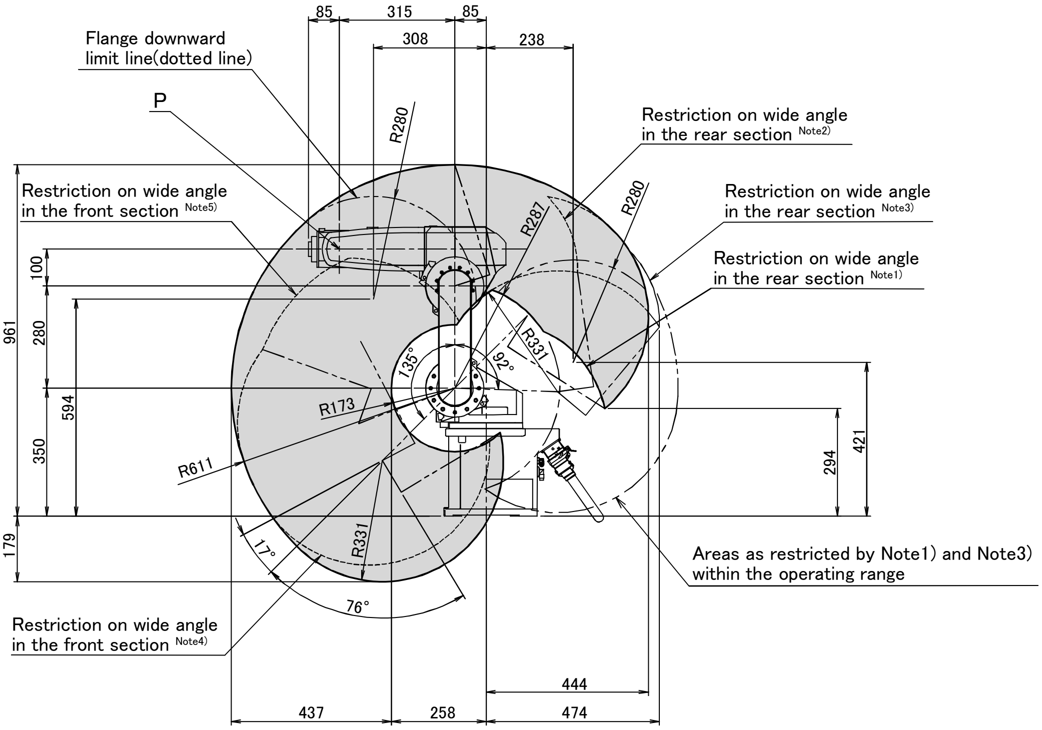
Inverse Kinematics - example

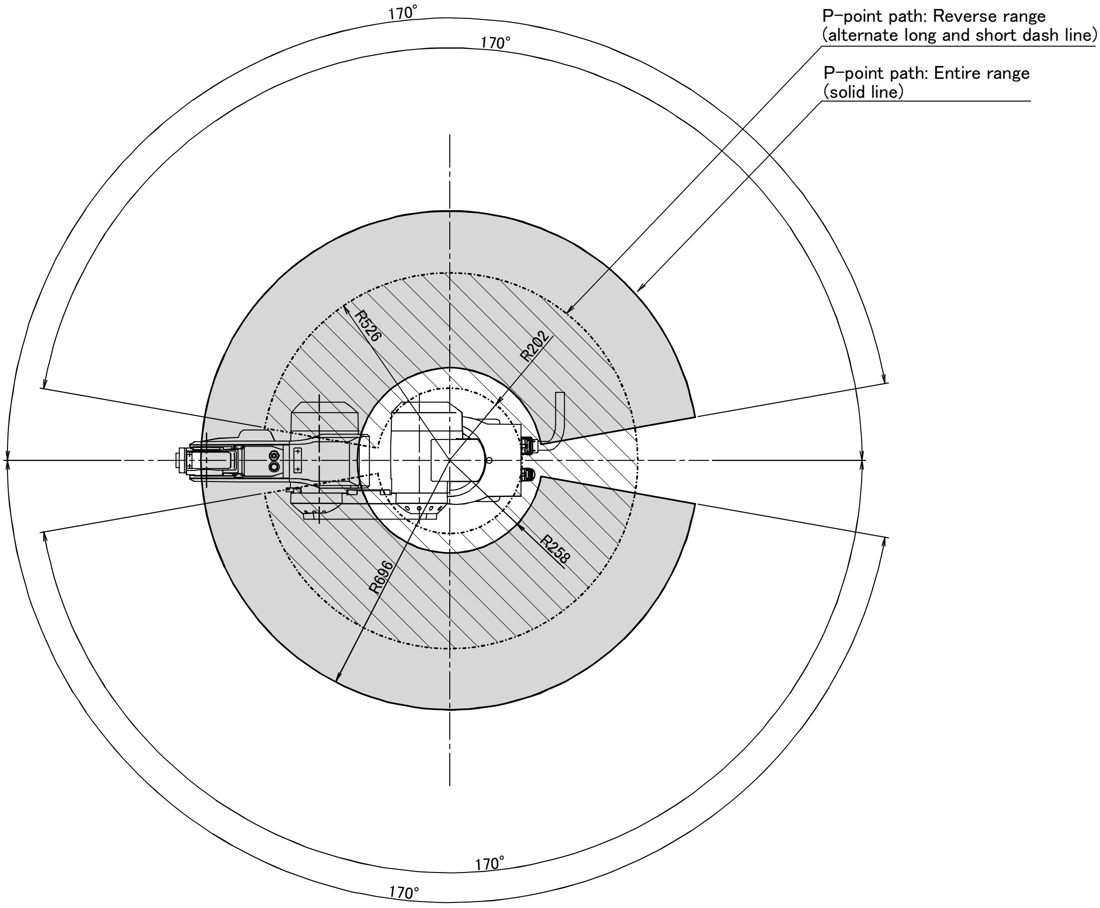


Inverse Kinematics - example

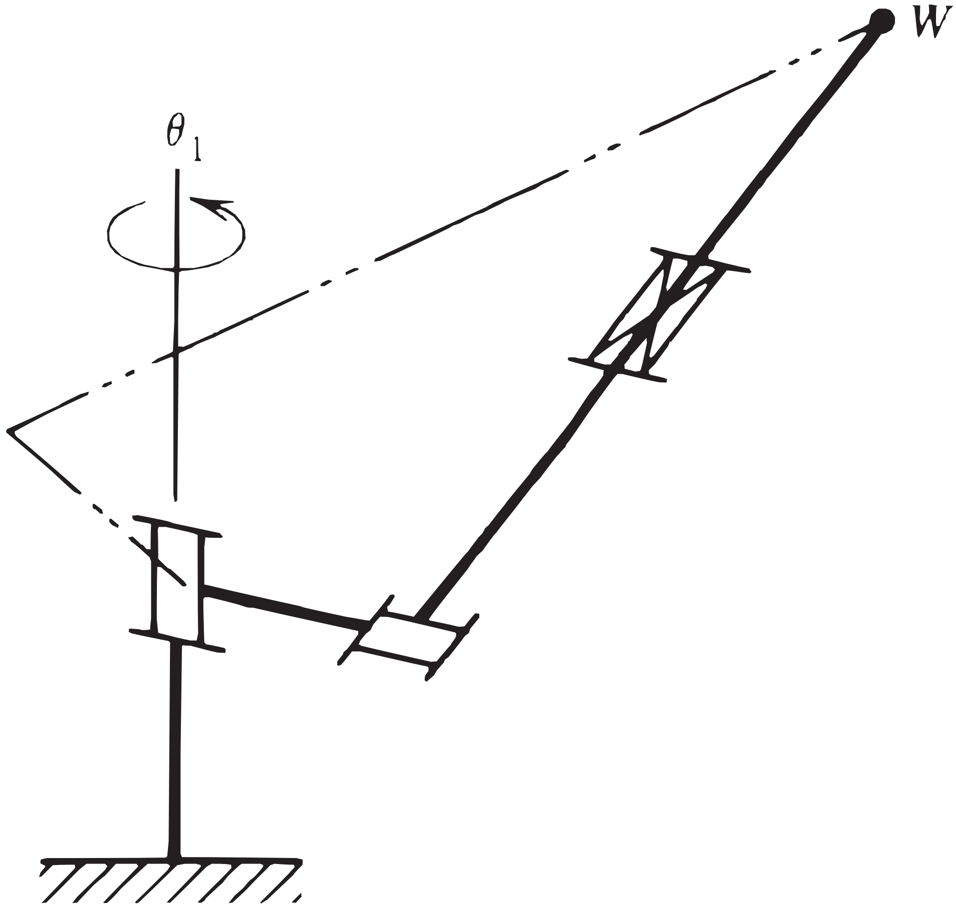


Restriction on wide angle in the rear section
Note1) J2+J3×2 ≧ −200 degree when −45 degree≦ J2 < 15 degree.
Note2) J2+J3 ≧ 8 degree when |J1| ≦ 75 degree, J2 < −45 degree.
Note3) J2+J3 ≧ −40 degree when |J1| > 75 degree, J2 < −45 degree.
Restriction on wide angle in the front section
Note4) J3 ≧ −40 degree when −105 degree≦ J1 ≦ 95 degree, J2 ≧ 123 degree.
Note5) J2 ≧ 110 degree when J1 < −105 degree, J1 < −95 degree.
However, J2 − J3 ≦ 150 degree when 85 degree J2 ≦ 110 degree.

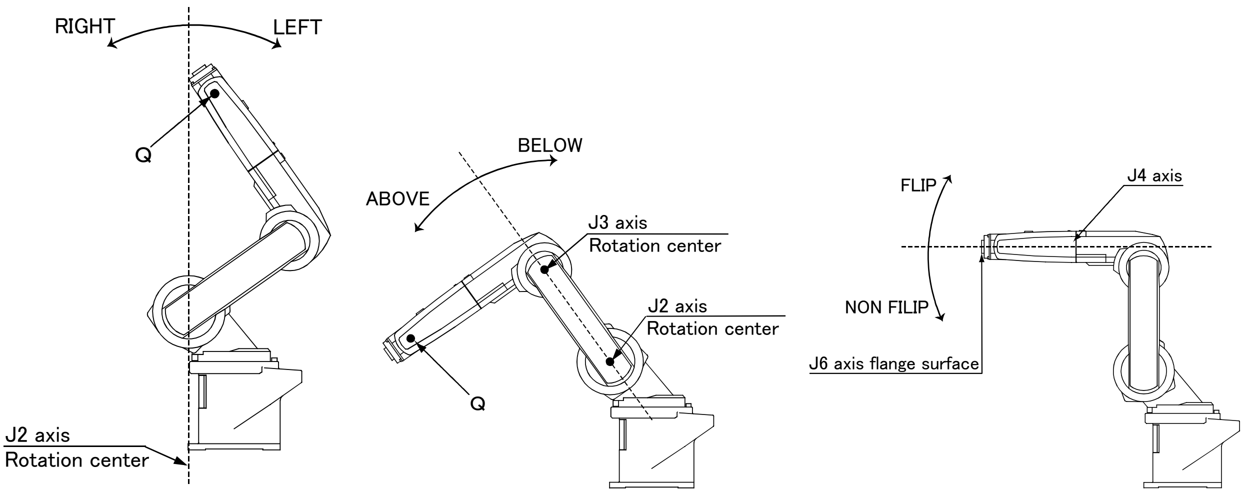
Inverse Kinematics - example



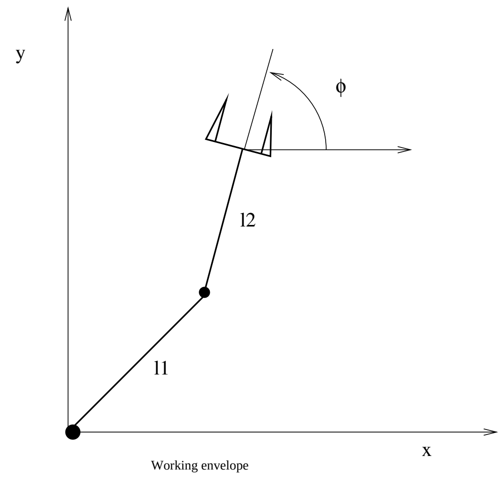
Multiple configurations in inverse kinematics

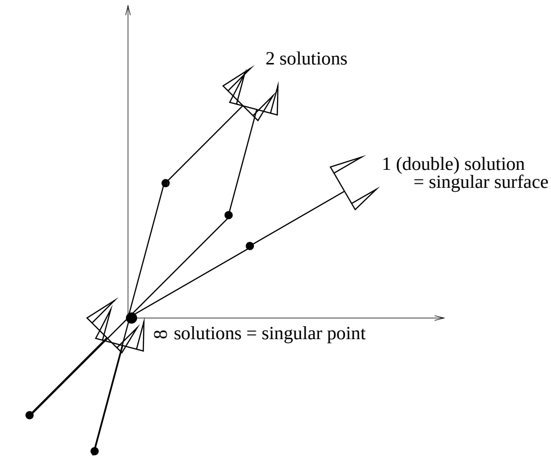


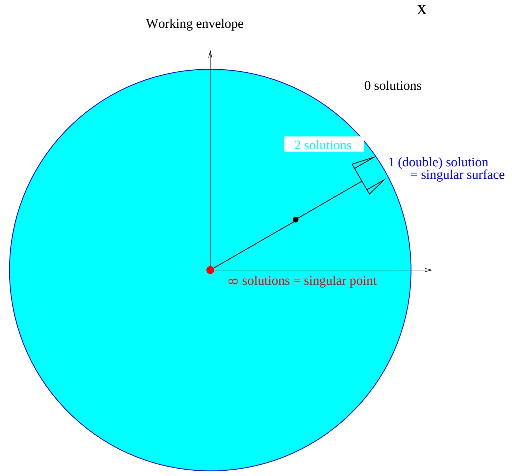
Inverse Kinematics - Multiple Configurations

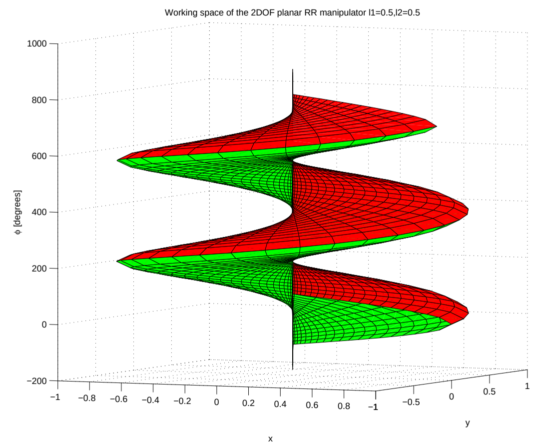


Example: Multiple configurations for simple planar manipulator







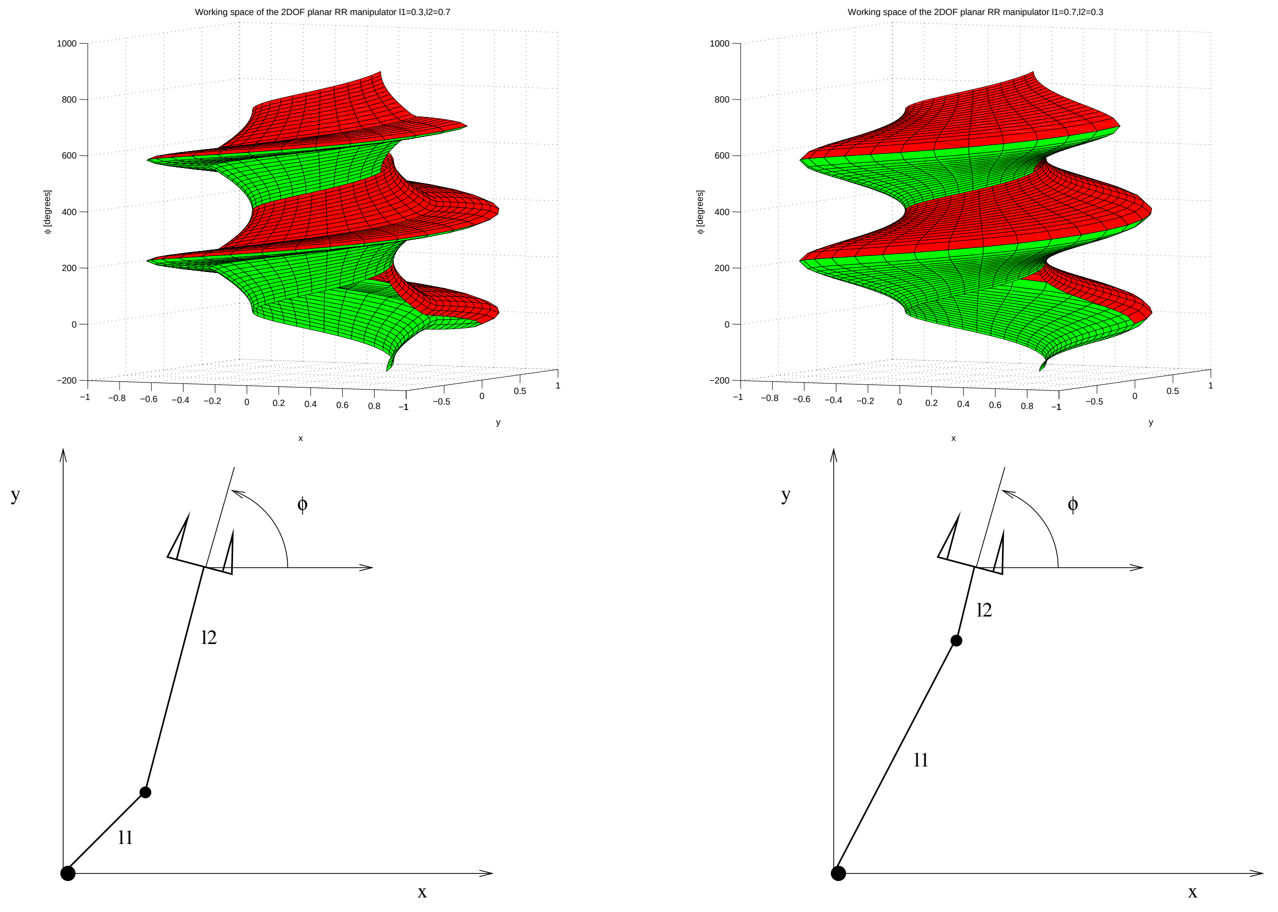


2DOF planar manipulator with 2 revolute joints have two solutions within the circle, where it can reach. On the border of the circle there is a single solution, where two solutions basically coincide (compare 1 solution of quadratic equation), this border is singular surface, where one configuration can switch to the other configuration. Outside of the circle there is no solution. There is infinitely many solutions in the center of the circle, this is another singular point of the robot.

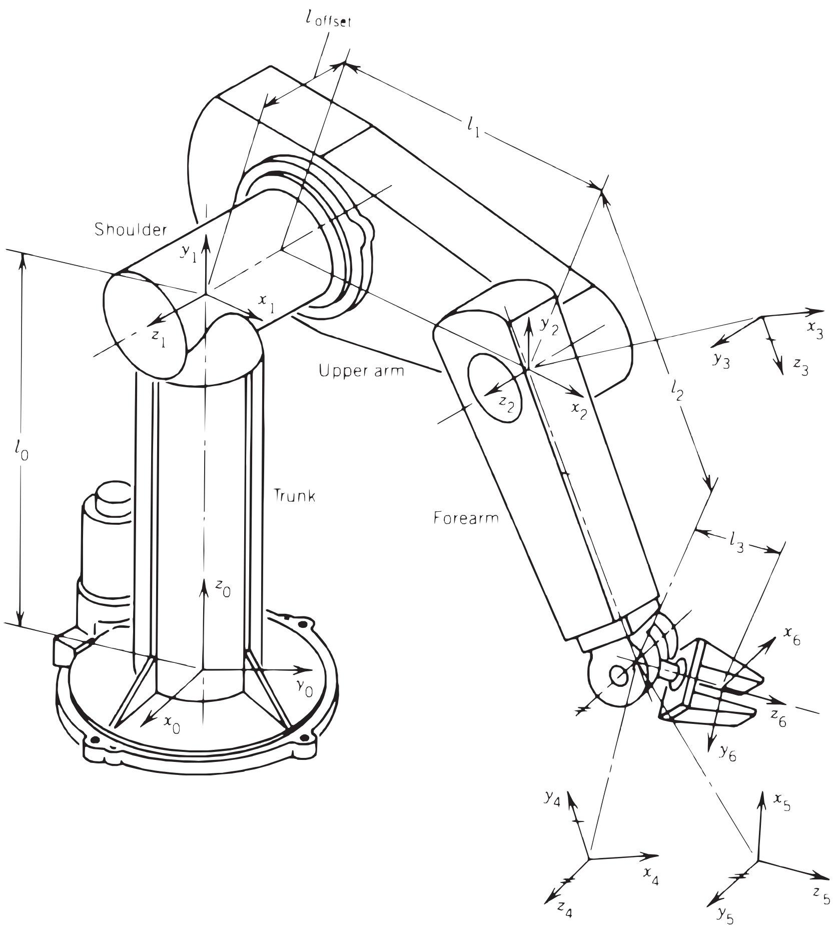
This planar manipulator has only 2 DOF but it operates in the 3D working space, that is e.g. x, y, and orientation of the gripper ϕ. DOF deficiency thus causes that only some points in the working space are reachable, that is only some combinations of (x, y, ϕ). The picture of the working space shows the reachable points, green color represent first configuration, red color representing second configuration. The ϕ axis is a singular point, where any orientation is reachable, the boundary between green and red surface is also singular, where both solutions meet. Working space is shown here in the interval $< 0, 720^o >$, the spiral is actually from $-\infty$ to ∞.

It shall be stressed that ideally the working space shall occupy some compact but dense region, where all orientations of the end effector could be reached in all locations. Visualisation of six dimensional working space of spatial manipulator is of course difficult.

Example: Multiple configurations for simple planar manipulator



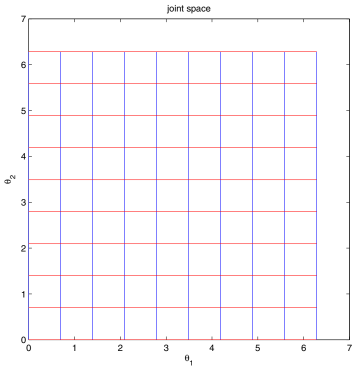
Manipulator with links of different length cannot reach near first joint.



l_{offset}
l_1
Shoulder
y_1
x_1
z_1
l_0
Upper arm
y_2
z_2
x_2
x_3
y_3
z_3
l_2
Trunk
Forearm
l_3
z_0
y_0
x_0
x_6
z_6
y_6
y_4
x_4
z_4
x_5
z_5
y_5

Mapping between Joint and Working Space

Joints space ⇔ Working (Cartezian) space



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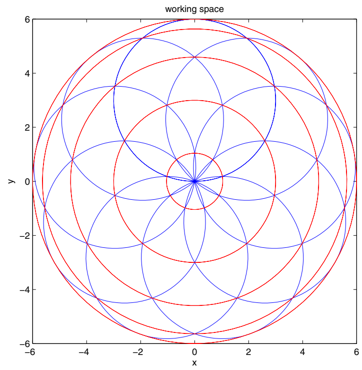

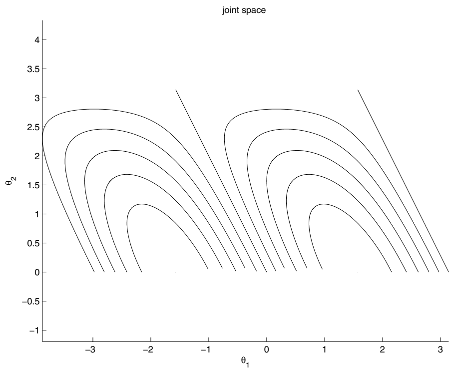

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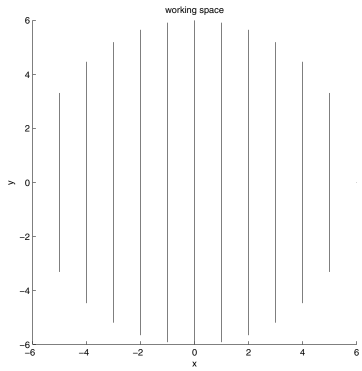
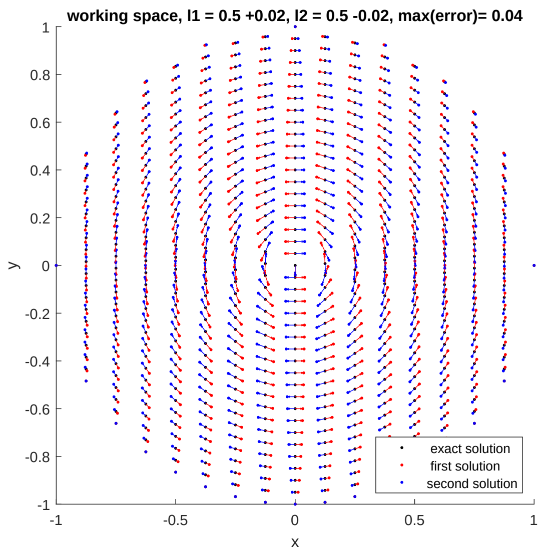


Mapping between joint space and working space is for robot with revolute joints quite nonlinear.

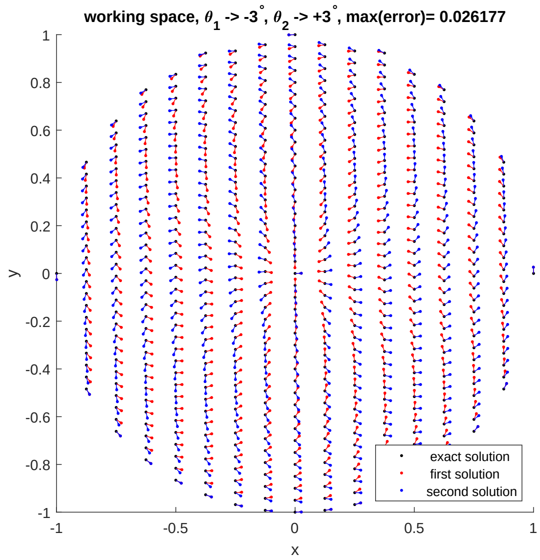
Innacurately Known Robot Model – Robot Accuracy

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31

Innacurate knowledge of links lengths



Innacurate knowledge of joints offsets



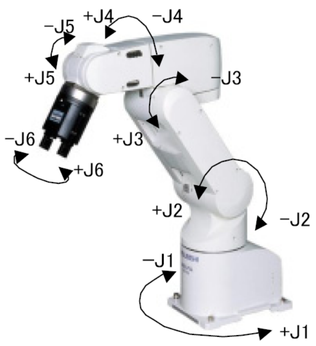
Mapping between joint space and working space is for robot with revolute joints quite nonlinear. The same holds for the impact of the robot model parameters, when they are not known accurately. Particularly important is that different robot configurations result in different position, so the robot positions compensated in one configuration will produce even worse position for other configuration. The resulting errors for various points in working space are shown for 2–D manipulator with two revolute joints. The resulting errors in position are demonstrated for incorectly known length of links and incorectly known joint offsets. Chosen 2–D manipulator cannot demonstrate the errors caused e.g by non-perpendicularity of the succesive joint axes or non-straight linear guidance of prismatic joint.

Forward and inverse kinematics - summary

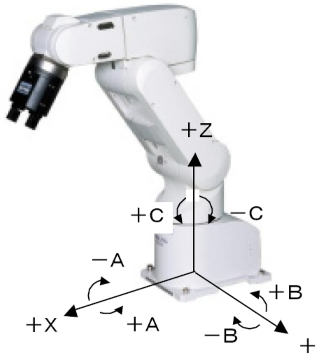
Kinematics	Structure	Solutions	Difficulty
	Serial	1	Easy
Forward	Hybrid	$0, 1, N, \infty$	Difficult
	Parallel	$0, 1, N, \infty$	Difficult
	Serial	$0, 1, N, \infty$	Difficult
Inverse	Hybrid	$0, 1, N, \infty$	Difficult
	Parallel	$0, 1, N, \infty$	Easier

Number of solutions and difficulty to solve the particular kinematics for particular robot is given by the mathematical nature of the problem, the tranformation is described by set of nonlinear equations, which has to be solved. The equations are basically polynomial in variables or their sines and cosines, goniometric functions causing the nonlinearity. The equtions have in some cases unique solution, e.g. forward kinematics of the open kinematic chain (serial manipulator) and are relatively easily solvable. In other cases the task is not solvable analytically or its solution is not known. Numerical methods are used in such cases or such structures are avoided altogether.

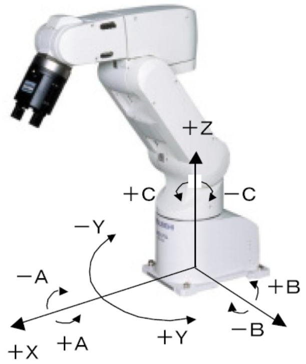
Motion in other coordinate systems



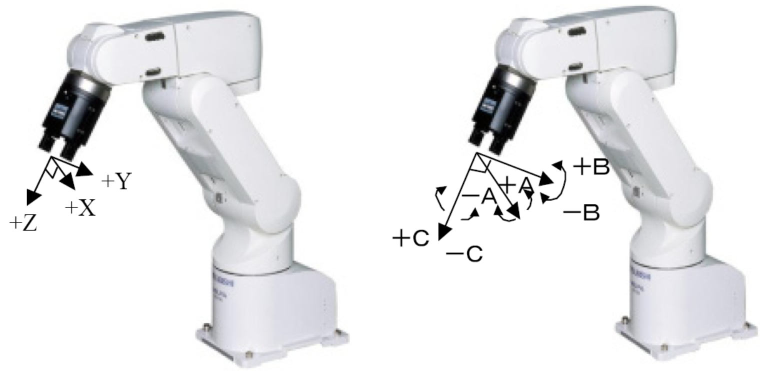
Joint coordinates



Cartezian world coordinates



Cylindrical world coordinates



Cartesian tool coordinates

The robot controller usually allows using a pendant interactive control of end-effector position in various coordinate systems:

- joint coordinates (standard),
- cartezian coordinates in world coordinate system (almost standard),
- cylindrical coordinate system in world coordinate system,
- cartesian coordinate system in end-effector coordinates system,...