Robotics

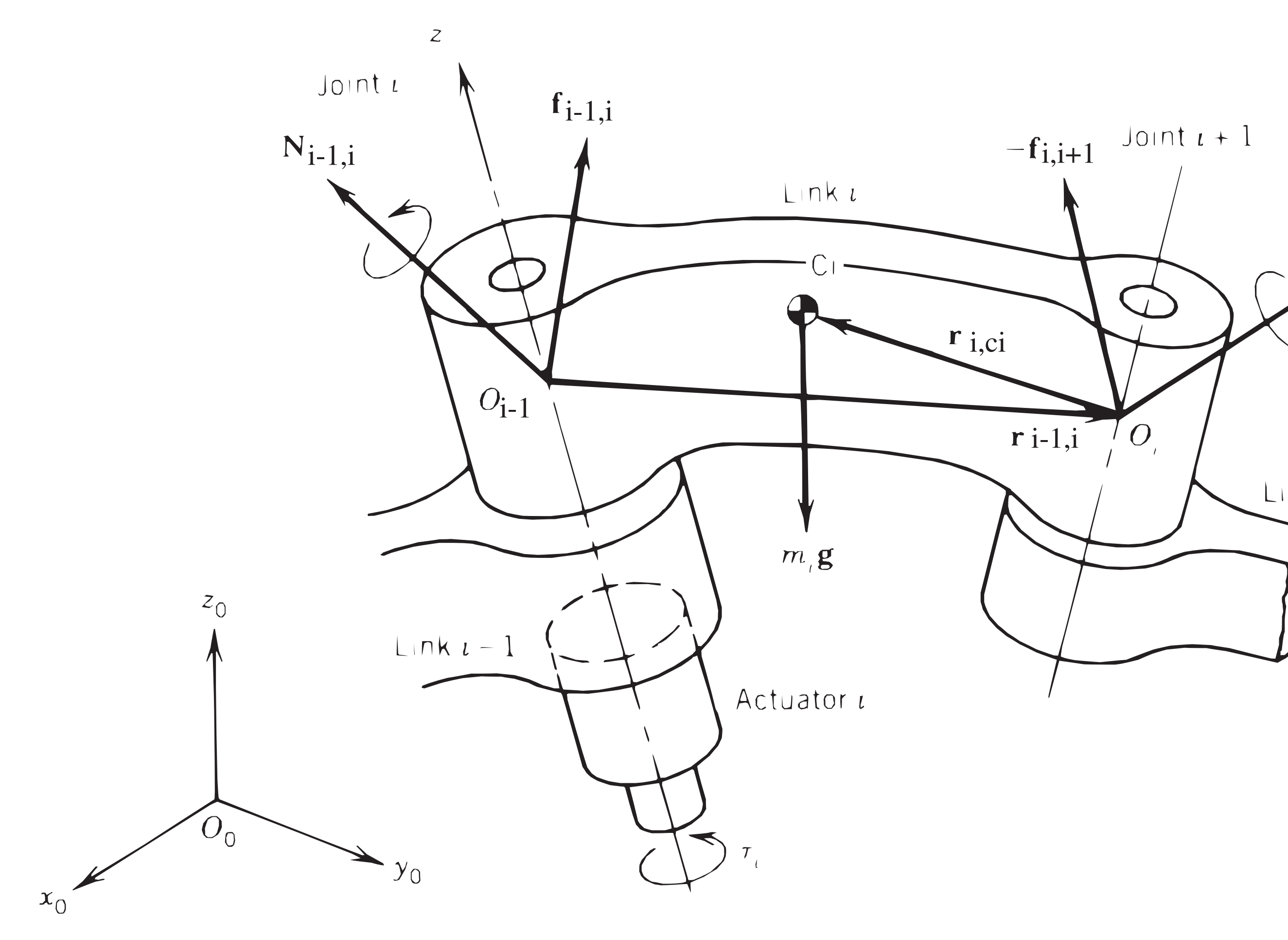
Serial Manipulator Statics

Vladimír Smutný

Center for Machine Perception

Czech Institute for Informatics, Robotics, and Cybernetics (CIIRC)

Czech Technical University in Prague



Statics

- Deals with forces and moments applied on the robot at rest.
- Takes into account weights of links and manipulated object.
- Takes into account the force and the moment robot applies on the environment.
- Considers finite stiffness of joints and links.

We will take into account only simplified model of the joint stiffness. The links are considered infinitely stiff. Open kinematic chain is discussed only.

See Fig. The forces acting on the i-th link are (placed to the origin of i-th coordinate system) $\mathbf{f}_{i-1,i}$, $-f_{i,i+1}$ the weight of a link is $m_i\mathbf{g}$ and moments are $\mathbf{N}_{i-1,i}$, $-\mathbf{N}_{i,i+1}$. Let us denote the vectors $\mathbf{r}_{i-1,i} = O_{i-1}O_i$, $\mathbf{r}_{i,ci} = O_iC_i$. Then the condition for static balance of the forces is:

$$\mathbf{f}_{i-1,i} - f_{i,i+1} + m_i\mathbf{g} = \mathbf{0},\ \ i = 1,\ldots,n\,. \tag{1}$$

Static balance of the moments expressed to the centroid:

$$\mathbf{N}_{i-1,i} - \mathbf{N}_{i,i+1} - (\mathbf{r}_{i-1,i} + \mathbf{r}_{i,ci}) \times \mathbf{f}_{i-1,i} + (-\mathbf{r}_{i,ci}) \times (-f_{i,i+1}) = \mathbf{0},\ \ i = 1,\ldots,n\,. \tag{2}$$

All variables (forces, moments, vectors) has to be expressed in a single coordinate system, e.g. in the base coordinate system.

Show as the exercise that the condition is independent on the reference point.

Manipulator with n DOF is described by a system of $2n$ equations with $2n+2$ unknown variables. To get unique solution we need to know the force and moment the robot is acting on the environment. For example robot manipulating the object acts by force $\mathbf{f}_{n,n+1}$ equal to the weight of the object and the moment is zero: $\mathbf{N}_{n,n+1} = 0$. Let us define the vector of forces in end effector the vector:

$$\mathbf{F} = \begin{bmatrix} \mathbf{f}_{n,n+1} \\ \mathbf{N}_{n,n+1} \end{bmatrix}. \tag{3}$$

The force and the moment, by which one link acts on the other, are compensated partially by the structure of the manipulator and partially by the joint force. The structure of prismatic joint absorbs all moments and two of three force components. Only the component acting in the direction of joint axis has to be compensated by the joint force

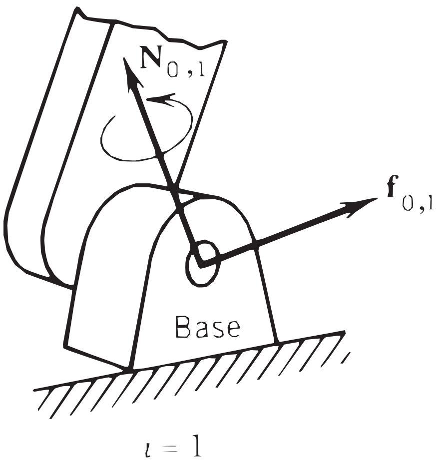
$$\tau_i = \mathbf{b}_{i-1}^T \mathbf{f}_{i-1,i}\,. \tag{4}$$

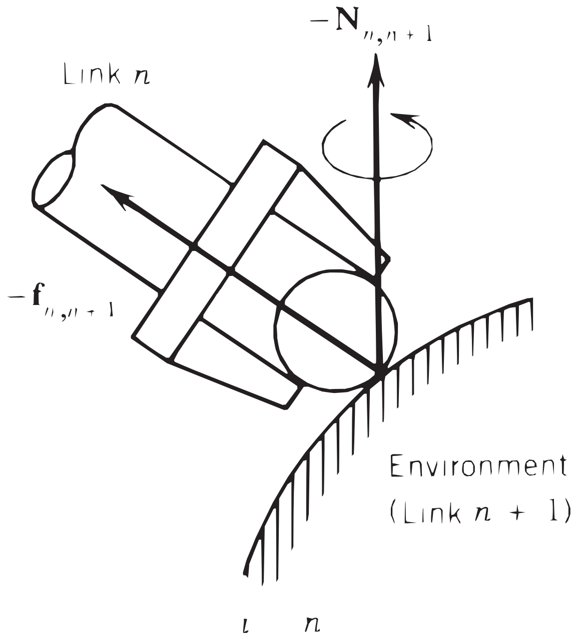
The joint moment for revolute joint can be derived from:

$$\tau_i = \mathbf{b}_{i-1}^T \mathbf{N}_{i-1,i}\,. \tag{5}$$

The components which are compensated by the structure do not produce the work because of zero path, on which they act.

Statics – Forces acting on the base and the end effector.





The vector of forces and moments produced by joints is called joint moments and denoted as:

$$\boldsymbol{\tau} = \begin{bmatrix} \tau_1 \\ \tau_2 \\ \vdots \\ \tau_n \end{bmatrix} . \tag{6}$$

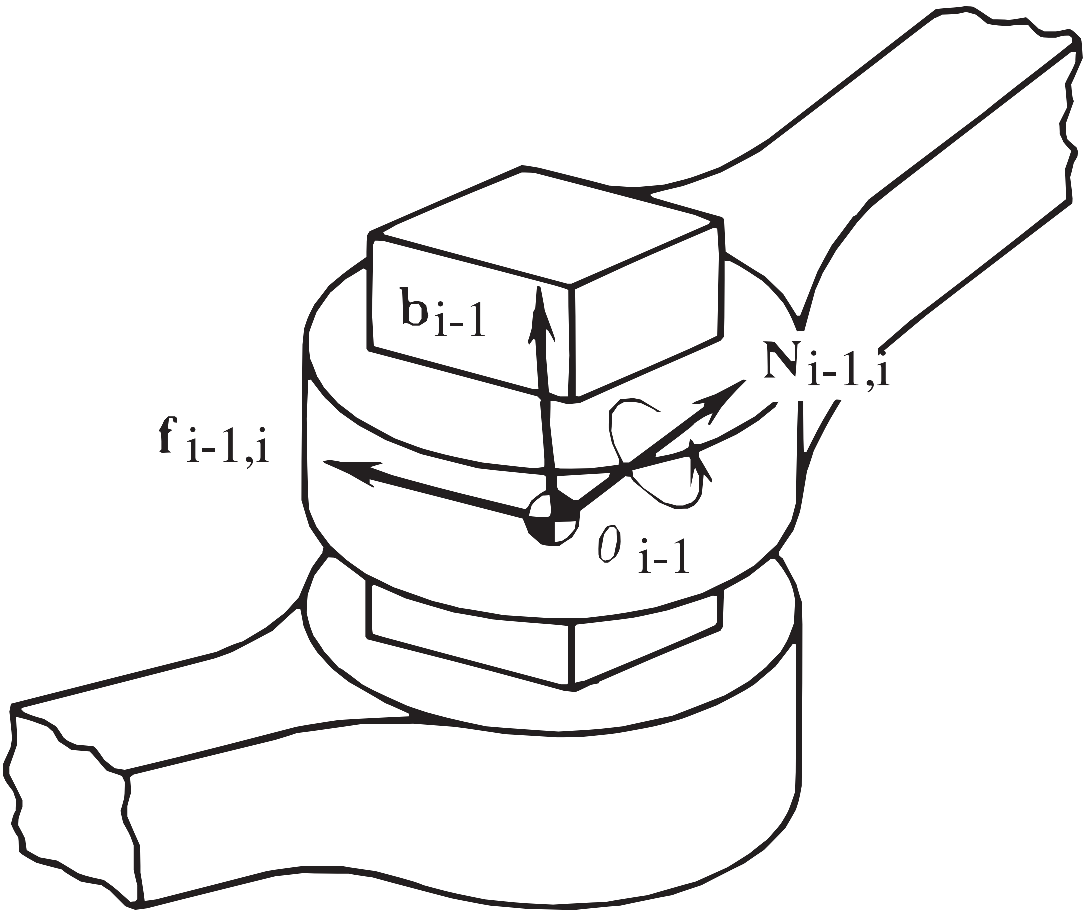
If the robot does not apply force to the environment $\mathbf{F} = 0$, then we can calculate the part of the joint moments corresponding to the supporting of the arm against gravity: $\boldsymbol{\tau}_G$. Component $\boldsymbol{\tau}_G$ will be omitted in following calculations.

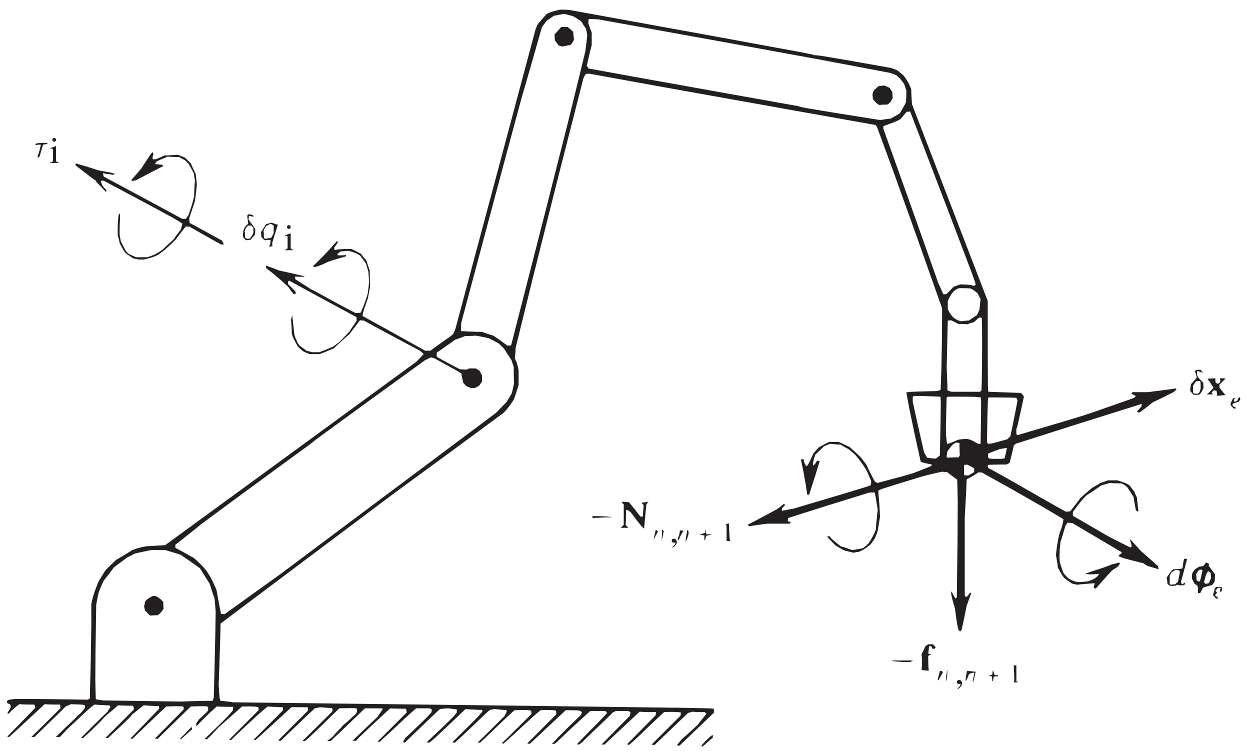
It can be shown that joint moments without the component supporting the robot weight can be calculated from the vector of forces in end effector by the formula:

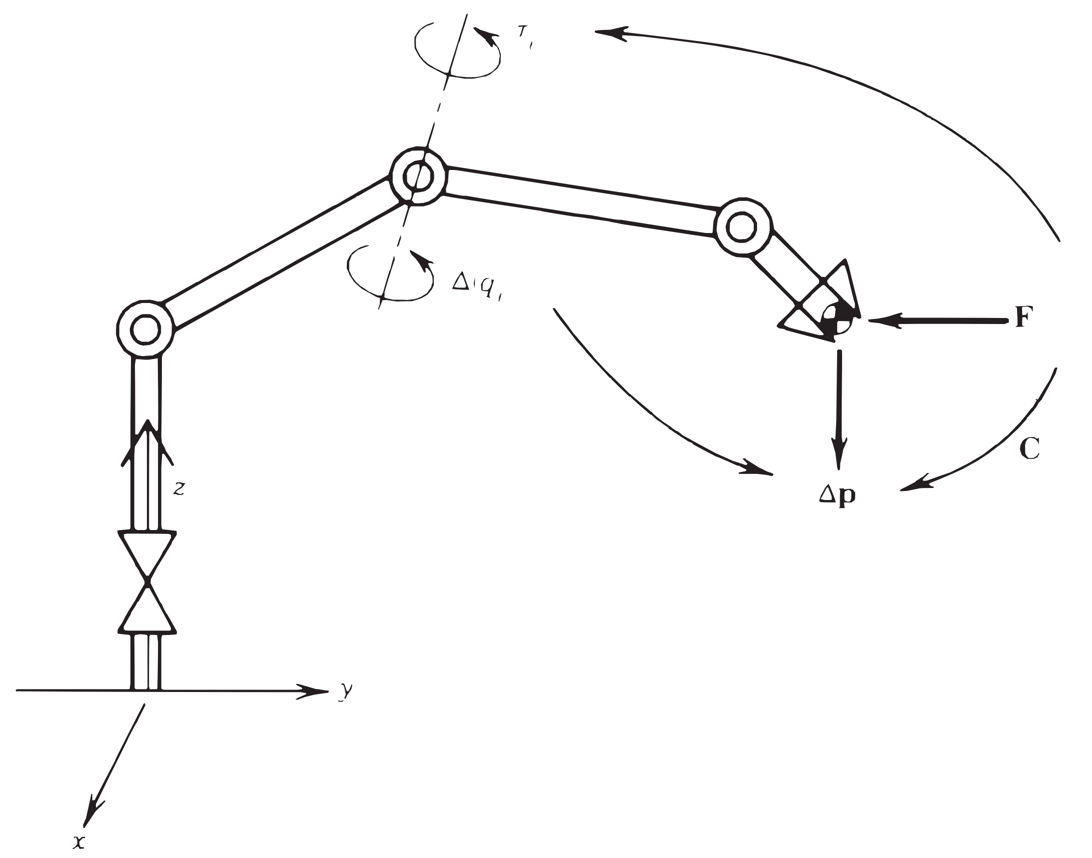
$$\boldsymbol{\tau} = \mathbf{J}^T \mathbf{F} , \tag{7}$$

where $\mathbf{J}$ is a Jacobian of the manipulator. $\boldsymbol{\tau}_G$. will be omitted in following calculations. Show as the exercise the above relationship (e.g. by use of Energy conservation law). Calculate $\boldsymbol{\tau}$for manipulator shown on the Fig.

Statics – The forces in prismatic joint.




τ_i
δq_i
$\delta \mathbf{x}_e$
$-\mathbf{N}_{n,n+1}$
$d\boldsymbol{\Phi}_e$
$-\mathbf{f}_{n,n+1}$



Stiffness modeling

The stiffness of the manipulator is influenced by various components: stiffness of the links, stiffness of the joints etc. The links are usually relatively stiff (the counterexample is the manipulator on the spaceshutle). The stiffness of the joints can be modeled e.g. by the spring between actuator and a link. This is a good assumption as the most of the error has actually this origin and some of the stiffness of link can be approximated in this way (the linear term in the Taylor expansion of the actual error). This model does not model the effects similar to the hysteresis. The force and moment τ_i is then proportional to the deviation of the joint position from the unloaded position Δq_i by the coefficient k_i:

$$\tau_i = k_i \Delta q_i \,.$$

In matrix notation:

$$\boldsymbol{\tau} = \mathbf{K} \Delta \mathbf{q} \,,$$

where

$$\mathbf{K} = \begin{bmatrix} k_1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & k_n \end{bmatrix} .$$

The compliance in joints results in compliance of end effector (see Fig. The force $\mathbf{F}$ acting on the end effector have to be compensated by the joint torques $\boldsymbol{\tau}$:

$$\boldsymbol{\tau} = \mathbf{J}^T \mathbf{F} \,. \tag{8}$$

The deviation in joints using the compliance model of joints is expressed as $\Delta \mathbf{q}$.

$$\boldsymbol{\tau} = \mathbf{K} \Delta \mathbf{q} \,. \tag{9}$$

The deviation in joints will cause the deviation in end effector position $\Delta \mathbf{p}$:

$$\Delta \mathbf{p} = \mathbf{J} \Delta \mathbf{q} \,. \tag{10}$$

When the stiffness of the manipulator are all nonzero, the matrix $\mathbf{K}$ is invertible and we can write

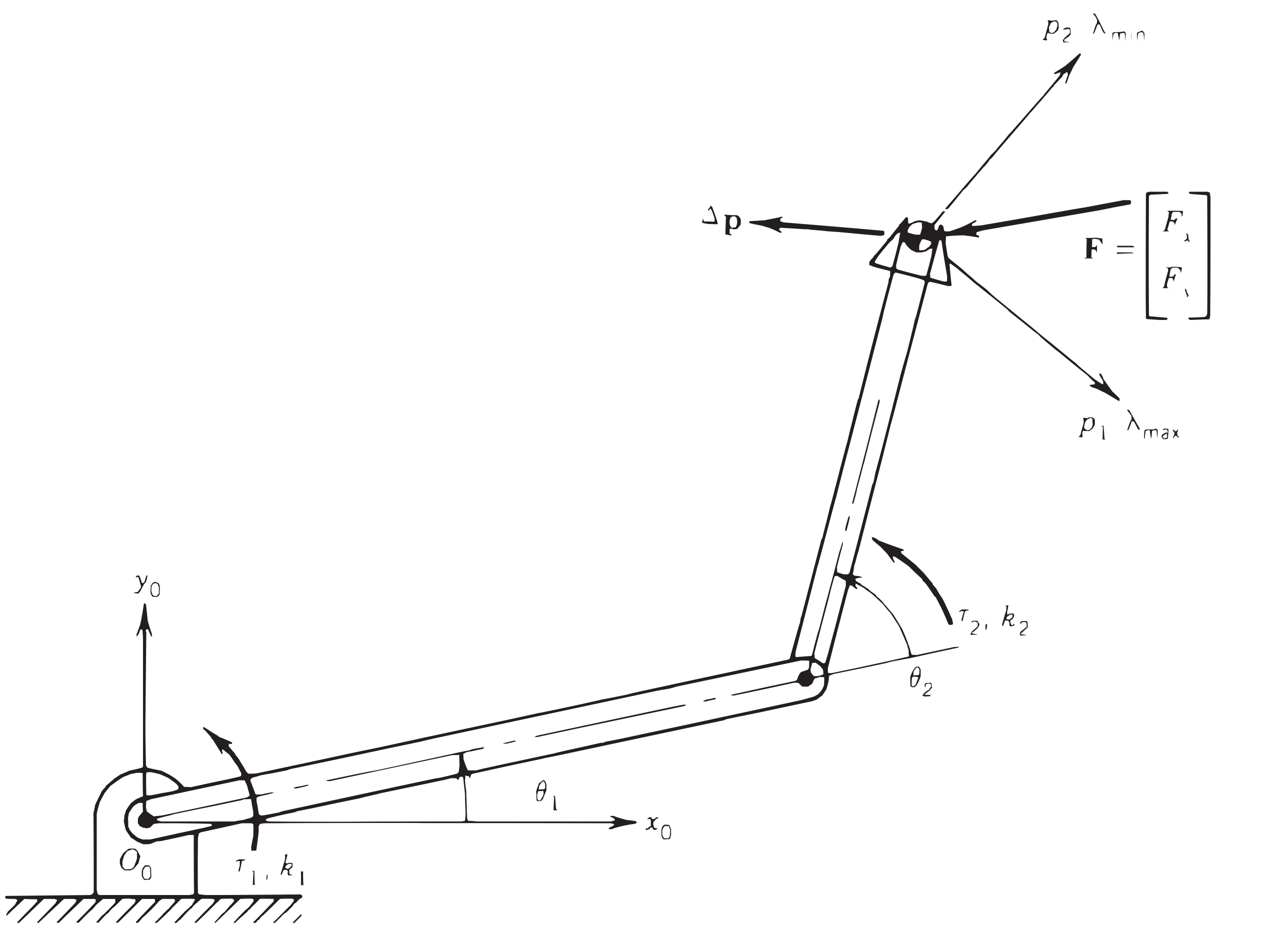
$$\Delta \mathbf{p} = \mathbf{J} \mathbf{K}^{-1} \mathbf{J}^T \mathbf{F} = \mathbf{C} \mathbf{F} \,. \tag{11}$$

The matrix $\mathbf{C}$ is called compliance matrix of the end effector. If the manipulator Jacobian has square size and regular then the matrix $\mathbf{C}$ can be inverted and its inversion is called the stiffness matrix:

$$\mathbf{F} = \mathbf{C}^{-1} \Delta \mathbf{p} \,. \tag{12}$$

If the manipulator Jacobian is singular, it exists a nonempty region S_2 (see Fig. and thus nonempty null space $N(\mathbf{J}^T)$. If the force acts in region $N(\mathbf{J}^T)$ no joint torques are generated and the manipulator appears as infinitely stiff. Note that manipulator Jacobian and thus the compliance matrix depends on the manipulator position.

Statics – Example of stiffness analysis.



Because the deviation of the end effector position depends on the manipulator position, the size and the direction of the forces acting on the manipulator, we can analyze the maximal and minimal deviation resulting from the force of a given size. The size of the deviation for unit size of force is:

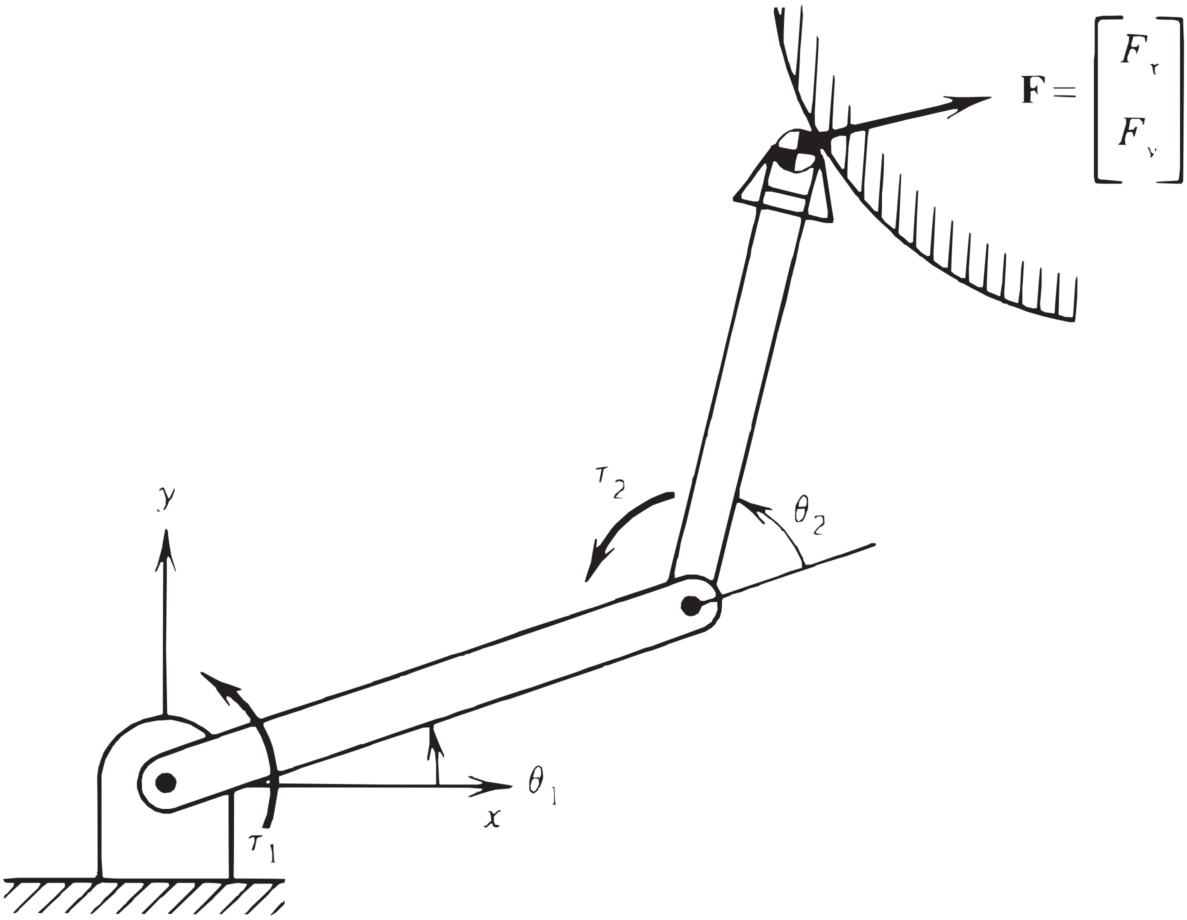
$$|\Delta \mathbf{p}|^2 = \Delta \mathbf{p}^T \Delta \mathbf{p} = \mathbf{F}^T \mathbf{C}^T \mathbf{C} \mathbf{F}, \tag{13}$$

when

$$|\mathbf{F}|^2 = \mathbf{F}^T \mathbf{F} = 1\,. \tag{14}$$

By optimization we can find that extreme values $|\Delta \mathbf{p}| = \sqrt{\lambda}$ are reached in the directions given by the vectors $\mathbf{e}$, where λ is minimal or maximal eigenvalue of the matrix $\mathbf{C}^2$ and a vector $\mathbf{e}$ is a corresponding eigenvector. The coordinate transformation generated by the matrix of eigenvectors is called the main transformation. If the forces act only in the direction of eigenvectors the deviation of end effector position is also in the direction of the eigenvectors and with the same orientation as acting force. Because the matrix $\mathbf{C}$ is symmetrical, the eigenvectors are perpendicular each to other. Let us note that the force $\mathbf{F}$ contains also moments and the end effector deviation $\mathbf{p}$ also rotation.

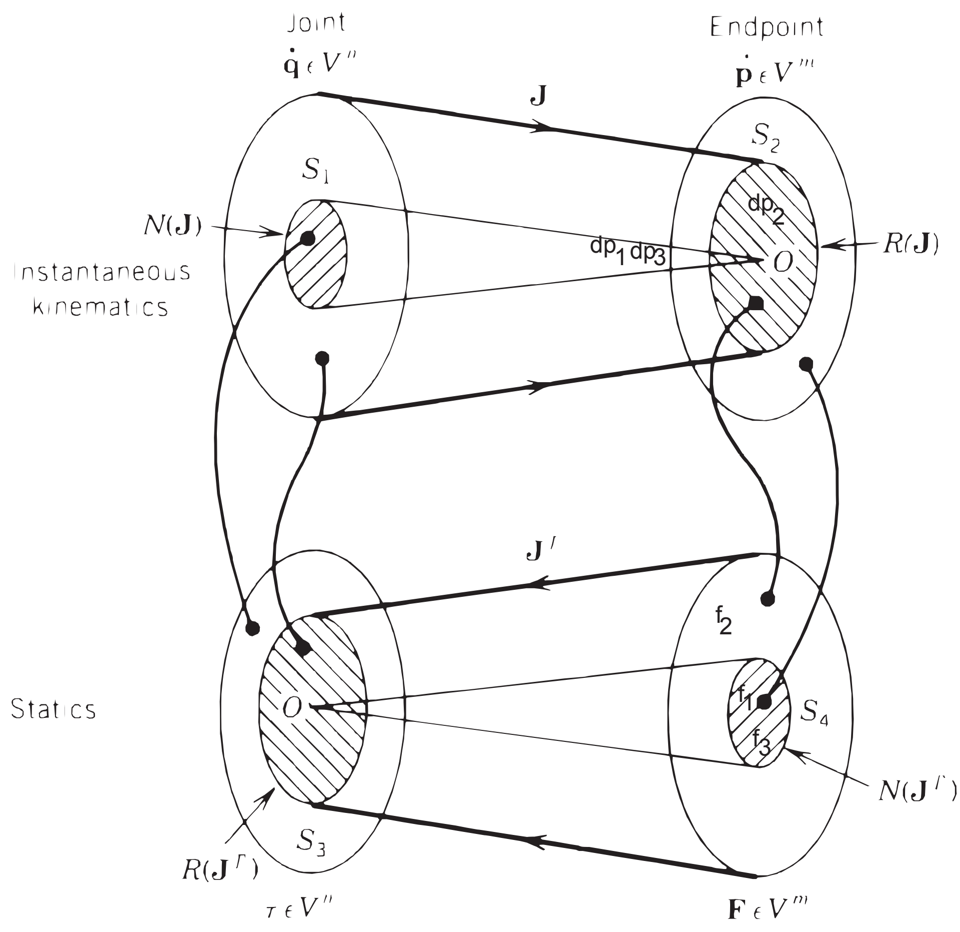
Example: Analyze the manipulator from Fig.



As exercise choose $\theta_2 = 0$ in Fig. form the Jacobian, determine the its dimension and show which directions correspond to which regions in the Fig.

Let us emphasize that the manipulator Jacobian and thus the its null space depends on the position of the manipulator $\mathbf{q}$.

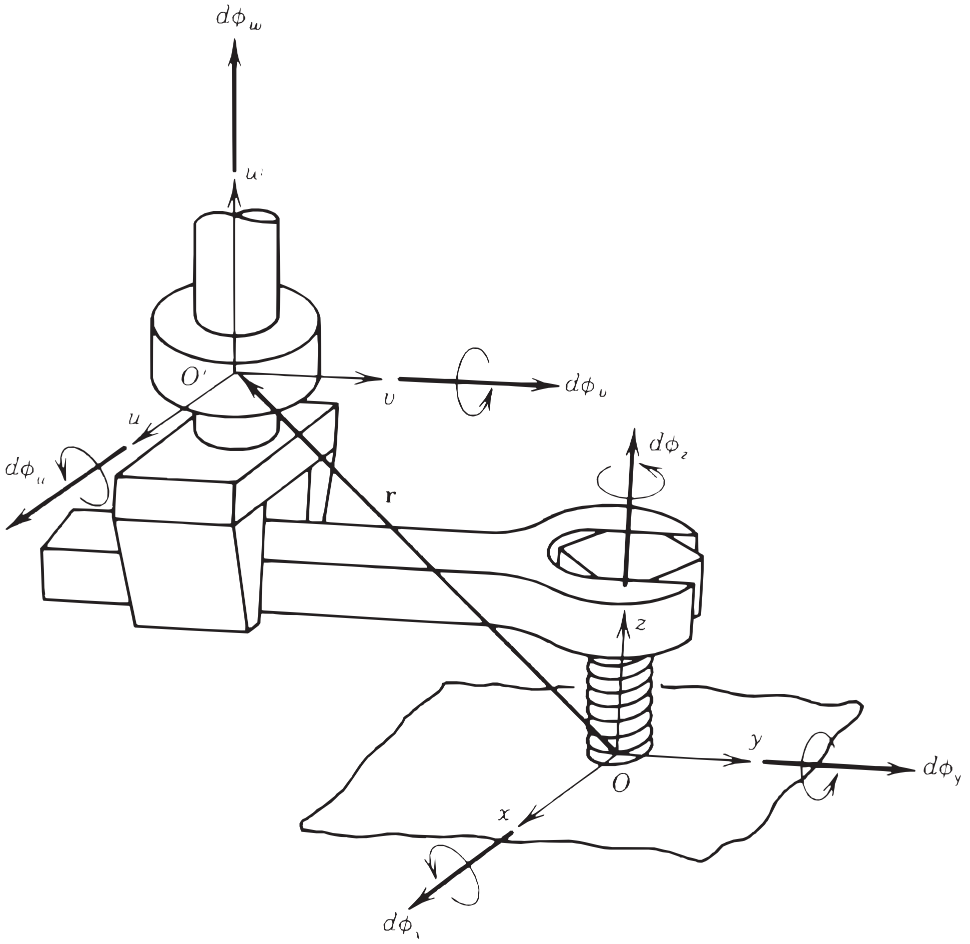
Statics – Mapping for kinematics and statics.



Let us discuss the scheme shown on Fig. In the upper part of the drawing is shown the transformation from velocity in joint coordinates into the velocities in Cartesian coordinates of end effector that is the differential kinematics analysis. In the bottom part it is shown the transformation from the forces in end effector vector to the joint moment, that is statics problem. The region $N(\mathbf{J})$ is the null space of the transformation contains the velocities or infinitesimal moves in joints which does not change the position of the end effector. The region $R(\mathbf{J})$ describes the region of infinitesimal motions, which can be obtained by infinitesimal motion of joints. The complement region S_2 denotes the infinitesimal motions of end effector which cannot be performed by the manipulator.

The region $N(\mathbf{J}^T)$, the null space of the transformation of the transposed Jacobian, is a set of all forces in end effector, which are compensated by the structure of the manipulator. The region $R(\mathbf{J}^T)$ represents the joint moments, which can compensate the forces in the end effector. The region S_3 denotes joint moments, which cannot be compensated by the forces in the end effector. No force nor the moment in the end effector can compensate the joint forces/moments from the region S_3 in the null space $N(\mathbf{J})$.

Statics – Example



Transformation of coordinates and forces

Let us have two coordinates systems. We can express the coordinate transformation of infinitesimal motions, forces and moments between them. Let us denote $\mathbf{q}$ and $\mathbf{p}$ coordinate in one and the other coordinate system. The relationship between infinitesimal motions can be expressed by:

$$d\mathbf{p} = \mathbf{J} d\mathbf{q}\,, \tag{15}$$

where $\mathbf{J}$ is a Jacobian of the corresponding transformation matrix. We can write for the transformation of generalized forces $\mathbf{P}$ and $\mathbf{Q}$

$$\mathbf{Q} = \mathbf{J}^T \mathbf{P}\,. \tag{16}$$

The use can be shown on the figure. The moment, which shall be applied on the screw is given and we want to calculate the force which should apply the end effector in point O'. We can place the force/torque sensor in the point O' and calculated value can be inputed to the feedback controller.

The wrench is described by $d\mathbf{q} = [dx, dy, dz, d\phi_x, d\phi_y, d\phi_z]^T$ in the coordinate system $O - xyz$ and by $d\mathbf{p} = [du, dv, dw, d\phi_u, d\phi_v, d\phi_w]^T$ in the coordinate system $O' - uvw$. It can be easily shown that:

$$d\mathbf{p} = \begin{bmatrix} du \\ dv \\ dw \\ d\phi_u \\ d\phi_v \\ d\phi_w \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & r_z & -r_y \\ 0 & 1 & 0 & -r_z & 0 & r_x \\ 0 & 0 & 1 & r_y & -r_x & 0 \\ & & & 1 & 0 & 0 \\ & \mathbf{0} & & 0 & 1 & 0 \\ & & & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} dx \\ dy \\ dz \\ d\phi_x \\ d\phi_y \\ d\phi_z \end{bmatrix} = \mathbf{J} d\mathbf{q} \tag{17}$$

If we choose the generalized forces in the same coordinate systems, that is: $\mathbf{Q} = [\mathbf{F}_x, \mathbf{F}_y, \mathbf{F}_z, \mathbf{M}_x, \mathbf{M}_y, \mathbf{M}_z]^T$ in the coordinate system $O - xyz$ and $\mathbf{P} = [\mathbf{F}_u, \mathbf{F}_v, \mathbf{F}_w, \mathbf{M}_u, \mathbf{M}_v, \mathbf{M}_w]^T$, then

$$d\mathbf{Q} = \begin{bmatrix} \mathbf{F}_x \\ \mathbf{F}_y \\ \mathbf{F}_z \\ \mathbf{M}_x \\ \mathbf{M}_y \\ \mathbf{M}_z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & & & \\ 0 & 1 & 0 & & \mathbf{0} & \\ 0 & 0 & 1 & & & \\ 0 & -r_z & r_y & 1 & 0 & 0 \\ r_z & 0 & -r_x & 0 & 1 & 0 \\ -r_y & r_x & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{F}_u \\ \mathbf{F}_v \\ \mathbf{F}_w \\ \mathbf{M}_u \\ \mathbf{M}_v \\ \mathbf{M}_w \end{bmatrix} \tag{18}$$

Solve the case when axis v passes through axis z for any position during screw driving as an exercise.