3D Computer Vision

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Open Informatics Master's Course

Part II

Perspective Camera

covered by

[H&Z] Secs: 2.1, 2.2, 3.1, 6.1, 6.2, 8.6, 2.5, Example: 2.19

▶Basic Geometric Entities, their Representation, and Notation

- entities have names and representations
- names and their components:

entity	in 2-space	in 3-space
point	$m = (u, v)$	$X = (x, y, z)$
line	n	O
plane		π, φ

- associated vector representations

$$\mathbf{m} = \begin{bmatrix} u \\ v \end{bmatrix} = [u, v]^\top, \quad \mathbf{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \mathbf{n}$$

will also be written in an 'in-line' form as $\mathbf{m} = (u, v)$, $\mathbf{X} = (x, y, z)$, etc.

- vectors are always meant to be columns $\mathbf{x} \in \mathbb{R}^{n,1}$
- associated homogeneous representations

$$\underline{\mathbf{m}} = [m_1, m_2, m_3]^\top, \quad \underline{\mathbf{X}} = [x_1, x_2, x_3, x_4]^\top, \quad \underline{\mathbf{n}}$$

'in-line' forms: $\underline{\mathbf{m}} = (m_1, m_2, m_3)$, $\underline{\mathbf{X}} = (x_1, x_2, x_3, x_4)$, etc.

- matrices are $\mathbf{Q} \in \mathbb{R}^{m,n}$, linear map of a $\mathbb{R}^{n,1}$ vector is $\mathbf{y} = \mathbf{Q}\mathbf{x}$

▶Image Line

finite line in the plane $a\,u + b\,v + c = 0$

corresponds to a (homogeneous) vector $\underline{\mathbf{n}} \simeq (a,\ b,\ c)$

and there is an equivalence class for $\lambda \in \mathbb{R},\ \lambda \neq 0$ $(\lambda a,\ \lambda b,\ \lambda c) \simeq (a,\ b,\ c)$

'Finite' lines

- standard representative for finite $\underline{\mathbf{n}} = (n_1, n_2, n_3)$ is $\lambda\underline{\mathbf{n}}$, where $\lambda = \frac{\mathbf{1}}{\sqrt{n_1^2+n_2^2}}$

assuming $n_1^2 + n_2^2 \neq 0$; $\mathbf{1}$ is the unit, usually $\mathbf{1} = 1$

'Infinite' lines

- we augment the set of lines for a special entity called the **Ideal Line** (line at infinity)

$$\underline{\mathbf{n}}_\infty \simeq (0, 0, 1)$$ (standard representative)

- the set of equivalence classes of vectors in $\mathbb{R}^3 \setminus (0,0,0)$ forms the projective space $\mathbb{P}^2$

a set of rays →20

- lines at infinity are a proper member of $\mathbb{P}^2$
- I may sometimes wrongly use $=$ instead of $\simeq$, if you are in doubt, ask me

▶Image Point

Finite point $\mathbf{m} = (u, v)$ is incident on a finite line $\underline{\mathbf{n}} = (a, b, c)$ iff this works both ways!

$$a\,u + b\,v + c = 0$$

can be rewritten as (with scalar product): $(u, v, \mathbf{1}) \cdot (a, b, c) = \underline{\mathbf{m}}^\top \underline{\mathbf{n}} = 0$

'Finite' points

- a finite point is also represented by a homogeneous vector $\underline{\mathbf{m}} \simeq (u, v, \mathbf{1})$
- the equivalence class for $\lambda \in \mathbb{R}, \lambda \neq 0$ is $(m_1, m_2, m_3) = \lambda\,\underline{\mathbf{m}} \simeq \underline{\mathbf{m}}$
- the standard representative for finite point $\underline{\mathbf{m}}$ is $\lambda\,\underline{\mathbf{m}}$, where $\lambda = \frac{\mathbf{1}}{m_3}$ assuming $m_3 \neq 0$
- when $\mathbf{1} = 1$ then units are pixels and $\lambda \underline{\mathbf{m}} = (u, v, 1)$
- when $\mathbf{1} = f$ then all components have a similar magnitude, $f \sim$ image diagonal

use $\mathbf{1} = 1$ unless you know what you are doing;
all entities participating in a formula must be expressed in the same units

'Infinite' points

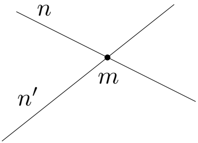
- we augment for **Ideal Points** (points at infinity) $\underline{\mathbf{m}}_\infty \simeq (m_1, m_2, 0)$

  proper members of $\mathbb{P}^2$
- all such points lie on the ideal line (line at infinity) $\underline{\mathbf{n}}_\infty \simeq (0, 0, 1)$, i.e. $\underline{\mathbf{m}}_\infty^\top \underline{\mathbf{n}}_\infty = 0$

▶Line Intersection and Point Join

The point of **intersection** m of image lines n and n', $n \not\simeq n'$ is

$$\underline{\mathbf{m}} \simeq \underline{\mathbf{n}} \times \underline{\mathbf{n}}'$$



proof: If $\underline{\mathbf{m}} = \underline{\mathbf{n}} \times \underline{\mathbf{n}}'$ is the intersection point, it must be incident on both lines. Indeed, using a known equivalence from vector algebra

$$\underline{\mathbf{n}}^\top \underbrace{(\underline{\mathbf{n}} \times \underline{\mathbf{n}}')}_{\underline{\mathbf{m}}} \equiv \underline{\mathbf{n}}'^\top \underbrace{(\underline{\mathbf{n}} \times \underline{\mathbf{n}}')}_{\underline{\mathbf{m}}} \equiv 0$$

The **join** n of two image points m and m', $m \not\simeq m'$ is

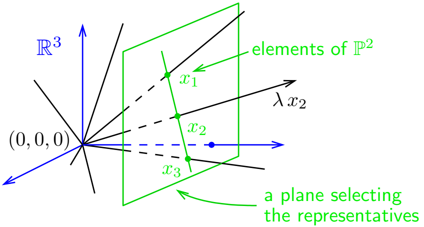
$$\underline{\mathbf{n}} \simeq \underline{\mathbf{m}} \times \underline{\mathbf{m}}'$$

Paralel lines intersect at the line at infinity $\underline{\mathbf{n}}_\infty \simeq (0, 0, 1)$

$$a\,u + b\,v + c = 0,$$
$$a\,u + b\,v + d = 0, \qquad d \neq c$$
$$(a, b, c) \times (a, b, d) \simeq (b, -a, 0)$$

- all such intersections lie on $\underline{\mathbf{n}}_\infty$
- line at infinity represents a set of directions in the plane
- Matlab: `m = cross(n, n_prime);`

▶Homography



Projective plane $\mathbb{P}^2$: Vector space of dimension 3 excluding the zero vector, $\mathbb{R}^3 \setminus (0,0,0)$, factorized to linear equivalence classes ('rays')

including 'points at infinity'

Homography: Non-singular linear mapping in $\mathbb{P}^2$

$$\underline{\mathbf{x}}' \simeq \mathbf{H}\,\underline{\mathbf{x}}, \quad \mathbf{H} \in \mathbb{R}^{3,3} \text{ non-singular}$$

defining properties

- collinear image points are mapped to collinear image points

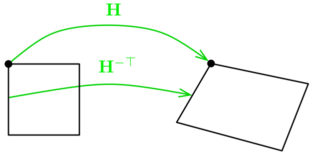
  lines of points are mapped to lines of points
- concurrent image lines are mapped to concurrent image lines

  concurrent = intersecting at a point
- and point-line incidence is preserved

  e.g. line intersection points mapped to line intersection points

- homogeneous matrix representant: $\det \mathbf{H} = 1$
- what we call homography here is often called 'projective collineation' in mathematics

▶Mapping Points and Lines by Homography



$$\underline{\mathbf{m}}' \simeq \mathbf{H}\,\underline{\mathbf{m}} \qquad \text{image point}$$

$$\underline{\mathbf{n}}' \simeq \mathbf{H}^{-\top}\underline{\mathbf{n}} \qquad \text{image line} \qquad \mathbf{H}^{-\top} = (\mathbf{H}^{-1})^{\top} = (\mathbf{H}^{\top})^{-1}$$

- incidence is preserved: $(\underline{\mathbf{m}}')^{\top}\underline{\mathbf{n}}' \simeq \underline{\mathbf{m}}^{\top}\mathbf{H}^{\top}\mathbf{H}^{-\top}\underline{\mathbf{n}} = \underline{\mathbf{m}}^{\top}\underline{\mathbf{n}} = 0$

1. $\mathbf{H}$ is a 3×3 matrix
2. homography has 8 DOF; it is given by 4 correspondences (points, lines) in a general position
3. extending pixel coordinates to homogeneous coordinates $\underline{\mathbf{m}} = (u, v, \mathbf{1})$
4. mapping by homography, eg. $\underline{\mathbf{m}}' = \mathbf{H}\,\underline{\mathbf{m}}$
5. conversion of the result $\underline{\mathbf{m}}' = (m'_1, m'_2, m'_3)$ to canonical coordinates (pixels):

$$u' = \frac{m'_1}{m'_3}\mathbf{1}, \qquad v' = \frac{m'_2}{m'_3}\mathbf{1}$$

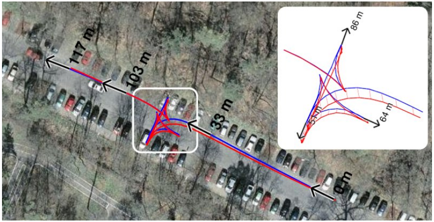
6. can use the unity for the homogeneous coordinate on one side of the equation only!

Some Homographic Tasters

Rectification of camera rotation: →59 (geometry), →121 (homography estimation)


$$\mathbf{H} \simeq \mathbf{K}\mathbf{R}^\top\mathbf{K}^{-1}$$ from image to facade

Homographic Mouse for Visual Odometry: [Mallis 2007]

illustrations courtesy of AMSL Racing Team, Meiji University and LIBVISO: Library for VISual Odometry

$$\mathbf{H} \simeq \mathbf{K}\left(\mathbf{R} - \frac{\mathbf{t}\mathbf{n}^\top}{d}\right)\mathbf{K}^{-1}$$ [H&Z, p. 327]

Elementary Decomposition of a Homography

Unique decompositions: $\mathbf{H} = \mathbf{H}_S\,\mathbf{H}_A\,\mathbf{H}_P \quad (= \mathbf{H}'_P\,\mathbf{H}'_A\,\mathbf{H}'_S)$

$$\mathbf{H}_S = \begin{bmatrix} s\,\mathbf{R} & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix} \qquad \text{similarity}$$

$$\mathbf{H}_A = \begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0}^\top & 1 \end{bmatrix} \qquad \text{special affine}$$

$$\mathbf{H}_P = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{v}^\top & w \end{bmatrix} \qquad \text{special projective}$$

$\mathbf{K}$ – upper triangular matrix with positive diagonal entries

$\mathbf{R}$ – orthogonal, $\mathbf{R}^\top\mathbf{R} = \mathbf{I}$, $\det\mathbf{R} = 1$

$s, w \in \mathbb{R}$, $s > 0$, $w \neq 0$

$$\mathbf{H} = \begin{bmatrix} s\mathbf{R}\mathbf{K} + \mathbf{t}\,\mathbf{v}^\top & w\,\mathbf{t} \\ \mathbf{v}^\top & w \end{bmatrix}$$

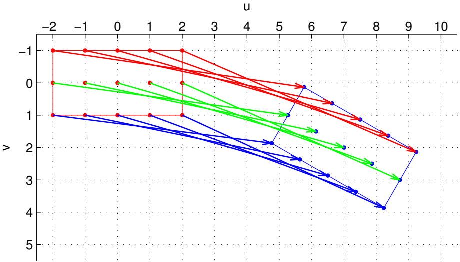
- must use 'thin' QR decomposition, which is unique [Golub & van Loan 2013, Sec. 5.2.6]
- $\mathbf{H}_S$, $\mathbf{H}_A$, $\mathbf{H}_P$ are homography subgroups
  (eg. $\mathbf{K} = \mathbf{K}_1\mathbf{K}_2$, $\mathbf{K}^{-1}$, $\mathbf{I}$ are all upper triangular with unit determinant, associativity holds)

▶Homography Subgroups: Euclidean Mapping

- Euclidean mapping: rotation, translation and their combination

$$\mathbf{H} = \begin{bmatrix} \cos\phi & -\sin\phi & t_x \\ \sin\phi & \cos\phi & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

- eigenvalues $\left(1,\, e^{-i\phi},\, e^{i\phi}\right)$



rotation by 30°, then translation by $(7,\ 2)$

1. $\det \mathbf{H} = 1$ ... **areas are preserved**
2. Let $\underline{\mathbf{x}}'_i = \mathbf{H}\underline{\mathbf{x}}_i$ (check we can use $=$ instead of $\simeq$). Let $(x_i)_3 = 1$, Then

$$\|\underline{\mathbf{x}}'_2 - \underline{\mathbf{x}}'_1\| = \|\mathbf{H}\underline{\mathbf{x}}_2 - \mathbf{H}\underline{\mathbf{x}}_1\| = \|\mathbf{H}(\underline{\mathbf{x}}_2 - \underline{\mathbf{x}}_1)\| = \cdots = \|\underline{\mathbf{x}}_2 - \underline{\mathbf{x}}_1\|$$

   and **lengths are preserved**
3. **angles are preserved**

- eigenvectors when $\phi \neq k\pi$, $k = 0, 1, \ldots$ (columnwise)

$$\mathbf{e}_1 \simeq \begin{bmatrix} t_x + t_y \cot\frac{\phi}{2} \\ t_y - t_x \cot\frac{\phi}{2} \\ 2 \end{bmatrix}, \quad \mathbf{e}_2 \simeq \begin{bmatrix} i \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{e}_3 \simeq \begin{bmatrix} -i \\ 1 \\ 0 \end{bmatrix}$$

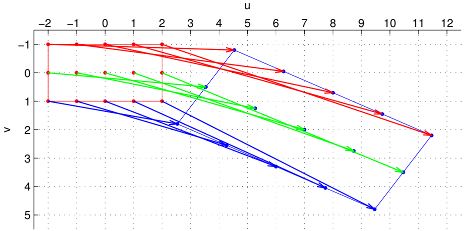
$\mathbf{e}_2$, $\mathbf{e}_3$ circular points

4. points at infinity $(i, 1, 0)$, $(-i, 1, 0)$: **circular points**; **are preserved** (by similarity)

- **similarity**: scaled Euclidean mapping (does not preserve lengths, areas)

▶Homography Subgroups: Affine Mapping

$$\mathbf{H} = \begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$$



rotation by 30°
then scaling by $\mathrm{diag}(1,\, 1.5,\, 1)$
then translation by $(7,\, 2)$

preserves

- parallelism
- ratio of areas
- ratio of lengths on parallel lines
- linear combinations of vectors (e.g. midpoints)
- convex hull
- line at infinity $\underline{\mathbf{n}}_\infty$ (not pointwise)

$$\mathbf{H}^\top \underline{\mathbf{n}}_\infty \simeq \underline{\mathbf{n}}_\infty \;\Rightarrow\; \underline{\mathbf{n}}_\infty \simeq \mathbf{H}^{-\top} \underline{\mathbf{n}}_\infty$$

does not preserve

- lengths
- angles
- areas
- circular points

Euclidean mappings preserve all properties affine mappings preserve, of course

►Homography Subgroups: General Homography

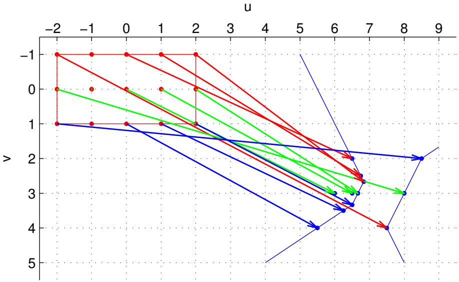
$$\mathbf{H} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$$

preserves only

- incidence and concurrency
- collinearity
- cross-ratio on the line   see later

does not preserve

- lengths
- areas
- parallelism
- ratio of areas
- ratio of lengths
- linear combinations of vectors (midpoints, etc.)
- convex hull
- line at infinity $\underline{\mathbf{n}}_\infty$

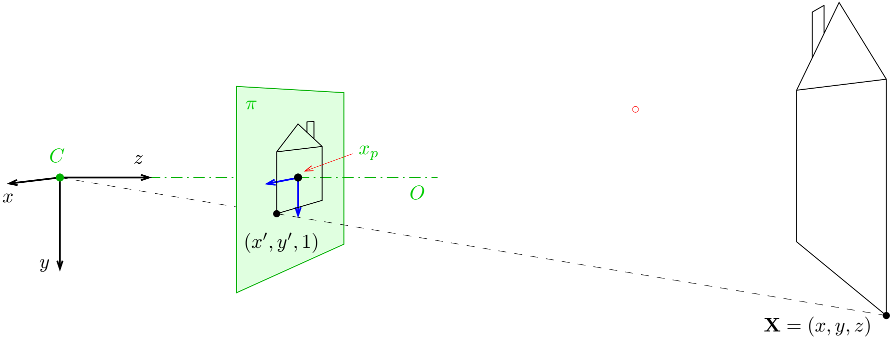


$$\mathbf{H} = \begin{bmatrix} 7 & -0.5 & 6 \\ 3 & 1 & 3 \\ 1 & 0 & 1 \end{bmatrix}$$

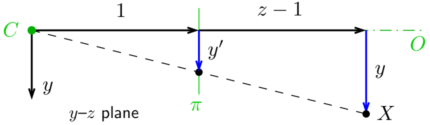
line $\underline{\mathbf{n}} = (1, 0, 1)$ is mapped to $\underline{\mathbf{n}}_\infty$: $\mathbf{H}^{-\top}\underline{\mathbf{n}} \simeq \underline{\mathbf{n}}_\infty$

(which line is it in the picture?)

►Canonical Perspective Camera (Pinhole Camera, Camera Obscura)



1. in this picture we are looking 'down the street'
2. right-handed canonical coordinate system (x, y, z)
3. origin = center of projection C
4. image plane π at <u>unit</u> distance from C
5. optical axis O is perpendicular to π
6. principal point x_p: intersection of O and π
7. perspective camera is given by C and π



projected point in the natural image coordinate system:

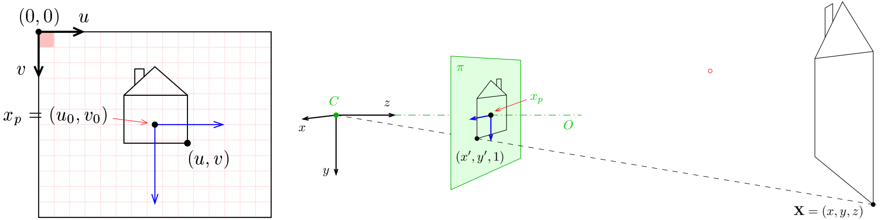
$$\frac{y'}{1} = y' = \frac{y}{1+z-1} = \frac{y}{z}, \qquad x' = \frac{x}{z}$$

▶Natural and Canonical Image Coordinate Systems

projected point **in canonical camera**

$$(x', y', 1) = \left(\frac{x}{z}, \frac{y}{z}, 1\right) = \frac{1}{z}(x, y, z) \simeq \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{\mathbf{P}_0} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \mathbf{P}_0 \, \underline{\mathbf{X}}$$

projected point **in scanned image** (scale by f and translate the coordinate system)



$$u = f\,\frac{x}{z} + u_0$$
$$v = f\,\frac{y}{z} + v_0$$

$$\frac{1}{z}\begin{bmatrix} f\,x + z\,u_0 \\ f\,y + z\,v_0 \\ z \end{bmatrix} \simeq \begin{bmatrix} f & 0 & u_0 \\ 0 & f & v_0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \mathbf{K}\mathbf{P}_0 \, \underline{\mathbf{X}} = \mathbf{P} \, \underline{\mathbf{X}}$$

- 'calibration' matrix $\mathbf{K}$ transforms canonical camera $\mathbf{P}_0$ to standard projective camera $\mathbf{P}$

►Computing with Perspective Camera Projection Matrix

$$\underline{\mathbf{m}} = \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} = \underbrace{\begin{bmatrix} f & 0 & u_0 & 0 \\ 0 & f & v_0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{\mathbf{P}} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \simeq \begin{bmatrix} f x + u_0 z \\ f y + v_0 z \\ z \end{bmatrix} \simeq \underbrace{\begin{bmatrix} x + \frac{z}{f} u_0 \\ y + \frac{z}{f} v_0 \\ \frac{z}{f} \end{bmatrix}}_{(a)}$$

$$\frac{m_1}{m_3} = \frac{f\,x}{z} + u_0 = u, \qquad \frac{m_2}{m_3} = \frac{f\,y}{z} + v_0 = v \quad \text{when} \quad m_3 \neq 0$$

f – 'focal length' – converts length ratios to pixels, $[f] = \mathrm{px}$, $f > 0$

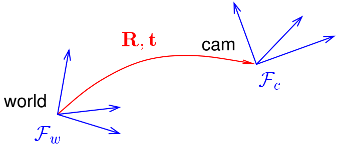
(u_0, v_0) – principal point in pixels

Perspective Camera:

1. dimension reduction — since $\mathbf{P} \in \mathbb{R}^{3,4}$
2. nonlinear unit change $\mathbf{1} \mapsto \mathbf{1} \cdot z/f$, see (a)
   for convenience we use $P_{11} = P_{22} = f$ rather than $P_{33} = 1/f$ and the u_0, v_0 in relative units
3. $m_3 = 0$ represents points at infinity in image plane π — i.e. points with $z = 0$

▶Changing The Outer (World) Reference Frame

A transformation of a point from the world to camera coordinate system:



$$\mathbf{X}_c = \mathbf{R}\,\mathbf{X}_w + \mathbf{t}$$

$\mathbf{R}$ – camera rotation matrix — world orientation in the camera coordinate frame

$\mathbf{t}$ – camera translation vector — world origin in the camera coordinate frame

$$\mathbf{P}\,\underline{\mathbf{X}}_c = \mathbf{K}\mathbf{P}_0 \begin{bmatrix} \mathbf{X}_c \\ 1 \end{bmatrix} = \mathbf{K}\mathbf{P}_0 \begin{bmatrix} \mathbf{R}\mathbf{X}_w + \mathbf{t} \\ 1 \end{bmatrix} = \mathbf{K}\mathbf{P}_0 \underbrace{\begin{bmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix}}_{\mathbf{T}} \begin{bmatrix} \mathbf{X}_w \\ 1 \end{bmatrix} = \mathbf{K}\begin{bmatrix} \mathbf{R} & \mathbf{t} \end{bmatrix} \underline{\mathbf{X}}_w$$

$\mathbf{P}_0$ selects the first 3 rows of $\mathbf{T}$ and discards the last row

- $\mathbf{R}$ is rotation, $\mathbf{R}^\top\mathbf{R} = \mathbf{I}$, $\det \mathbf{R} = +1$ — $\mathbf{I} \in \mathbb{R}^{3,3}$ identity matrix
- 6 **extrinsic parameters**: 3 rotation angles (Euler theorem), 3 translation components
- alternative, often used, camera representations

$$\mathbf{P} = \mathbf{K}\begin{bmatrix} \mathbf{R} & \mathbf{t} \end{bmatrix} = \mathbf{K}\mathbf{R}\begin{bmatrix} \mathbf{I} & -\mathbf{C} \end{bmatrix}$$

$\mathbf{C}$ – camera position in the world reference frame — $\mathbf{t} = -\mathbf{R}\mathbf{C}$

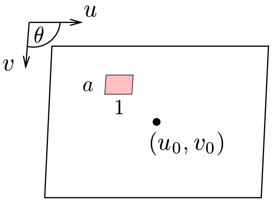
$\mathbf{r}_3^\top$ – optical axis in the world reference frame — third row of $\mathbf{R}$: $\mathbf{r}_3 = \mathbf{R}^{-1}[0, 0, 1]^\top$

- we can save some conversion and computation by noting that $\mathbf{K}\mathbf{R}\begin{bmatrix} \mathbf{I} & -\mathbf{C} \end{bmatrix} \underline{\mathbf{X}} = \mathbf{K}\mathbf{R}(\mathbf{X} - \mathbf{C})$

▶Changing the Inner (Image) Reference Frame

The general form of calibration matrix $\mathbf{K}$ includes

- skew angle θ of the digitization raster
- pixel aspect ratio a



$$\mathbf{K} = \begin{bmatrix} f & -f\cot\theta & u_0 \\ 0 & f/(a\sin\theta) & v_0 \\ 0 & 0 & 1 \end{bmatrix}$$

units: $[f] = \mathrm{px}$, $[u_0] = \mathrm{px}$, $[v_0] = \mathrm{px}$, $[a] = 1$

⊛ H1; 2pt: Verify this $\mathbf{K}$. Hints: express point $\mathbf{x}$ as $\mathbf{x} = u'\mathbf{e}_{u'} + v'\mathbf{e}_{v'} = u\mathbf{e}_u + v\mathbf{e}_v$, $\mathbf{e}_u$, $\mathbf{e}_v$ etc. are basis vectors, $\mathbf{K}$ maps from an orthogonal system to a skewed system $[w'u', w'v', w']^\top = \mathbf{K}[u, v, 1]^\top$; map first by skew then by sampling scale then shift by u_0, v_0 deadline LD+2 wk

general finite perspective camera has 11 parameters:

- 5 intrinsic parameters: f, u_0, v_0, a, θ
- 6 extrinsic parameters: $\mathbf{t}$, $\mathbf{R}(\alpha, \beta, \gamma)$

finite camera: $\det \mathbf{K} \neq 0$

$$\underline{\mathbf{m}} \simeq \mathbf{P}\underline{\mathbf{X}}, \qquad \mathbf{P} = \begin{bmatrix}\mathbf{Q} & \mathbf{q}\end{bmatrix} = \mathbf{K}\begin{bmatrix}\mathbf{R} & \mathbf{t}\end{bmatrix} = \mathbf{K}\mathbf{R}\begin{bmatrix}\mathbf{I} & -\mathbf{C}\end{bmatrix}$$

a recipe for filling $\mathbf{P}$

Representation Theorem: The set of projection matrices $\mathbf{P}$ of finite projective cameras is isomorphic to the set of homogeneous 3×4 matrices with the left hand 3×3 submatrix $\mathbf{Q}$ non-singular.

►Projection Matrix Decomposition

$$\mathbf{P} = \begin{bmatrix}\mathbf{Q} & \mathbf{q}\end{bmatrix} \longrightarrow \mathbf{K}\mathbf{R}\begin{bmatrix}\mathbf{I} & -\mathbf{C}\end{bmatrix} = \mathbf{K}\begin{bmatrix}\mathbf{R} & \mathbf{t}\end{bmatrix}$$

$\mathbf{Q} \in \mathbb{R}^{3,3}$	full rank	(if finite perspective camera)
$\mathbf{K} \in \mathbb{R}^{3,3}$	upper triangular with positive diagonal entries	
$\mathbf{R} \in \mathbb{R}^{3,3}$	rotation: $\mathbf{R}^\top\mathbf{R} = \mathbf{I}$ and $\det \mathbf{R} = +1$	

1. $\begin{bmatrix}\mathbf{Q} & \mathbf{q}\end{bmatrix} = \mathbf{Q}\begin{bmatrix}\mathbf{I} & \mathbf{Q}^{-1}\mathbf{q}\end{bmatrix} = \mathbf{K}\mathbf{R}\begin{bmatrix}\mathbf{I} & -\mathbf{C}\end{bmatrix} = \mathbf{K}\begin{bmatrix}\mathbf{R} & -\mathbf{R}\mathbf{C}\end{bmatrix} = \mathbf{K}\begin{bmatrix}\mathbf{R} & \mathbf{t}\end{bmatrix}$ also →34
2. RQ decomposition of $\mathbf{Q} = \mathbf{K}\mathbf{R}$ using three Givens rotations [H&Z, p. 579]

$$\mathbf{K} = \mathbf{Q}\,\underbrace{\mathbf{R}_{32}\mathbf{R}_{31}\mathbf{R}_{21}}_{\mathbf{R}^{-1}}$$

$\mathbf{R}_{ij}$ zeroes element ij in $\mathbf{Q}$ affecting only columns i and j and the sequence preserves previously zeroed elements, e.g.

$$\mathbf{R}_{32} = \begin{bmatrix}1 & 0 & 0\\ 0 & c & -s\\ 0 & s & c\end{bmatrix} \text{ gives } \begin{matrix} c^2+s^2=1 \\ 0 = k_{32} = c\,q_{32} + s\,q_{33}\end{matrix} \Rightarrow c = \frac{q_{33}}{\sqrt{q_{32}^2+q_{33}^2}} \quad s = \frac{-q_{32}}{\sqrt{q_{32}^2+q_{33}^2}}$$

⊛ P1; 1pt: Multiply known matrices $\mathbf{K}$, $\mathbf{R}$ and then decompose back; discuss numerical errors

- RQ decomposition nonuniqueness: $\mathbf{K}\mathbf{R} = \mathbf{K}\mathbf{T}^{-1}\mathbf{T}\mathbf{R}$, where $\mathbf{T} = \operatorname{diag}(-1,-1,1)$ is also a rotation, we must correct the result so that the diagonal elements of $\mathbf{K}$ are all positive
  'thin' RQ decomposition
- care must be taken to avoid overflow, see [Golub & van Loan 2013, sec. 5.2]

RQ Decomposition Step

```
Q = Array [q#1,#2 &, {3, 3}];
R32 = {{1, 0, 0}, {0, c, -s}, {0, s, c}}; R32 // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & c & -s \\ 0 & s & c \end{pmatrix}$$

```
Q1 = Q.R32 ; Q1 // MatrixForm
```

$$\begin{pmatrix} q_{1,1} & c\, q_{1,2} + s\, q_{1,3} & -s\, q_{1,2} + c\, q_{1,3} \\ q_{2,1} & c\, q_{2,2} + s\, q_{2,3} & -s\, q_{2,2} + c\, q_{2,3} \\ q_{3,1} & c\, q_{3,2} + s\, q_{3,3} & -s\, q_{3,2} + c\, q_{3,3} \end{pmatrix}$$

```
s1 = Solve [{Q1[[3]][[2]] == 0, c^2 + s^2 == 1}, {c, s}][[2]]
```

$$\left\{ c \to \frac{q_{3,3}}{\sqrt{q_{3,2}^2 + q_{3,3}^2}},\ s \to -\frac{q_{3,2}}{\sqrt{q_{3,2}^2 + q_{3,3}^2}} \right\}$$

```
Q1 /. s1 // Simplify // MatrixForm
```

$$\begin{pmatrix} q_{1,1} & \frac{-q_{1,3}\, q_{3,2} + q_{1,2}\, q_{3,3}}{\sqrt{q_{3,2}^2 + q_{3,3}^2}} & \frac{q_{1,2}\, q_{3,2} + q_{1,3}\, q_{3,3}}{\sqrt{q_{3,2}^2 + q_{3,3}^2}} \\ q_{2,1} & \frac{-q_{2,3}\, q_{3,2} + q_{2,2}\, q_{3,3}}{\sqrt{q_{3,2}^2 + q_{3,3}^2}} & \frac{q_{2,2}\, q_{3,2} + q_{2,3}\, q_{3,3}}{\sqrt{q_{3,2}^2 + q_{3,3}^2}} \\ q_{3,1} & 0 & \sqrt{q_{3,2}^2 + q_{3,3}^2} \end{pmatrix}$$

▶Center of Projection

Observation: finite $\mathbf{P}$ has a non-trivial right null-space — rank 3 but 4 columns

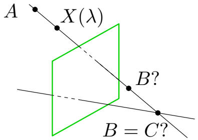
Theorem

Let there be $\underline{\mathbf{B}} \neq \mathbf{0}$ s.t. $\mathbf{P}\,\underline{\mathbf{B}} = \mathbf{0}$. Then $\underline{\mathbf{B}}$ is equal to the projection center $\underline{\mathbf{C}}$ (in world coordinate frame).

Proof.

1. Consider spatial line $A\,B$ (B is given). We can write

$$\underline{\mathbf{X}}(\lambda) \simeq \underline{\mathbf{A}} + \lambda\,\underline{\mathbf{B}}, \qquad \lambda \in \mathbb{R}$$



2. it images to

$$\mathbf{P}\underline{\mathbf{X}}(\lambda) \simeq \mathbf{P}\,\underline{\mathbf{A}} + \lambda\,\mathbf{P}\,\underline{\mathbf{B}} = \mathbf{P}\,\underline{\mathbf{A}}$$

- the whole line images to a single point $\Rightarrow$ it must pass through the optical center of $\mathbf{P}$
- this holds for all choices of A $\Rightarrow$ the only common point of the lines is the C, i.e. $\underline{\mathbf{B}} \simeq \underline{\mathbf{C}}$

□

Hence

$$\mathbf{0} = \mathbf{P}\,\underline{\mathbf{C}} = \begin{bmatrix}\mathbf{Q} & \mathbf{q}\end{bmatrix}\begin{bmatrix}\mathbf{C} \\ 1\end{bmatrix} = \mathbf{Q}\,\mathbf{C} + \mathbf{q} \Rightarrow \boxed{\mathbf{C} = -\mathbf{Q}^{-1}\mathbf{q}}$$

$\underline{\mathbf{C}} = (c_j)$, where $c_j = (-1)^j \det \mathbf{P}^{(j)}$, in which $\mathbf{P}^{(j)}$ is $\mathbf{P}$ with column j dropped

Matlab: `C_homo = null(P);` or `C = -Q\q;`

▶Optical Ray

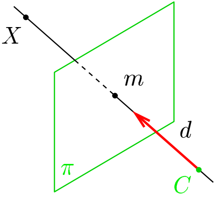
Optical ray: Spatial line that projects to a single image point.

1. consider line

   $\mathbf{d}$ unit line direction vector, $\|\mathbf{d}\| = 1$, $\lambda \in \mathbb{R}$, Cartesian representation

$$\mathbf{X} = \mathbf{C} + \lambda\,\mathbf{d}$$

2. the image of the (finite) point X is



$$\underline{\mathbf{m}} \simeq \begin{bmatrix}\mathbf{Q} & \mathbf{q}\end{bmatrix} \begin{bmatrix}\mathbf{X} \\ 1\end{bmatrix} = \mathbf{Q}(\mathbf{C} + \lambda\mathbf{d}) + \mathbf{q} = \lambda\,\mathbf{Q}\,\mathbf{d} =$$

$$= \lambda \begin{bmatrix}\mathbf{Q} & \mathbf{q}\end{bmatrix} \begin{bmatrix}\mathbf{d} \\ 0\end{bmatrix}$$

...which is also the image of a point at infinity in $\mathbb{P}^3$

- optical ray line corresponding to image point m is

$$\boxed{\mathbf{X} = \mathbf{C} + (\lambda\mathbf{Q})^{-1}\underline{\mathbf{m}}, \qquad \lambda \in \mathbb{R}}$$

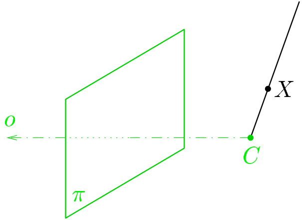
- optical ray may be represented by a point at infinity $(\mathbf{d}, 0)$ in $\mathbb{P}^3$

▶Optical Axis

Optical axis: The line through C that is perpendicular to image plane π

1. a line parallel to π images to line at infinity in π:

$$\begin{bmatrix} u \\ v \\ 0 \end{bmatrix} \simeq \mathbf{P}\underline{\mathbf{X}} = \begin{bmatrix} \mathbf{q}_1^\top & q_{14} \\ \mathbf{q}_2^\top & q_{24} \\ \mathbf{q}_3^\top & q_{34} \end{bmatrix} \begin{bmatrix} \mathbf{X} \\ 1 \end{bmatrix}$$



2. therefore the set of points X is parallel to π iff

$$\mathbf{q}_3^\top \mathbf{X} + q_{34} = 0$$

3. this is a plane with $\pm\mathbf{q}_3$ as the normal vector
4. optical axis direction: substitution $\mathbf{P} \mapsto \lambda\mathbf{P}$ must not change the direction
5. we select (assuming $\det(\mathbf{R}) > 0$)

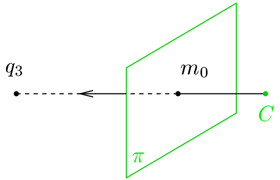
$$\boxed{\mathbf{o} = \det(\mathbf{Q})\,\mathbf{q}_3}$$

if $\mathbf{P} \mapsto \lambda\mathbf{P}$ then $\det(\mathbf{Q}) \mapsto \lambda^3 \det(\mathbf{Q})$ and $\mathbf{q}_3 \mapsto \lambda\,\mathbf{q}_3$ [H&Z, p. 161]

►Principal Point

Principal point: The intersection of image plane and the optical axis

1. as we saw, $\mathbf{q}_3$ is the directional vector of optical axis
2. we take point at infinity on the optical axis that must project to principal point m_0



3. then

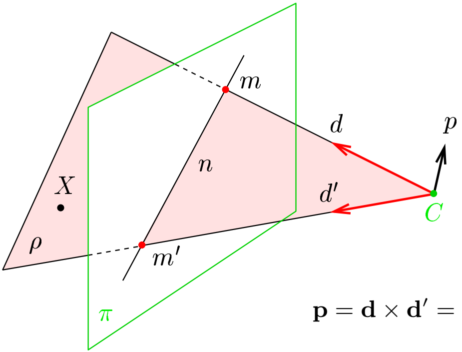
$$\underline{\mathbf{m}}_0 \simeq \begin{bmatrix}\mathbf{Q} & \mathbf{q}\end{bmatrix} \begin{bmatrix}\mathbf{q}_3 \\ 0\end{bmatrix} = \mathbf{Q}\,\mathbf{q}_3$$

principal point: $\boxed{\underline{\mathbf{m}}_0 \simeq \mathbf{Q}\,\mathbf{q}_3}$

- principal point is also the center of radial distortion (see →**??**)

►Optical Plane

A spatial plane with normal p passing through optical center C and a given image line n.



optical ray given by m $\quad \mathbf{d} = \mathbf{Q}^{-1}\underline{\mathbf{m}}$

optical ray given by m' $\quad \mathbf{d}' = \mathbf{Q}^{-1}\underline{\mathbf{m}}'$

$$\mathbf{p} = \mathbf{d} \times \mathbf{d}' = (\mathbf{Q}^{-1}\underline{\mathbf{m}}) \times (\mathbf{Q}^{-1}\underline{\mathbf{m}}') = \mathbf{Q}^{\top}(\underline{\mathbf{m}} \times \underline{\mathbf{m}}') = \mathbf{Q}^{\top}\underline{\mathbf{n}}$$

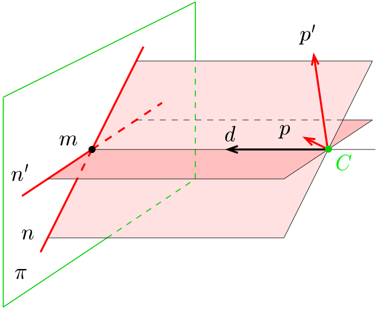
• note the way $\mathbf{Q}$ factors out!

hence, $0 = \mathbf{p}^{\top}(\mathbf{X} - \mathbf{C}) = \underline{\mathbf{n}}^{\top}\mathbf{Q}(\mathbf{X} - \mathbf{C}) = \underline{\mathbf{n}}^{\top}\mathbf{P}\underline{\mathbf{X}} = (\mathbf{P}^{\top}\underline{\mathbf{n}})^{\top}\underline{\mathbf{X}}$ for every X in plane ρ

see →30

optical plane is given by n: $\quad \boxed{\boldsymbol{\rho} \simeq \mathbf{P}^{\top}\underline{\mathbf{n}}}$ $\qquad \rho_1\, x + \rho_2\, y + \rho_3\, z + \rho_4 = 0$

Cross-Check: Optical Ray as Optical Plane Intersection



optical plane normal given by n $\qquad$ $\mathbf{p} = \mathbf{Q}^\top \underline{\mathbf{n}}$

optical plane normal given by n' $\qquad$ $\mathbf{p}' = \mathbf{Q}^\top \underline{\mathbf{n}}'$

$$\mathbf{d} = \mathbf{p} \times \mathbf{p}' = (\mathbf{Q}^\top \underline{\mathbf{n}}) \times (\mathbf{Q}^\top \underline{\mathbf{n}}') = \mathbf{Q}^{-1}(\underline{\mathbf{n}} \times \underline{\mathbf{n}}') = \mathbf{Q}^{-1}\underline{\mathbf{m}}$$

▶Summary: Optical Center, Ray, Axis, Plane

General finite camera

$$\mathbf{P} = \begin{bmatrix}\mathbf{Q} & \mathbf{q}\end{bmatrix} = \begin{bmatrix}\mathbf{q}_1^\top & q_{14} \\ \mathbf{q}_2^\top & q_{24} \\ \mathbf{q}_3^\top & q_{34}\end{bmatrix} = \mathbf{K}\begin{bmatrix}\mathbf{R} & \mathbf{t}\end{bmatrix} = \mathbf{K}\,\mathbf{R}\begin{bmatrix}\mathbf{I} & -\mathbf{C}\end{bmatrix}$$

$\underline{\mathbf{C}} \simeq \operatorname{rnull}(\mathbf{P})$	optical center (world coords.)
$\mathbf{d} = \mathbf{Q}^{-1}\,\underline{\mathbf{m}}$	optical ray direction (world coords.)
$\det(\mathbf{Q})\,\mathbf{q}_3$	outward optical axis (world coords.)
$\mathbf{Q}\,\mathbf{q}_3$	principal point (in image plane)
$\boldsymbol{\rho} = \mathbf{P}^\top\,\underline{\mathbf{n}}$	optical plane (world coords.)
$\mathbf{K} = \begin{bmatrix} f & -f\cot\theta & u_0 \\ 0 & f/(a\sin\theta) & v_0 \\ 0 & 0 & 1\end{bmatrix}$	camera (calibration) matrix (f, u_0, v_0 in pixels)
$\mathbf{R}$	camera rotation matrix (cam coords.)
$\mathbf{t}$	camera translation vector (cam coords.)

What Can We Do with An 'Uncalibrated' Perspective Camera?

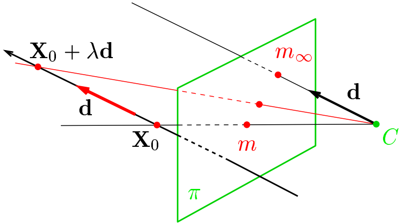
How far is the engine?

distance between sleepers (ties) 0.806m but we cannot count them, resolution is too low

We will review some life-saving theory. . .
. . . and build a bit of geometric intuition. . .

▶Vanishing Point

Vanishing point: the limit of the projection of a point that moves along a space line infinitely in one direction. the image of the point at infinity on the line



$$\underline{\mathbf{m}}_\infty \simeq \lim_{\lambda\to\pm\infty} \mathbf{P} \begin{bmatrix} \mathbf{X}_0 + \lambda\mathbf{d} \\ 1 \end{bmatrix} = \cdots = \mathbf{Q}\,\mathbf{d}$$

⊛ P1; 1pt: Derive or prove

- V.P. is independent on line position, it depends on its orientation only
  all parallel lines have the same V.P.
- the image of the V.P. of a spatial line with direction vector $\mathbf{d}$ is $\underline{\mathbf{m}} = \mathbf{Q}\,\mathbf{d}$
- V.P. m corresponds to spatial direction $\mathbf{d} = \mathbf{Q}^{-1}\underline{\mathbf{m}}$ optical ray through m

- V.P. is the image of a point at infinity on any line, not just the optical ray as on →35

Some Vanishing Point Applications

where is the sun?

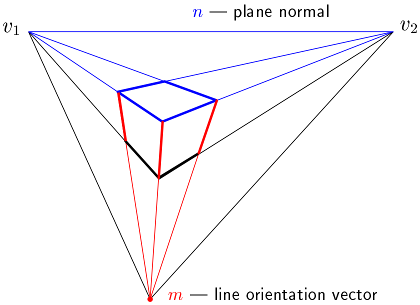
what is the wind direction?
(must have video)

fly above the lane,
at constant altitude!

►Vanishing Line

Vanishing line: The set of vanishing points of all lines in a plane

the image of the line at infinity in the plane
and in all parallel planes



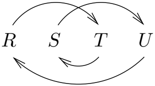
- V.L. n corresponds to space plane of normal vector $\mathbf{p} = \mathbf{Q}^\top \underline{\mathbf{n}}$

  because this is the normal vector of a parallel optical plane (!) →38
- a space plane of normal vector $\mathbf{p}$ has a V.L. represented by $\underline{\mathbf{n}} = \mathbf{Q}^{-\top}\mathbf{p}$.

▶Cross Ratio

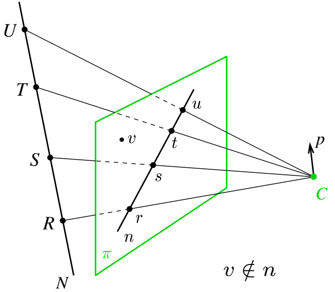
Four distinct collinear space points R, S, T, U define cross-ratio

$$[RSTU] = \frac{|\overrightarrow{RT}|}{|\overrightarrow{UR}|}\,\frac{|\overrightarrow{SU}|}{|\overrightarrow{TS}|}$$



a mnemonic

$|\overrightarrow{RT}|$ – signed distance from R to T (w.r.t. a fixed line orientation)



$$[SRUT] = [RSTU],\ [RSUT] = \frac{1}{[RSTU]},\ [RTSU] = 1 - [RSTU]$$

Obs: $$[RSTU] = \frac{|\underline{\mathbf{r}}\ \ \underline{\mathbf{t}}\ \ \underline{\mathbf{v}}|}{|\underline{\mathbf{r}}\ \ \underline{\mathbf{u}}\ \ \underline{\mathbf{v}}|} \cdot \frac{|\underline{\mathbf{s}}\ \ \underline{\mathbf{u}}\ \ \underline{\mathbf{v}}|}{|\underline{\mathbf{s}}\ \ \underline{\mathbf{t}}\ \ \underline{\mathbf{v}}|}, \qquad |\underline{\mathbf{r}}\ \ \underline{\mathbf{t}}\ \ \underline{\mathbf{v}}| = \det\left[\underline{\mathbf{r}}\ \ \underline{\mathbf{t}}\ \ \underline{\mathbf{v}}\right] = (\underline{\mathbf{r}} \times \underline{\mathbf{t}})^\top \underline{\mathbf{v}} \qquad (1)$$

Corollaries:

- cross ratio is invariant under homographies $\underline{\mathbf{x}}' \simeq \mathbf{H}\underline{\mathbf{x}}$ plug $\mathbf{H}\underline{\mathbf{x}}$ in (1): $(\mathbf{H}^{-\top}(\underline{\mathbf{r}} \times \underline{\mathbf{t}}))^\top \mathbf{H}\underline{\mathbf{v}}$
- cross ratio is invariant under perspective projection: $[RSTU] = [r\,s\,t\,u]$

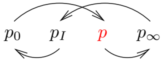
- 4 collinear points: any perspective camera will "see" the same cross-ratio of their images
- we measure the same cross-ratio in image as on the world line
- one of the points R, S, T, U may be at infinity (we take the limit, in effect $\frac{\infty}{\infty} = 1$)

▶1D Projective Coordinates

The 1-D projective coordinate of a point P is defined by the following cross-ratio:

$$[P] = [P_\infty \, P_0 \, P_I \, P] = [p_\infty \, p_0 \, p_I \, p] = \frac{|\overrightarrow{p_0\, p}|}{|\overrightarrow{p_I\, p_0}|} \, \frac{|\overrightarrow{p_\infty\, p_I}|}{|\overrightarrow{p\, p_\infty}|} = [p]$$

the mnemonic now is '∞'

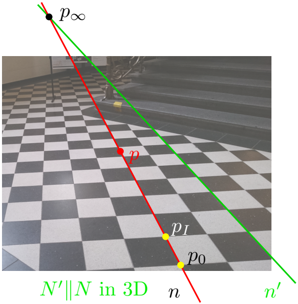


naming convention:

P_0 – the origin	$[P_0] = 0$
P_I – the unit point	$[P_I] = 1$
P_∞ – the supporting point	$[P_\infty] = \pm\infty$

$[P]$ is equal to Euclidean coordinate along N

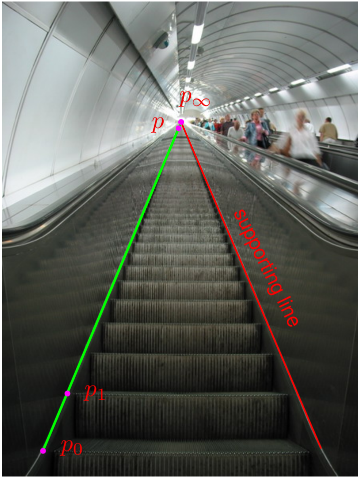
$[p]$ is its measurement in the image plane



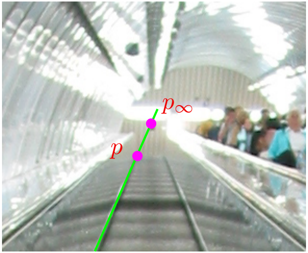
Applications

- Given the image of a 3D line N, the origin, the unit point, and the vanishing point, then the Euclidean coordinate of any point $P \in N$ can be determined →47
- Finding v.p. of a line through a regular object →48

Application: Counting Steps



- Namesti Miru underground station in Prague

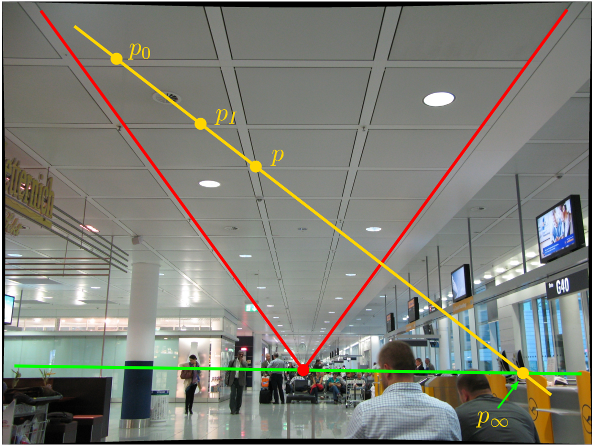


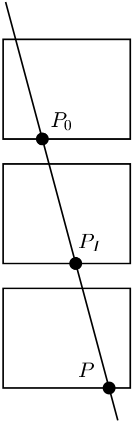
detail around the vanishing point

Result: $[P] = 214$ steps (correct answer is 216 steps)

4Mpx camera

Application: Finding the Horizon from Repetitions





in 3D: $|P_0 P| = 2|P_0 P_I|$ then [H&Z, p. 218]⊛ P1; 1pt: How high is the camera above the floor?

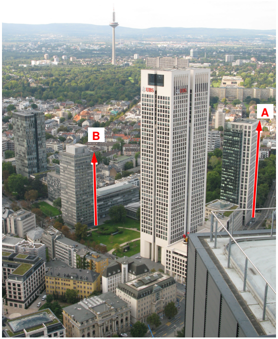
$$[P_\infty P_0 P_I P] = \frac{|P_0 P|}{|P_0 P_I|} = 2 \quad \Rightarrow \quad |p_\infty p_0| = \frac{|p_0 p_I| \cdot |p_0 p|}{|p_0 p| - 2|p_0 p_I|}$$

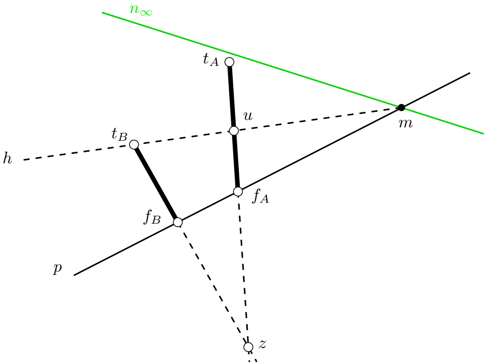
- could be applied to counting steps (→47)

Homework Problem

⊛ H2; 3pt: What is the ratio of heights of Building A to Building B?

- expected: conceptual solution; use notation from this figure
- deadline: LD+2 weeks

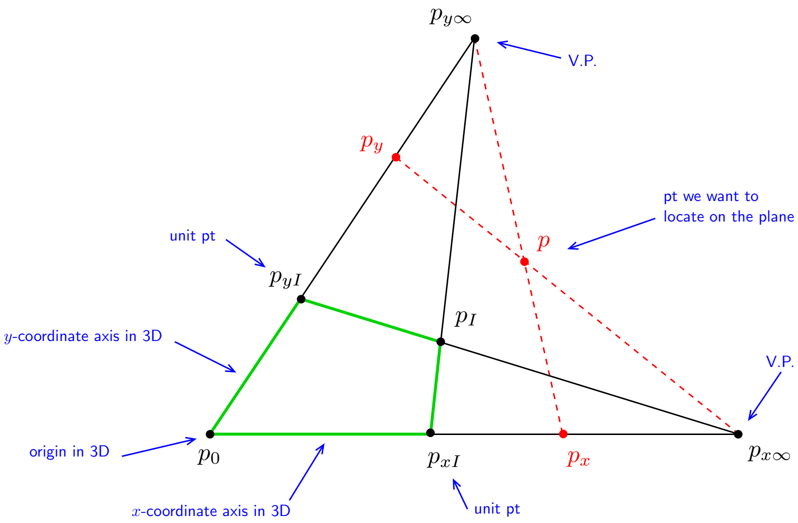




Hints

1. what are the properties of line h connecting the top of Buiding B t_B with the point m at which the horizon is intersected with the line p joining the foots f_A, f_B of both buildings? [1 point]
2. how do we actually get the horizon n_∞? (we do not see it directly, there are hills there) [1 point]
3. what tool measures the length ratio? [formula = 1 point]

2D Projective Coordinates



$$[P_x] = [P_{x\infty}\ P_0\ P_{xI}\ P_x] \qquad [P_y] = [P_{y\infty}\ P_0\ P_{yI}\ P_y]$$

Application: Measuring on the Floor (Wall, etc)

San Giovanni in Laterano, Rome

- measuring distances on the floor in terms of tile units
- what are the dimensions of the seal? Is it circular (assuming square tiles)?
- needs no explicit camera calibration

because we can see the calibrating object (vanishing points)

Part III

Computing with a Single Camera

3.1 Calibration: Internal Camera Parameters from Vanishing Points and Lines

3.2 Camera Resection: Projection Matrix from 6 Known Points

3.3 Exterior Orientation: Camera Rotation and Translation from 3 Known Points

covered by

[1] [H&Z] Secs: 8.6, 7.1, 22.1

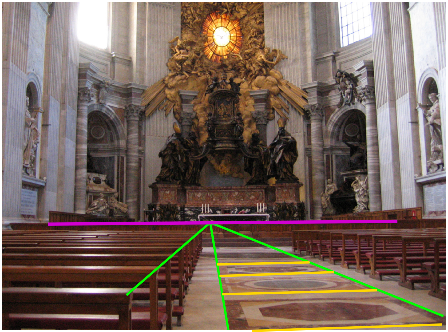
[2] Fischler, M.A. and Bolles, R.C . Random Sample Consensus: A Paradigm for Model Fitting with Applications to Image Analysis and Automated Cartography. *Communications of the ACM* 24(6):381–395, 1981

[3] [Golub & van Loan 2013, Sec. 2.5]

Obtaining Vanishing Points and Lines

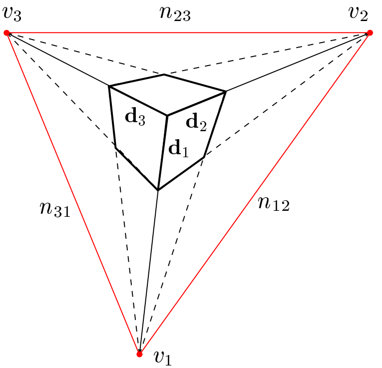
- orthogonal direction pairs can be collected from more images by camera rotation

- vanishing line can be obtained without vanishing points ($\rightarrow$48)



▶Camera Calibration from Vanishing Points and Lines

Problem: Given finite vanishing points and/or vanishing lines, compute $\mathbf{K}$



$$\mathbf{d}_i = \mathbf{Q}^{-1}\underline{\mathbf{v}}_i, \qquad i = 1,2,3 \qquad \rightarrow 42$$
$$\mathbf{p}_{ij} = \mathbf{Q}^{\top}\underline{\mathbf{n}}_{ij}, \quad i,j = 1,2,3,\ i \neq j \qquad \rightarrow 38 \tag{2}$$

- naive method: solve linear eqs. (2)

Constraints

1. orthogonal rays $\mathbf{d}_1 \perp \mathbf{d}_2$ in space then
$$0 = \mathbf{d}_1^\top \mathbf{d}_2 = \underline{\mathbf{v}}_1^\top \mathbf{Q}^{-\top}\mathbf{Q}^{-1}\underline{\mathbf{v}}_2 = \underline{\mathbf{v}}_1^\top \underbrace{(\mathbf{K}\mathbf{K}^\top)^{-1}}_{\boldsymbol{\omega}\ \text{(IAC)}} \underline{\mathbf{v}}_2$$
2. orthogonal planes $\mathbf{p}_{ij} \perp \mathbf{p}_{ik}$ in space
$$0 = \mathbf{p}_{ij}^\top \mathbf{p}_{ik} = \underline{\mathbf{n}}_{ij}^\top \mathbf{Q}\mathbf{Q}^\top \underline{\mathbf{n}}_{ik} = \underline{\mathbf{n}}_{ij}^\top \boldsymbol{\omega}^{-1} \underline{\mathbf{n}}_{ik}$$
3. orthogonal ray and plane $\mathbf{d}_k \parallel \mathbf{p}_{ij}$, $k \neq i,j$ normal parallel to optical ray
$$\mathbf{p}_{ij} \simeq \mathbf{d}_k \quad \Rightarrow \quad \mathbf{Q}^\top \underline{\mathbf{n}}_{ij} = \lambda \mathbf{Q}^{-1}\underline{\mathbf{v}}_k \quad \Rightarrow \quad \underline{\mathbf{n}}_{ij} = \lambda \mathbf{Q}^{-\top}\mathbf{Q}^{-1}\underline{\mathbf{v}}_k = \lambda \boldsymbol{\omega}\,\underline{\mathbf{v}}_k, \qquad \lambda \neq 0$$

- n_{ij} may be constructed from non-orthogonal v_i and v_j, e.g. using the cross-ratio
- $\boldsymbol{\omega}$ is a symmetric, positive definite 3×3 matrix

IAC = Image of Absolute Conic

►cont'd

	condition	equation	# constraints
(3)	orthogonal v.p.	$\underline{\mathbf{v}}_i^\top \boldsymbol{\omega}\, \underline{\mathbf{v}}_j = 0$	1
(4)	orthogonal v.l.	$\underline{\mathbf{n}}_{ij}^\top \boldsymbol{\omega}^{-1} \underline{\mathbf{n}}_{ik} = 0$	1
(5)	v.p. orthogonal to v.l.	$\underline{\mathbf{n}}_{ij} = \lambda \boldsymbol{\omega}\, \underline{\mathbf{v}}_k$	2
(6)	orthogonal raster $\theta = \pi/2$	$\omega_{12} = \omega_{21} = 0$	1
(7)	unit aspect $a = 1$ when $\theta = \pi/2$	$\omega_{11} - \omega_{22} = 0$	1
(8)	known principal point $u_0 = v_0 = 0$	$\omega_{13} = \omega_{31} = \omega_{23} = \omega_{32} = 0$	2

- these are homogeneous linear equations for the 5 parameters in $\boldsymbol{\omega}$ in the form $\mathbf{D}\mathbf{w} = \mathbf{0}$
  λ can be eliminated from (5)
- we need at least 5 constraints for full $\boldsymbol{\omega}$
  symmetric 3×3
- we get $\mathbf{K}$ from $\boldsymbol{\omega}^{-1} = \mathbf{K}\mathbf{K}^\top$ by Choleski decomposition
  the decomposition returns a positive definite upper triangular matrix
  one avoids solving an explicit set of quadratic equations for the parameters in $\mathbf{K}$
- unlike in the naive method solving (2), we can introduce constraints on $\mathbf{K}$

Examples

Assuming orthogonal raster, unit aspect (ORUA): $\theta = \pi/2$, $a = 1$

$$\boldsymbol{\omega} \simeq \begin{bmatrix} 1 & 0 & -u_0 \\ 0 & 1 & -v_0 \\ -u_0 & -v_0 & f^2 + u_0^2 + v_0^2 \end{bmatrix}$$

Ex 1:

Assuming ORUA and known $m_0 = (u_0, v_0)$, two <u>finite</u> orthogonal vanishing points give f

$$\underline{\mathbf{v}}_1^\top \boldsymbol{\omega}\, \underline{\mathbf{v}}_2 = 0 \quad \Rightarrow \quad f^2 = \left|(\mathbf{v}_1 - \mathbf{m}_0)^\top (\mathbf{v}_2 - \mathbf{m}_0)\right|$$

in this formula, $\mathbf{v}_i$, $\mathbf{m}_0$ are not homogeneous!

Ex 2:

Non-orthogonal vanishing points $\mathbf{v}_i$, $\mathbf{v}_j$, known angle ϕ: $\cos\phi = \dfrac{\underline{\mathbf{v}}_i^\top \boldsymbol{\omega} \underline{\mathbf{v}}_j}{\sqrt{\underline{\mathbf{v}}_i^\top \boldsymbol{\omega} \underline{\mathbf{v}}_i}\sqrt{\underline{\mathbf{v}}_j^\top \boldsymbol{\omega} \underline{\mathbf{v}}_j}}$

- leads to polynomial equations
- e.g. ORUA and $u_0 = v_0 = 0$ gives

$$(f^2 + \mathbf{v}_i^\top \mathbf{v}_j)^2 = (f^2 + \|\mathbf{v}_i\|^2) \cdot (f^2 + \|\mathbf{v}_j\|^2) \cdot \cos^2\phi$$

In[1]:=
```
K = {{f, s, u[0]}, {0, a * f, v[0]}, {0, 0, 1}};
K // MatrixForm
```

Out[2]//MatrixForm=

$$\begin{pmatrix} f & s & u[0] \\ 0 & a\,f & v[0] \\ 0 & 0 & 1 \end{pmatrix}$$

In[4]:=
```
ω = Inverse[K.Transpose[K]] * Det[K]^2;
ω // Simplify // MatrixForm
```

Out[5]//MatrixForm=

$$\begin{pmatrix} a^2 f^2 & -a f s & a f (-a f u[0] + s v[0]) \\ -a f s & f^2 + s^2 & a f s u[0] - (f^2 + s^2) v[0] \\ a f (-a f u[0] + s v[0]) & a f s u[0] - (f^2 + s^2) v[0] & a^2 f^2 (f^2 + u[0]^2) - 2 a f s u[0] v[0] + (f^2 + s^2) v[0]^2 \end{pmatrix}$$

In[8]:=
```
ω / f^2 /. s → 0 // Simplify // MatrixForm
```

Out[8]//MatrixForm=

$$\begin{pmatrix} a^2 & 0 & -a^2 u[0] \\ 0 & 1 & -v[0] \\ -a^2 u[0] & -v[0] & a^2 (f^2 + u[0]^2) + v[0]^2 \end{pmatrix}$$

In[10]:=
```
ω /. {u[0] → 0, v[0] → 0} // MatrixForm
```

Out[10]//MatrixForm=

$$\begin{pmatrix} a^2 f^2 & -a f s & 0 \\ -a f s & f^2 + s^2 & 0 \\ 0 & 0 & a^2 f^4 \end{pmatrix}$$

In[17]:=
```
ω / f^2 /. {a → 1, s → 0} // Simplify // MatrixForm
```

Out[17]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & -u[0] \\ 0 & 1 & -v[0] \\ -u[0] & -v[0] & f^2 + u[0]^2 + v[0]^2 \end{pmatrix}$$

▶Camera Orientation from Two Finite Vanishing Points

Problem: Given $\mathbf{K}$ and two vanishing points corresponding to two known orthogonal directions $\mathbf{d}_1$, $\mathbf{d}_2$, compute camera orientation $\mathbf{R}$ with respect to the plane.

- 3D coordinate system choice, e.g.:

$$\mathbf{d}_1 = (1, 0, 0), \quad \mathbf{d}_2 = (0, 1, 0)$$

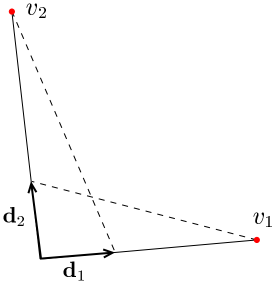
- we know that

$$\mathbf{d}_i \simeq \mathbf{Q}^{-1}\underline{\mathbf{v}}_i = (\mathbf{K}\mathbf{R})^{-1}\underline{\mathbf{v}}_i = \mathbf{R}^{-1}\underbrace{\mathbf{K}^{-1}\underline{\mathbf{v}}_i}_{\underline{\mathbf{w}}_i}$$

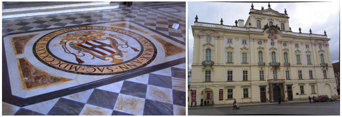
$$\mathbf{R}\mathbf{d}_i \simeq \underline{\mathbf{w}}_i$$

- knowing $\mathbf{d}_{1,2}$ we conclude that $\underline{\mathbf{w}}_i/\|\underline{\mathbf{w}}_i\|$ is the i-th column $\mathbf{r}_i$ of $\mathbf{R}$
- the third column is orthogonal: $\mathbf{r}_3 = \mathbf{r}_1 \times \mathbf{r}_2$

$$\mathbf{R} = \begin{bmatrix} \dfrac{\underline{\mathbf{w}}_1}{\|\underline{\mathbf{w}}_1\|} & \dfrac{\underline{\mathbf{w}}_2}{\|\underline{\mathbf{w}}_2\|} & \dfrac{\underline{\mathbf{w}}_1 \times \underline{\mathbf{w}}_2}{\|\underline{\mathbf{w}}_1 \times \underline{\mathbf{w}}_2\|} \end{bmatrix}$$



some suitable scenes



Application: Planar Rectification

Principle: Rotate camera parallel to the plane of interest.

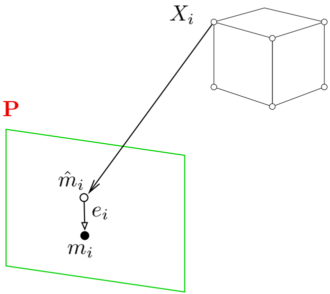
$$\underline{\mathbf{m}} \simeq \mathbf{K}\mathbf{R}\begin{bmatrix}\mathbf{I} & -\mathbf{C}\end{bmatrix}\underline{\mathbf{X}} \qquad\qquad \underline{\mathbf{m}}' \simeq \mathbf{K}\begin{bmatrix}\mathbf{I} & -\mathbf{C}\end{bmatrix}\underline{\mathbf{X}}$$

$$\underline{\mathbf{m}}' \simeq \mathbf{K}(\mathbf{K}\mathbf{R})^{-1}\,\underline{\mathbf{m}} = \mathbf{K}\mathbf{R}^\top\mathbf{K}^{-1}\,\underline{\mathbf{m}} = \mathbf{H}\,\underline{\mathbf{m}}$$

- $\mathbf{H}$ is the rectifying homography
- both $\mathbf{K}$ and $\mathbf{R}$ can be calibrated from two finite vanishing points assuming ORUA →56
- not possible when one (or both) of them are infinite
- without ORUA we would need 4 additional views as on →53

►Camera Resection

Camera calibration and orientation from a known set of $k \geq 6$ reference points and their images $\{(X_i, m_i)\}_{i=1}^{6}$.

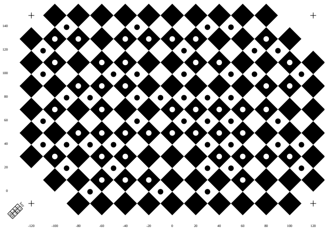


- X_i are considered exact
- m_i is a measurement subject to detection error

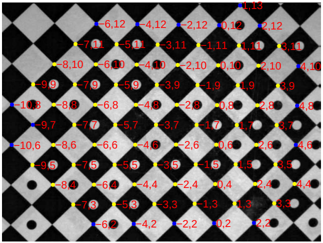
$$\mathbf{m}_i = \hat{\mathbf{m}}_i + \mathbf{e}_i \qquad \text{Cartesian}$$

- where $\underline{\hat{\mathbf{m}}}_i \simeq \mathbf{P}\underline{\mathbf{X}}_i$

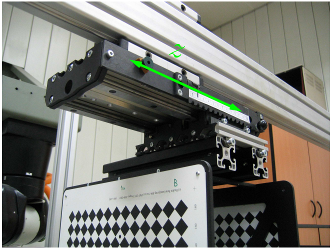
Resection Targets



calibration chart



automatic calibration point detection



resection target with translation stage

- target translated at least once
- by a calibrated (known) translation
- X_i point locations looked up in a table based on their code

▶The Minimal Problem for Camera Resection

Problem: Given $k = 6$ corresponding pairs $\{(X_i, m_i)\}_{i=1}^{k}$, find $\mathbf{P}$

$$\lambda_i \underline{\mathbf{m}}_i = \mathbf{P}\underline{\mathbf{X}}_i, \qquad \mathbf{P} = \begin{bmatrix} \mathbf{q}_1^\top & q_{14} \\ \mathbf{q}_2^\top & q_{24} \\ \mathbf{q}_3^\top & q_{34} \end{bmatrix} \qquad \begin{aligned} \underline{\mathbf{X}}_i &= (x_i, y_i, z_i, 1), \quad i = 1, 2, \ldots, k, \ k = 6 \\ \underline{\mathbf{m}}_i &= (u_i, v_i, 1), \quad \lambda_i \in \mathbb{R}, \ \lambda_i \neq 0 \end{aligned}$$

easy to modify for infinite points X_i

expanded: $\lambda_i u_i = \mathbf{q}_1^\top \mathbf{X}_i + q_{14}, \quad \lambda_i v_i = \mathbf{q}_2^\top \mathbf{X}_i + q_{24}, \quad \lambda_i = \mathbf{q}_3^\top \mathbf{X}_i + q_{34}$

after elimination of λ_i: $(\mathbf{q}_3^\top \mathbf{X}_i + q_{34}) u_i = \mathbf{q}_1^\top \mathbf{X}_i + q_{14}, \quad (\mathbf{q}_3^\top \mathbf{X}_i + q_{34}) v_i = \mathbf{q}_2^\top \mathbf{X}_i + q_{24}$

Then

$$\mathbf{A}\,\mathbf{q} = \begin{bmatrix} \mathbf{X}_1^\top & 1 & \mathbf{0}^\top & 0 & -u_1 \mathbf{X}_1^\top & -u_1 \\ \mathbf{0}^\top & 0 & \mathbf{X}_1^\top & 1 & -v_1 \mathbf{X}_1^\top & -v_1 \\ \vdots & & & & & \vdots \\ \mathbf{X}_k^\top & 1 & \mathbf{0}^\top & 0 & -u_k \mathbf{X}_k^\top & -u_k \\ \mathbf{0}^\top & 0 & \mathbf{X}_k^\top & 1 & -v_k \mathbf{X}_k^\top & -v_k \end{bmatrix} \cdot \begin{bmatrix} \mathbf{q}_1 \\ q_{14} \\ \mathbf{q}_2 \\ q_{24} \\ \mathbf{q}_3 \\ q_{34} \end{bmatrix} = \mathbf{0} \tag{9}$$

- we need 11 indepedent parameters for $\mathbf{P}$
- $\mathbf{A} \in \mathbb{R}^{2k,12}$, $\mathbf{q} \in \mathbb{R}^{12}$
- 6 points in a general position give $\operatorname{rank} \mathbf{A} = 12$ and there is no non-trivial null space
- drop one row to get rank 11 matrix, then the basis vector of the null space of $\mathbf{A}$ gives $\mathbf{q}$

▶The Jack-Knife Solution for $k = 6$

- given the 6 correspondences, we have 12 equations for the 11 parameters
- can we use all the information present in the 6 points?

Jack-knife estimation

1. $n := 0$
2. for $i = 1, 2, \ldots, 2k$ do
   a) delete i-th row from $\mathbf{A}$, this gives $\mathbf{A}_i$
   b) if $\dim \operatorname{null} \mathbf{A}_i > 1$ continue with the next i
   c) $n := n + 1$
   d) compute the right null-space $\mathbf{q}_i$ of $\mathbf{A}_i$ — e.g. by 'economy-size' SVD
   e) $\hat{\mathbf{q}}_i := \mathbf{q}_i$ normalized by q_{11} and dimension-reduced — assuming finite camera with $P_{3,3} = 1$
3. from all n vectors $\hat{\mathbf{q}}_i$ collected in Step 1d compute

$$\mathbf{q} = \frac{1}{n} \sum_{i=1}^{n} \hat{\mathbf{q}}_i, \qquad \operatorname{var}[\mathbf{q}] = \frac{n-1}{n} \operatorname{diag} \sum_{i-1}^{n} (\hat{\mathbf{q}}_i - \mathbf{q})(\hat{\mathbf{q}}_i - \mathbf{q})^\top$$

regular for $n \geq 11$

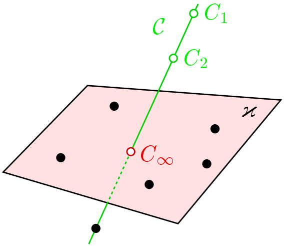
- have a solution + an error estimate, per individual elements of $\mathbf{P}$
- at least 5 points must be in a general position (→64)
- large error indicates near degeneracy
- computation not efficient with $k > 6$ points, needs $\binom{2k}{11}$ draws, e.g. $k = 7 \Rightarrow 364$ draws
- better error estimation method: decompose $\mathbf{P}_i$ to $\mathbf{K}_i$, $\mathbf{R}_i$, $\mathbf{t}_i$ (→32), represent $\mathbf{R}_i$ with 3 parameters (e.g. Euler angles, or in Cayley representation →136) and compute the errors for the parameters

▶Degenerate (Critical) Configurations for Camera Resection

Let $\mathcal{X} = \{X_i;\ i = 1, \ldots\}$ be a set of points and $\mathbf{P}_1 \not\simeq \mathbf{P}_j$ be two regular (rank-3) cameras. Then two configurations $(\mathbf{P}_1, \mathcal{X})$ and $(\mathbf{P}_j, \mathcal{X})$ are image-equivalent if

$$\mathbf{P}_1 \underline{\mathbf{X}}_i \simeq \mathbf{P}_j \underline{\mathbf{X}}_i \quad \text{for all} \quad X_i \in \mathcal{X}$$

there is a non-trivial set of other cameras that see the same image



Case 4

- **importantly**: If all calibration points $X_i \in \mathcal{X}$ lie on a plane $\varkappa$ then camera resection is non-unique and all image-equivalent camera centers lie on a spatial line $\mathcal{C}$ with the $C_\infty = \varkappa \cap \mathcal{C}$ excluded

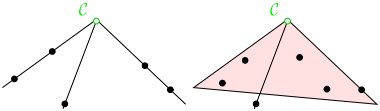
  this also means we cannot resect if all X_i are infinite
- by adding points $X_i \in \mathcal{X}$ to $\mathcal{C}$ we gain nothing
- there are additional image-equivalent configurations, see next

proof sketch in [H&Z, Sec. 22.1.2]

Note that if $\mathbf{Q}$, $\mathbf{T}$ are suitable homographies then $\mathbf{P}_1 \simeq \mathbf{Q}\mathbf{P}_0\mathbf{T}$, where $\mathbf{P}_0$ is canonical and the analysis can be made with $\hat{\mathbf{P}}_j \simeq \mathbf{Q}^{-1}\mathbf{P}_j$

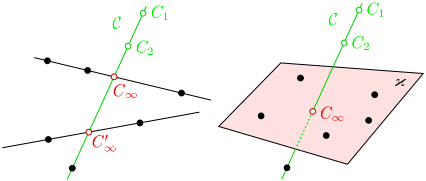
$$\mathbf{P}_0 \underbrace{\mathbf{T}\underline{\mathbf{X}}_i}_{\underline{\mathbf{Y}}_i} \simeq \hat{\mathbf{P}}_j \underbrace{\mathbf{T}\underline{\mathbf{X}}_i}_{\underline{\mathbf{Y}}_i} \quad \text{for all} \quad Y_i \in \mathcal{Y}$$

cont'd (all cases)



Case 5 Case 6

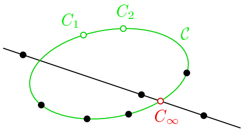
- cameras C_1, C_2 co-located at point $\mathcal{C}$
- points on three optical rays or one optical ray and one optical plane
- Case 5: camera sees 3 isolated point images
- Case 6: cam. sees a line of points and an isolated point



Case 3 **Case 4**

- cameras lie on a line $\mathcal{C} \setminus \{C_\infty, C'_\infty\}$
- points lie on $\mathcal{C}$ and
  1. on two lines meeting $\mathcal{C}$ at C_∞, C'_∞
  2. or on a plane meeting $\mathcal{C}$ at C_∞
- Case 3: camera sees 2 lines of points

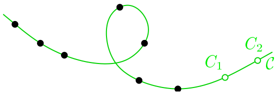
Case 2



- cameras lie on a planar conic $\mathcal{C} \setminus \{C_\infty\}$

  not necessarily an ellipse
- points lie on $\mathcal{C}$ and an additional line meeting the conic at C_∞
- Case 2: camera sees 2 lines of points

Case 1



- cameras and points all lie on a twisted cubic $\mathcal{C}$
- Case 1: camera sees a conic

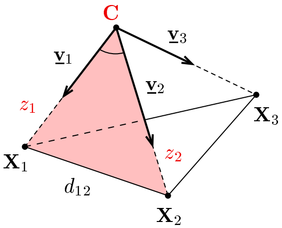
►Three-Point Exterior Orientation Problem (P3P)

Calibrated camera rotation and translation from Perspective images of 3 reference Points.

Problem: Given $\mathbf{K}$ and three corresponding pairs $\left\{(m_i,\, X_i)\right\}_{i=1}^{3}$, find $\mathbf{R}$, $\mathbf{C}$ by solving

$$\lambda_i \underline{\mathbf{m}}_i = \mathbf{K}\mathbf{R}\left(\mathbf{X}_i - \mathbf{C}\right), \qquad i = 1,2,3$$

configuration w/o rotation in (11)



1. Transform $\underline{\mathbf{v}}_i \stackrel{\text{def}}{=} \mathbf{K}^{-1}\underline{\mathbf{m}}_i$. Then

$$\lambda_i \underline{\mathbf{v}}_i = \mathbf{R}\left(\mathbf{X}_i - \mathbf{C}\right). \tag{10}$$

2. Eliminate $\mathbf{R}$ by taking — rotation preserves length: $\|\mathbf{R}\mathbf{x}\| = \|\mathbf{x}\|$

$$|\lambda_i| \cdot \|\underline{\mathbf{v}}_i\| = \|\mathbf{X}_i - \mathbf{C}\| \stackrel{\text{def}}{=} z_i \tag{11}$$

3. Consider only angles among $\underline{\mathbf{v}}_i$ and apply Cosine Law per triangle $(\mathbf{C}, \mathbf{X}_i, \mathbf{X}_j)$ $i,j = 1,2,3,\ i \neq j$

$$d_{ij}^2 = z_i^2 + z_j^2 - 2\, z_i\, z_j\, c_{ij},$$

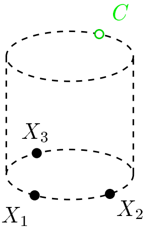
$$z_i = \|\mathbf{X}_i - \mathbf{C}\|,\ d_{ij} = \|\mathbf{X}_j - \mathbf{X}_i\|,\ c_{ij} = \cos(\angle \underline{\mathbf{v}}_i\, \underline{\mathbf{v}}_j)$$

4. Solve system of 3 quadratic eqs in 3 unknowns z_i [Fischler & Bolles, 1981]
   there may be no real root; there are up to 4 solutions that cannot be ignored
   (verify on additional points)

5. Compute $\mathbf{C}$ by trilateration (3-sphere intersection) from $\mathbf{X}_i$ and z_i; then λ_i from (11) and $\mathbf{R}$ from (10)

Similar problems (P4P with unknown f) at `http://cmp.felk.cvut.cz/minimal/` (with code)

Degenerate (Critical) Configurations for Exterior Orientation



unstable solution

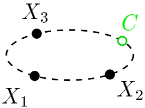
- center of projection C located on the orthogonal circular cylinder with base circumscribing the three points X_i

unstable: a small change of X_i results in a large change of C
can be detected by error propagation

degenerate

- camera C is coplanar with points (X_1, X_2, X_3) but is not on the circumscribed circle of (X_1, X_2, X_3) camera sees a line

---



no solution

1. C cocyclic with (X_1, X_2, X_3) camera sees a line

- additional critical configurations depend on the method to solve the quadratic equations

[Haralick et al. IJCV 1994]

►Populating A Little ZOO of Minimal Geometric Problems in CV

problem	given	unknown	slide
camera resection	6 world–img correspondences $\left\{(X_i,\, m_i)\right\}_{i=1}^{6}$	$\mathbf{P}$	62
exterior orientation	$\mathbf{K}$, 3 world–img correspondences $\left\{(X_i,\, m_i)\right\}_{i=1}^{3}$	$\mathbf{R}$, $\mathbf{C}$	66

- camera resection and exterior orientation are similar problems in a sense:
  - we do resectioning when our camera is uncalibrated
  - we do orientation when our camera is calibrated
- more problems to come

Part IV

Computing with a Camera Pair

covered by

[1] [H&Z] Secs: 9.1, 9.2, 9.6, 11.1, 11.2, 11.9, 12.2, 12.3, 12.5.1

[2] H. Li and R. Hartley. Five-point motion estimation made easy. In *Proc ICPR* 2006, pp. 630–633

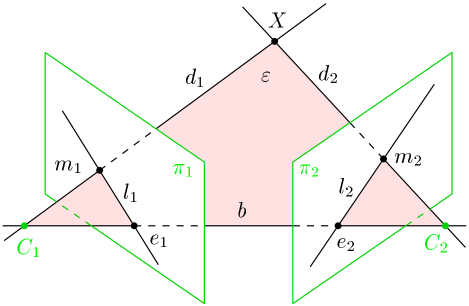
additional references

H. Longuet-Higgins. A computer algorithm for reconstructing a scene from two projections. *Nature*, 293 (5828):133–135, 1981.

▶Geometric Model of a Camera Pair

Epipolar geometry:

- brings constraints necessary for inter-image matching
- its parametric form encapsulates information about the relative pose of two cameras



two-camera setup

Description

- baseline b joins projection centers C_1, C_2

$$\mathbf{b} = \mathbf{C}_2 - \mathbf{C}_1$$

- epipole $e_i \in \pi_i$ is the image of C_j:

$$\underline{\mathbf{e}}_1 \simeq \mathbf{P}_1 \underline{\mathbf{C}}_2, \quad \underline{\mathbf{e}}_2 \simeq \mathbf{P}_2 \underline{\mathbf{C}}_1$$

- $l_i \in \pi_i$ is the image of epipolar plane

$$\varepsilon = (C_2, X, C_1)$$

- l_j is the epipolar line in image π_j induced by m_i in image π_i

Epipolar constraint: d_2, b, d_1 are coplanar

a necessary condition, see →82

Epipolar Geometry Example: Forward Motion

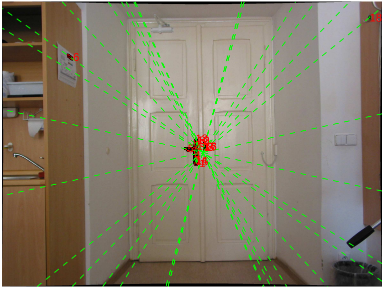


image 1

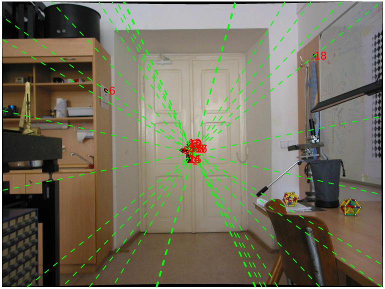
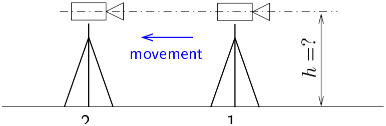


image 2

- red: correspondences
- green: epipolar line pairs per correspondence

click on the image to see their IDs
same ID in both images

How high was the camera above the floor?



▶Cross Products and Maps by Skew-Symmetric 3×3 Matrices

- There is an equivalence $\mathbf{b} \times \mathbf{m} = [\mathbf{b}]_\times \mathbf{m}$, where $[\mathbf{b}]_\times$ is a 3×3 skew-symmetric matrix

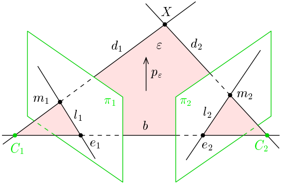
$$[\mathbf{b}]_\times = \begin{bmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{bmatrix}, \qquad \text{assuming} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Some properties

1. $[\mathbf{b}]_\times^\top = -[\mathbf{b}]_\times$ — the general antisymmetry property
2. $\mathbf{A}$ is skew-symmetric iff $\mathbf{x}^\top \mathbf{A} \mathbf{x} = 0$ for all $\mathbf{x}$
3. $[\mathbf{b}]_\times^3 = -\|\mathbf{b}\|^2 \cdot [\mathbf{b}]_\times$
4. $\|[\mathbf{b}]_\times\|_F = \sqrt{2}\,\|\mathbf{b}\|$ — Frobenius norm ($\|\mathbf{A}\|_F^2 = \sum_{i,j} |a_{ij}|^2$)
5. $[\mathbf{b}]_\times \mathbf{b} = \mathbf{0}$
6. $\operatorname{rank} [\mathbf{b}]_\times = 2$ iff $\|\mathbf{b}\| > 0$ — check minors of $[\mathbf{b}]_\times$
7. eigenvalues of $[\mathbf{b}]_\times$ are $(0, \lambda, -\lambda)$
8. for any regular $\mathbf{B}$: $[\mathbf{B}\mathbf{z}]_\times \mathbf{B} = \det \mathbf{B} \cdot \mathbf{B}^{-\top} [\mathbf{z}]_\times$ — follows from the factoring on →38
9. special case: if $\mathbf{R}\mathbf{R}^\top = \mathbf{I}$ then $[\mathbf{R}\mathbf{b}]_\times \mathbf{R} = \mathbf{R}\,[\mathbf{b}]_\times$

- note that if $\mathbf{R}_b$ is rotation about $\mathbf{b}$ then $\mathbf{R}_b \mathbf{b} = \mathbf{b}$
- note $[\mathbf{b}]_\times$ is not a homography — it is singular

▶Expressing Epipolar Constraint Algebraically



$$\mathbf{P}_i = \begin{bmatrix}\mathbf{Q}_i & \mathbf{q}_i\end{bmatrix} = \mathbf{K}_i \begin{bmatrix}\mathbf{R}_i & \mathbf{t}_i\end{bmatrix}, \quad i = 1, 2$$

$\mathbf{R}_{21}$ – relative camera rotation, $\mathbf{R}_{21} = \mathbf{R}_2 \mathbf{R}_1^\top$

$\mathbf{t}_{21}$ – relative camera translation, $\mathbf{t}_{21} = \mathbf{t}_2 - \mathbf{R}_{21}\mathbf{t}_1 = -\mathbf{R}_2 \mathbf{b}$

$\mathbf{b}$ – baseline (world coordinate system)

remember: $\mathbf{C} = -\mathbf{Q}^{-1}\mathbf{q} = -\mathbf{R}^\top \mathbf{t}$ →32 and 34

$$\boxed{0 = \mathbf{d}_2^\top \underbrace{\mathbf{p}_\varepsilon}_{\text{normal of } \varepsilon} \simeq \underbrace{(\mathbf{Q}_2^{-1}\underline{\mathbf{m}}_2)^\top}_{\text{optical ray}} \underbrace{\mathbf{Q}_1^\top \underline{\mathbf{l}}_1}_{\text{optical plane}} = \underline{\mathbf{m}}_2^\top \underbrace{\mathbf{Q}_2^{-\top}\mathbf{Q}_1^\top(\underline{\mathbf{e}}_1 \times \underline{\mathbf{m}}_1)}_{\text{image of } \varepsilon \text{ in } \pi_2} = \underline{\mathbf{m}}_2^\top \underbrace{\left(\mathbf{Q}_2^{-\top}\mathbf{Q}_1^\top [\underline{\mathbf{e}}_1]_\times\right)}_{\text{fundamental matrix } \mathbf{F}} \underline{\mathbf{m}}_1}$$

Epipolar constraint $\underline{\mathbf{m}}_2^\top \mathbf{F}\, \underline{\mathbf{m}}_1 = 0$ is a point-line incidence constraint

- point $\underline{\mathbf{m}}_2$ is incident on epipolar line $\underline{\mathbf{l}}_2 \simeq \mathbf{F}\underline{\mathbf{m}}_1$
- point $\underline{\mathbf{m}}_1$ is incident on epipolar line $\underline{\mathbf{l}}_1 \simeq \mathbf{F}^\top \underline{\mathbf{m}}_2$
- $\mathbf{F}\underline{\mathbf{e}}_1 = \mathbf{F}^\top \underline{\mathbf{e}}_2 = \mathbf{0}$ (non-trivially)
- all epipolars meet at the epipole

$$\underline{\mathbf{e}}_1 \simeq \mathbf{Q}_1 \mathbf{C}_2 + \mathbf{q}_1 = \mathbf{Q}_1 \mathbf{C}_2 - \mathbf{Q}_1 \mathbf{C}_1 = \mathbf{K}_1 \mathbf{R}_1 \mathbf{b} = -\mathbf{K}_1 \mathbf{R}_1 \mathbf{R}_2^\top \mathbf{t}_{21} = -\mathbf{K}_1 \mathbf{R}_{21}^\top \mathbf{t}_{21}$$

$$\mathbf{F} = \mathbf{Q}_2^{-\top}\mathbf{Q}_1^\top [\underline{\mathbf{e}}_1]_\times = \mathbf{Q}_2^{-\top}\mathbf{Q}_1^\top [\mathbf{K}_1 \mathbf{R}_1 \mathbf{b}]_\times = \overset{\circledast\, 1}{\cdots} \simeq \mathbf{K}_2^{-\top}[-\mathbf{t}_{21}]_\times \mathbf{R}_{21} \mathbf{K}_1^{-1} \quad \text{fundamental}$$

$$\mathbf{E} = [-\mathbf{t}_{21}]_\times \mathbf{R}_{21} = \underbrace{[\mathbf{R}_2 \mathbf{b}]_\times}_{\text{baseline in Cam 2}} \mathbf{R}_{21} = \mathbf{R}_{21} \underbrace{[\mathbf{R}_1 \mathbf{b}]_\times}_{\text{baseline in Cam 1}} = \mathbf{R}_{21}[-\mathbf{R}_{21}\mathbf{t}_{21}]_\times \quad \text{essential}$$

▶Key Properties of the Fundamental Matrix

$$\mathbf{F} = \mathbf{K}_2^{-\top} \underbrace{[-\mathbf{t}_{21}]_\times \mathbf{R}_{21}}_{\text{essential matrix } \mathbf{E}} \mathbf{K}_1^{-1}$$

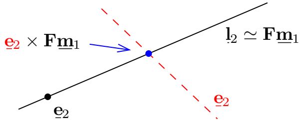
1. $\mathbf{E}$ captures relative camera pose only [Longuet-Higgins 1981]
   (the change of the world coordinate system does not change $\mathbf{E}$)

$$\begin{bmatrix}\mathbf{R}_i' & \mathbf{t}_i'\end{bmatrix} = \begin{bmatrix}\mathbf{R}_i & \mathbf{t}_i\end{bmatrix} \cdot \begin{bmatrix}\mathbf{R} & \mathbf{t} \\ \mathbf{0}^\top & 1\end{bmatrix} = \begin{bmatrix}\mathbf{R}_i\mathbf{R} & \mathbf{R}_i\mathbf{t} + \mathbf{t}_i\end{bmatrix},$$

   then

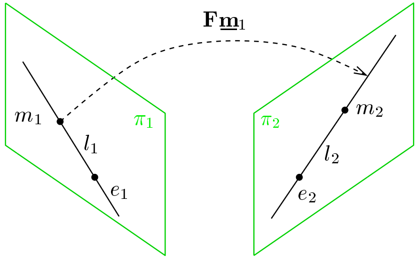
$$\mathbf{R}_{21}' = \mathbf{R}_2'\mathbf{R}_1'^\top = \cdots = \mathbf{R}_{21}$$
$$\mathbf{t}_{21}' = \mathbf{t}_2' - \mathbf{R}_{21}'\mathbf{t}_1' = \cdots = \mathbf{t}_{21}$$

2. the translation length $\mathbf{t}_{21}$ is lost since $\mathbf{E}$ is homogeneous
3. $\mathbf{F}$ maps points to lines and it is not a homography
4. $\underline{\mathbf{e}}_2 \times (\underline{\mathbf{e}}_2 \times \mathbf{F}\underline{\mathbf{m}}_1) \simeq \mathbf{F}\underline{\mathbf{m}}_1$, in general $\mathbf{F} \simeq [\underline{\mathbf{e}}_2]_\times^{2a} \mathbf{F} [\underline{\mathbf{e}}_1]_\times^{2b}$ for any $a, b \in \mathbb{N}$



- by point/line transmutation (left)
- point $\underline{\mathbf{e}}_2$ does not lie on line $\underline{\mathbf{e}}_2$ (dashed): $\underline{\mathbf{e}}_2^\top \underline{\mathbf{e}}_2 \neq 0$

▶Some Mappings by the Fundamental Matrix

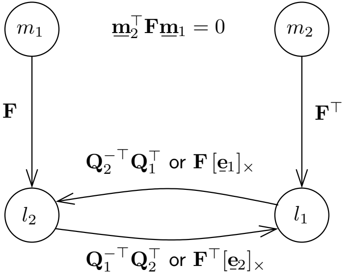


$$0 = \underline{\mathbf{m}}_2^\top \mathbf{F}\, \underline{\mathbf{m}}_1$$

$$\underline{\mathbf{e}}_1 \simeq \operatorname{null}(\mathbf{F}), \qquad \underline{\mathbf{e}}_2 \simeq \operatorname{null}(\mathbf{F}^\top)$$

$$\underline{\mathbf{l}}_2 = \mathbf{F}\underline{\mathbf{m}}_1 \qquad \underline{\mathbf{l}}_1 = \mathbf{F}^\top \underline{\mathbf{m}}_2$$

$$\underline{\mathbf{l}}_2 = \mathbf{F}[\underline{\mathbf{e}}_1]_\times \underline{\mathbf{l}}_1 \qquad \underline{\mathbf{l}}_1 = \mathbf{F}^\top[\underline{\mathbf{e}}_2]_\times \underline{\mathbf{l}}_2$$



- $\underline{\mathbf{l}}_2 \simeq \mathbf{F}\,[\underline{\mathbf{e}}_1]_\times\, \underline{\mathbf{l}}_1$: by 'transmutation' →74
- $\mathbf{F}[\underline{\mathbf{e}}_1]_\times$ maps lines to lines but it is not a homography
- $\mathbf{H}^{-\top} = \mathbf{Q}_2^{-\top}\mathbf{Q}_1^\top$ is the epipolar homography mapping epipolar lines to epipolar lines, hence

$$\mathbf{H} = \mathbf{Q}_2\mathbf{Q}_1^{-1} = \mathbf{K}_2\mathbf{R}_{21}\mathbf{K}_1^{-1}$$

you have seen this →59

▶Representation Theorem for Fundamental Matrices

Theorem

Every 3×3 matrix of rank 2 is a fundamental matrix.

Proof.

Converse: By the definition $\mathbf{F} = \mathbf{H}[\underline{\mathbf{e}}_1]_\times$ is a 3×3 matrix of rank 2.

Direct:

1. let $\mathbf{A} = \mathbf{U}\mathbf{D}\mathbf{V}^\top$ be the SVD of a 3×3 matrix $\mathbf{A}$ of rank 2; then $\mathbf{D} = \mathrm{diag}(\lambda_1, \lambda_2, 0)$
2. we can write $\mathbf{D} = \mathbf{B}\mathbf{C}$, where $\mathbf{B} = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$, $\mathbf{C} = \mathrm{diag}(1, 1, 0)$, $\lambda_3 \neq 0$
3. then $\mathbf{A} = \mathbf{U}\mathbf{B}\mathbf{C}\underbrace{\mathbf{W}\mathbf{W}^\top}_{\mathbf{I}}\mathbf{V}^\top$
4. we look for rotation $\mathbf{W}$ that maps $\mathbf{C}$ to skew-symmetric $\mathbf{S}$
5. then $\mathbf{W} = \begin{bmatrix} 0 & \alpha & 0 \\ -\alpha & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $|\alpha| = 1$, and $\mathbf{S} = [\mathbf{s}]_\times$, $\mathbf{s} = (0, 0, 1)$
6. we can write
$$\mathbf{A} = \mathbf{U}\mathbf{B}[\mathbf{s}]_\times \mathbf{W}^\top \mathbf{V}^\top \overset{\circledast 1}{=} \cdots = \underbrace{\mathbf{U}\mathbf{B}(\mathbf{V}\mathbf{W})^\top}_{\mathbf{H}^{-\top}} [\mathbf{v}_3]_\times, \qquad \mathbf{v}_3 \text{ – 3rd column of } \mathbf{V} \tag{12}$$
7. $\mathbf{H}$ regular $\Rightarrow$ $\mathbf{A}$ does the job of a fundamental matrix, with epipole $\mathbf{v}_3$ and epipolar homography $\mathbf{H}^{-\top}$

- we also got a (non-unique: λ_3, $\alpha = \pm 1$) decomposition formula for fundamental matrices □

▶Representation Theorem for Essential Matrices

Theorem

Let $\mathbf{E}$ *be a* 3×3 *matrix with SVD* $\mathbf{E} = \mathbf{U}\mathbf{D}\mathbf{V}^\top$. *Then* $\mathbf{E}$ *is essential iff* $\mathbf{D} \simeq \mathrm{diag}(1, 1, 0)$.

Proof.

Direct:

If $\mathbf{E}$ is an essential matrix, then $\mathbf{U}\mathbf{B}(\mathbf{V}\mathbf{W})^\top$ in (12) must be orthogonal, hence $\mathbf{B} = \lambda\mathbf{I}$.

Converse:

$\mathbf{E}$ is fundamental with $\mathbf{D} = \lambda\,\mathrm{diag}(1, 1, 0)$ then we do not need $\mathbf{B}$ (as if $\mathbf{B} = \lambda\mathbf{I}$) and $\mathbf{U}(\mathbf{V}\mathbf{W})^\top$ is orthogonal, as required.

□

▶Essential Matrix Decomposition

We are decomposing $\mathbf{E}$ to $\mathbf{E} = [-\mathbf{t}_{21}]_\times \mathbf{R}_{21} = \mathbf{R}_{21}[-\mathbf{R}_{21}^\top \mathbf{t}]_\times$ [H&Z, sec. 9.6]

1. compute SVD of $\mathbf{E} = \mathbf{U}\mathbf{D}\mathbf{V}^\top$ and verify $\mathbf{D} = \lambda\,\mathrm{diag}(1,1,0)$
2. if $\det \mathbf{U} < 0$ transform it to $-\mathbf{U}$, do the same for $\mathbf{V}$ the overall sign is dropped
3. compute

$$\mathbf{R}_{21} = \mathbf{U}\underbrace{\begin{bmatrix} 0 & \alpha & 0 \\ -\alpha & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{W}}\mathbf{V}^\top, \quad \mathbf{t}_{21} = -\beta\,\mathbf{u}_3, \qquad |\alpha| = 1, \quad \beta \neq 0 \tag{13}$$

Notes

- $\mathbf{U}(\mathbf{V}\mathbf{W})^\top \mathbf{v}_3 = \cdots = \mathbf{u}_3$
- $\mathbf{t}_{21}$ is recoverable up to scale β and direction $\mathrm{sign}\,\beta$
- the result for $\mathbf{R}_{21}$ is unique up to $\alpha = \pm 1$ despite non-uniqueness of SVD
- change of sign in $\mathbf{W}$ rotates the solution by 180° about $\mathbf{t}$

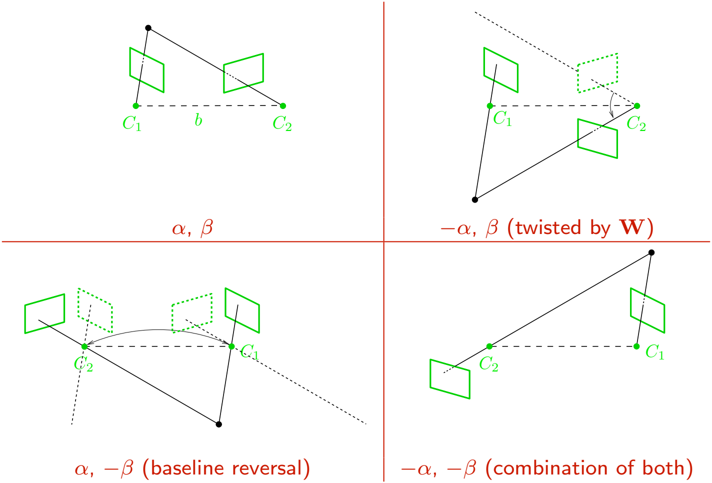
  $\mathbf{R}_1 = \mathbf{U}\mathbf{W}\mathbf{V}^\top$, $\mathbf{R}_2 = \mathbf{U}\mathbf{W}^\top\mathbf{V}^\top \Rightarrow \mathbf{T} = \mathbf{R}_2\mathbf{R}_1^\top = \cdots = \mathbf{U}\,\mathrm{diag}(-1,-1,1)\mathbf{U}^\top$ which is a rotation by 180° about $\mathbf{u}_3 = \mathbf{t}_{21}$:

$$\mathbf{U}\,\mathrm{diag}(-1,-1,1)\mathbf{U}^\top\mathbf{u}_3 = \mathbf{U}\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \mathbf{u}_3$$

- 4 solution sets for 4 sign combinations of α, β see next for geometric interpretation

►Four Solutions to Essential Matrix Decomposition

Transform the world coordinate system so that the origin is in Camera 2. Then $\mathbf{t}_{21} = -\mathbf{b}$ and $\mathbf{W}$ rotates about the baseline $\mathbf{b}$. →73



α, β

$-\alpha$, β (twisted by $\mathbf{W}$)

α, $-\beta$ (baseline reversal)

$-\alpha$, $-\beta$ (combination of both)

- chirality constraint: all 3D points are in front of both cameras
- this singles-out the upper left case [H&Z, Sec. 9.6.3]

►7-Point Algorithm for Estimating Fundamental Matrix

Problem: Given a set $\{(x_i, y_i)\}_{i=1}^{k}$ of $k = 7$ correspondences, estimate f. m. $\mathbf{F}$.

$$\underline{\mathbf{y}}_i^\top \mathbf{F}\, \underline{\mathbf{x}}_i = 0, \quad i = 1, \ldots, k, \qquad \underline{\text{known}}\text{:} \quad \underline{\mathbf{x}}_i = (u_i^1, v_i^1, 1), \quad \underline{\mathbf{y}}_i = (u_i^2, v_i^2, 1)$$

terminology: correspondence = truth, later: match = algorithm's result; hypothesized corresp.

Solution:

$$\mathbf{D} = \begin{bmatrix} u_1^1 u_1^2 & u_1^1 v_1^2 & u_1^1 & u_1^2 v_1^1 & v_1^1 v_1^2 & v_1^1 & u_1^2 & v_1^2 & 1 \\ u_2^1 u_2^2 & u_2^1 v_2^2 & u_2^1 & u_2^2 v_2^1 & v_2^1 v_2^2 & v_2^1 & u_2^2 & v_2^2 & 1 \\ u_3^1 u_3^2 & u_3^1 v_3^2 & u_3^1 & u_3^2 v_3^1 & v_3^1 v_3^2 & v_3^1 & u_3^2 & v_3^2 & 1 \\ \vdots & & & & & & & & \vdots \\ u_k^1 u_k^2 & u_k^1 v_k^2 & u_k^1 & u_k^2 v_k^1 & v_k^1 v_k^2 & v_k^1 & u_k^2 & v_k^2 & 1 \end{bmatrix} \qquad \mathbf{D} \in \mathbb{R}^{k,9}$$

$$\mathbf{D}\,\mathrm{vec}(\mathbf{F}) = \mathbf{0}, \qquad \mathrm{vec}(\mathbf{F}) = \begin{bmatrix} f_{11} & f_{21} & f_{31} & \ldots & f_{33} \end{bmatrix}^\top, \qquad \mathrm{vec}(\mathbf{F}) \in \mathbb{R}^9,$$

- for $k = 7$ we have a rank-deficient system, the null-space of $\mathbf{D}$ is 2-dimensional
- but we know that $\det \mathbf{F} = 0$, hence
  1. find a basis of the null space of $\mathbf{D}$: $\mathbf{F}_1$, $\mathbf{F}_2$ — by SVD or QR factorization
  2. get up to 3 real solutions for α_i from

     $$\det(\alpha \mathbf{F}_1 + (1 - \alpha)\mathbf{F}_2) = 0 \qquad \text{cubic equation in } \alpha$$

  3. get up to 3 fundamental matrices $\mathbf{F} = \alpha_i \mathbf{F}_1 + (1 - \alpha_i)\mathbf{F}_2$ — (check $\operatorname{rank} \mathbf{F} = 2$)

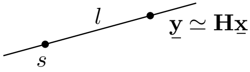
- the result may depend on image transformations
- normalization improves conditioning →87
- this gives a good starting point for the full algorithm →104
- dealing with mismatches need not be a part of the 7-point algorithm →105

▶Degenerate Configurations for Fundamental Matrix Estimation

When is $\mathbf{F}$ not uniquely determined from any number of correspondences? [H&Z, Sec. 11.9]

1. when images are related by homography
   a) camera centers coincide $C_1 = C_2$: $\quad \mathbf{H} = \mathbf{K}_2 \mathbf{R}_{21} \mathbf{K}_1^{-1}$
   b) camera moves but all 3D points lie in a plane $(\mathbf{n}, d)$: $\quad \mathbf{H} = \mathbf{K}_2 (\mathbf{R}_{21} - \mathbf{t}_{21} \mathbf{n}^\top / d) \mathbf{K}_1^{-1}$

   - in both cases: epipolar geometry is not defined
   - we do get an $\mathbf{F}$ from the 7-point algorithm but it is of the form of $\mathbf{F} = [\underline{\mathbf{s}}]_\times \mathbf{H}$ with $\underline{\mathbf{s}}$ arbitrary (nonzero) note that $[\underline{\mathbf{s}}]_\times \mathbf{H} \simeq \mathbf{H}' [\underline{\mathbf{s}}']_\times$ →72



   - correspondence $x \leftrightarrow y$
   - y is the image of x: $\underline{\mathbf{y}} \simeq \mathbf{H}\underline{\mathbf{x}}$
   - a necessary condition: $y \in l$, $\quad \underline{\mathbf{l}} \simeq \underline{\mathbf{s}} \times \mathbf{H}\underline{\mathbf{x}}$ arbitrary $\underline{\mathbf{s}}$

$$0 = \underline{\mathbf{y}}^\top (\underline{\mathbf{s}} \times \mathbf{H}\underline{\mathbf{x}}) = \underline{\mathbf{y}}^\top [\underline{\mathbf{s}}]_\times \mathbf{H}\underline{\mathbf{x}}$$

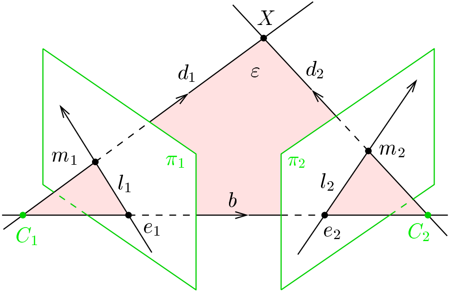
2. both camera centers and all 3D points lie on a ruled quadric
   hyperboloid of one sheet, cones, cylinders, two planes
   - there are 3 solutions for $\mathbf{F}$

notes

- estimation of $\mathbf{E}$ can deal with planes: $[\underline{\mathbf{s}}]_\times \mathbf{H} = [\underline{\mathbf{s}}]_\times (\mathbf{R}_{21} - \mathbf{t}_{21} \mathbf{n}^\top / d)$ has equal eigenvalues iff $\underline{\mathbf{s}} = \mathbf{t}_{21}$, the decomposition works (nonunique, as before) ⊛ P1; 1pt for a proof
- a complete treatment with additional degenerate configurations in [H&Z, sec. 22.2]
- a stronger epipolar constraint could reject some configurations

A Note on Oriented Epipolar Constraint

- a tighter epipolar constraint preserves orientations
- requires all points and cameras be on the same side of the plane at infinity



$$\underline{\mathbf{e}}_2 \times \underline{\mathbf{m}}_2 \overset{+}{\sim} \mathbf{F}\,\underline{\mathbf{m}}_1$$

notation: $\underline{\mathbf{m}} \overset{+}{\sim} \underline{\mathbf{n}}$ means $\underline{\mathbf{m}} = \lambda \underline{\mathbf{n}}$, $\lambda > 0$

- note that the constraint is not invariant to the change of either sign of $\underline{\mathbf{m}}_i$
- all 7 correspondence in 7-point alg. must have the same sign — see later
- this may help reject some wrong matches, see →105 — [Chum et al. 2004]
- an even more tight constraint: scene points in front of both cameras — expensive

this is called chirality constraint

▶5-Point Algorithm for Relative Camera Orientation

Problem: Given $\{m_i,\, m_i'\}_{i=1}^{5}$ corresponding image points and calibration matrix $\mathbf{K}$, recover the camera motion $\mathbf{R}$, $\mathbf{t}$.

Obs:

1. $\mathbf{E}$ – 8 numbers
2. $\mathbf{R}$ – 3DOF, $\mathbf{t}$ – we can recover 2DOF only, in total 5 DOF $\rightarrow$ we need 3 constraints on $\mathbf{E}$
3. $\mathbf{E}$ essential iff it has two equal singular values and the third is zero

This gives an equation system:

$$\underline{\mathbf{v}}_i^\top \mathbf{E}\, \underline{\mathbf{v}}_i' = 0 \qquad \text{5 linear constraints } (\underline{\mathbf{v}} \simeq \mathbf{K}^{-1}\underline{\mathbf{m}})$$

$$\det \mathbf{E} = 0 \qquad \text{1 cubic constraint}$$

$$\mathbf{E}\mathbf{E}^\top\mathbf{E} - \frac{1}{2}\operatorname{tr}(\mathbf{E}\mathbf{E}^\top)\mathbf{E} = \mathbf{0} \qquad \text{9 cubic constraints, 2 independent}$$

⊛ P1; 1pt: verify this equation from $\mathbf{E} = \mathbf{U}\mathbf{D}\mathbf{V}^\top$, $\mathbf{D} = \lambda\,\mathrm{diag}(1,1,0)$

1. estimate $\mathbf{E}$ by SVD from $\underline{\mathbf{v}}_i^\top \mathbf{E}\, \underline{\mathbf{v}}_i' = 0$ by the null-space method,
2. this gives $\mathbf{E} = x\mathbf{E}_1 + y\mathbf{E}_2 + z\mathbf{E}_3 + \mathbf{E}_4$
3. at most 10 (complex) solutions for x, y, z from the cubic constraints

- when all 3D points lie on a plane: at most 2 solutions (twisted-pair) can be disambiguated in 3 views or by chirality constraint (→79) unless all 3D points are closer to one camera
- 6-point problem for unknown f [Kukelova et al. BMVC 2008]
- resources at `http://cmp.felk.cvut.cz/minimal/5_pt_relative.php`

▶The Triangulation Problem

Problem: Given cameras $\mathbf{P}_1$, $\mathbf{P}_2$ and a correspondence $x \leftrightarrow y$ compute a 3D point $\mathbf{X}$ projecting to x and y

$$\lambda_1\,\underline{\mathbf{x}} = \mathbf{P}_1\underline{\mathbf{X}}, \qquad \lambda_2\,\underline{\mathbf{y}} = \mathbf{P}_2\underline{\mathbf{X}}, \qquad \underline{\mathbf{x}} = \begin{bmatrix} u^1 \\ v^1 \\ 1 \end{bmatrix}, \qquad \underline{\mathbf{y}} = \begin{bmatrix} u^2 \\ v^2 \\ 1 \end{bmatrix}, \qquad \mathbf{P}_i = \begin{bmatrix} (\mathbf{p}_1^i)^\top \\ (\mathbf{p}_2^i)^\top \\ (\mathbf{p}_3^i)^\top \end{bmatrix}$$

Linear triangulation method

$$u^1\,(\mathbf{p}_3^1)^\top\underline{\mathbf{X}} = (\mathbf{p}_1^1)^\top\underline{\mathbf{X}}, \qquad u^2\,(\mathbf{p}_3^2)^\top\underline{\mathbf{X}} = (\mathbf{p}_1^2)^\top\underline{\mathbf{X}},$$

$$v^1\,(\mathbf{p}_3^1)^\top\underline{\mathbf{X}} = (\mathbf{p}_2^1)^\top\underline{\mathbf{X}}, \qquad v^2\,(\mathbf{p}_3^2)^\top\underline{\mathbf{X}} = (\mathbf{p}_2^2)^\top\underline{\mathbf{X}},$$

Gives

$$\mathbf{D}\underline{\mathbf{X}} = \mathbf{0}, \qquad \mathbf{D} = \begin{bmatrix} u^1\,(\mathbf{p}_3^1)^\top - (\mathbf{p}_1^1)^\top \\ v^1\,(\mathbf{p}_3^1)^\top - (\mathbf{p}_2^1)^\top \\ u^2\,(\mathbf{p}_3^2)^\top - (\mathbf{p}_1^2)^\top \\ v^2\,(\mathbf{p}_3^2)^\top - (\mathbf{p}_2^2)^\top \end{bmatrix}, \qquad \mathbf{D} \in \mathbb{R}^{4,4}, \quad \underline{\mathbf{X}} \in \mathbb{R}^4 \tag{14}$$

- back-projected rays will generally not intersect due to image error, see next
- using Jack-knife (→63) not recommended — sensitive to small error
- we will use SVD (→85)
- but the result will not be invariant to projective frame
  replacing $\mathbf{P}_1 \mapsto \mathbf{P}_1\mathbf{H}$, $\mathbf{P}_2 \mapsto \mathbf{P}_2\mathbf{H}$ does not always result in $\underline{\mathbf{X}} \mapsto \mathbf{H}^{-1}\underline{\mathbf{X}}$
- note the homogeneous form in (14) can represent points at infinity

▶The Least-Squares Triangulation by SVD

- if $\mathbf{D}$ is full-rank we may minimize the algebraic least-squares error

$$\boldsymbol{\varepsilon}^2(\underline{\mathbf{X}}) = \|\mathbf{D}\underline{\mathbf{X}}\|^2 \quad \text{s.t.} \quad \|\underline{\mathbf{X}}\| = 1, \qquad \underline{\mathbf{X}} \in \mathbb{R}^4$$

- let $\mathbf{D}_i$ be the i-th row of $\mathbf{D}$, then

$$\|\mathbf{D}\underline{\mathbf{X}}\|^2 = \sum_{i=1}^{4} (\mathbf{D}_i\,\underline{\mathbf{X}})^2 = \sum_{i=1}^{4} \underline{\mathbf{X}}^\top \mathbf{D}_i^\top \mathbf{D}_i\,\underline{\mathbf{X}} = \underline{\mathbf{X}}^\top \mathbf{Q}\,\underline{\mathbf{X}}, \text{ where } \underline{\mathbf{Q}} = \sum_{i=1}^{4} \mathbf{D}_i^\top \mathbf{D}_i = \mathbf{D}^\top \mathbf{D} \;\in \mathbb{R}^{4,4}$$

- we write the SVD of $\mathbf{Q}$ as $\mathbf{Q} = \sum_{j=1}^{4} \sigma_j^2\,\mathbf{u}_j \mathbf{u}_j^\top$, in which [Golub & van Loan 2013, Sec. 2.5]

$$\sigma_1^2 \geq \cdots \geq \sigma_4^2 \geq 0 \quad \text{and} \quad \mathbf{u}_l^\top \mathbf{u}_m = \begin{cases} 0 & \text{if } l \neq m \\ 1 & \text{otherwise} \end{cases}$$

- then $\underline{\mathbf{X}} = \arg\min_{\mathbf{q},\|\mathbf{q}\|=1} \mathbf{q}^\top \mathbf{Q}\,\mathbf{q} = \mathbf{u}_4$

Proof (by contradiction).

$\mathbf{q}^\top \mathbf{Q}\,\mathbf{q} = \sum_{j=1}^{4} \sigma_j^2\,\mathbf{q}^\top \mathbf{u}_j\,\mathbf{u}_j^\top \mathbf{q} = \sum_{j=1}^{4} \sigma_j^2\,(\mathbf{u}_j^\top \mathbf{q})^2$ is a sum of non-negative elements $0 \leq (\mathbf{u}_j^\top \mathbf{q})^2 \leq 1$

Let $\mathbf{q} = \mathbf{u}_4 \cos\alpha + \bar{\mathbf{q}} \sin\alpha$ s.t. $\bar{\mathbf{q}} \perp \mathbf{u}_4$ and $\bar{\mathbf{q}} = 1$, then $\|\mathbf{q}\| = 1$ and

$$\mathbf{q}^\top \mathbf{Q}\,\mathbf{q} = \cdots = \sigma_4^2 \cos^2\alpha + \sin^2\alpha\,\underbrace{\sum_{j=1}^{3} \sigma_j^2\,(\mathbf{u}_j^\top \bar{\mathbf{q}})^2}_{\geq \sigma_4^2} \geq \sigma_4^2$$

▶cont'd

- if $\sigma_4 \ll \sigma_3$, there is a unique solution $\underline{\mathbf{X}} = \mathbf{u}_4$ with residual error $(\mathbf{D}\,\underline{\mathbf{X}})^2 = \sigma_4^2$
  the quality (conditioning) of the solution may be expressed as $q = \sigma_3/\sigma_4$ (greater is better)

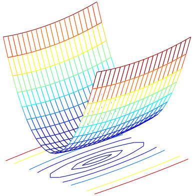
Matlab code for the least-squares solver:

```
[U,O,V] = svd(D);
X = V(:,end);
q = O(3,3)/O(4,4);
```

⊛ P1; 1pt: Why did we decompose $\mathbf{D}$ and not $\mathbf{Q} = \mathbf{D}^\top\mathbf{D}$?

►Numerical Conditioning

- The equation $\mathbf{D}\underline{\mathbf{X}} = \mathbf{0}$ in (14) may be ill-conditioned for numerical computation, which results in a poor estimate for $\underline{\mathbf{X}}$.



Why: on a row of $\mathbf{D}$ there are big entries together with small entries, e.g. of orders projection centers in mm, image points in px

$$\begin{bmatrix} 10^3 & 0 & 10^3 & 10^6 \\ 0 & 10^3 & 10^3 & 10^6 \\ 10^3 & 0 & 10^3 & 10^6 \\ 0 & 10^3 & 10^3 & 10^6 \end{bmatrix}$$

Quick fix:

1. re-scale the problem by a regular diagonal conditioning matrix $\mathbf{S} \in \mathbb{R}^{4,4}$

$$\mathbf{0} = \mathbf{D}\,\mathbf{q} = \mathbf{D}\,\mathbf{S}\,\mathbf{S}^{-1}\mathbf{q} = \bar{\mathbf{D}}\,\bar{\mathbf{q}}$$

choose $\mathbf{S}$ to make the entries in $\hat{\mathbf{D}}$ all smaller than unity in absolute value:

$$\mathbf{S} = \mathrm{diag}(10^{-3}, 10^{-3}, 10^{-3}, 10^{-6})$$

```
S = diag(1./max(abs(D), 1))
```

2. solve for $\bar{\mathbf{q}}$ as before
3. get the final solution as $\mathbf{q} = \mathbf{S}\,\bar{\mathbf{q}}$

- when SVD is used in camera resection, conditioning is essential for success →62

Algebraic Error vs Reprojection Error

- algebraic error (c – camera index, (u^c, v^c) – image coordinates) from SVD →86

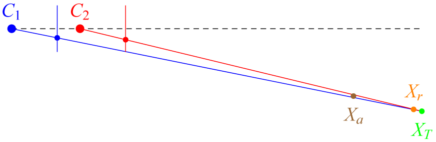
$$\varepsilon^2 = \sigma_4^2 = \sum_{c=1}^{2} \left[\left(u^c (\mathbf{p}_3^c)^\top \underline{\mathbf{X}} - (\mathbf{p}_1^c)^\top \underline{\mathbf{X}} \right)^2 + \left(v^c (\mathbf{p}_3^c)^\top \underline{\mathbf{X}} - (\mathbf{p}_2^c)^\top \underline{\mathbf{X}} \right)^2 \right]$$

- reprojection error

$$e^2 = \sum_{c=1}^{2} \left[\left(u^c - \frac{(\mathbf{p}_1^c)^\top \underline{\mathbf{X}}}{(\mathbf{p}_3^c)^\top \underline{\mathbf{X}}} \right)^2 + \left(v^c - \frac{(\mathbf{p}_2^c)^\top \underline{\mathbf{X}}}{(\mathbf{p}_3^c)^\top \underline{\mathbf{X}}} \right)^2 \right]$$

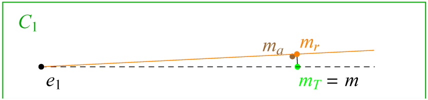
- algebraic error zero ⇒ reprojection error zero — $\sigma_4 = 0 \Rightarrow$ non-trivial null space
- epipolar constraint satisfied ⇒ equivalent results
- in general: minimizing algebraic error cheap but it gives inferior results
- minimizing reprojection error expensive but it gives good results
- the midpoint of the common perpendicular to both optical rays gives about 50% greater error in 3D
- the golden standard method – deferred to →100

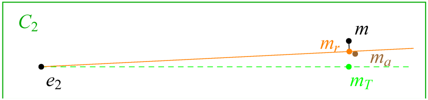
Ex:



- forward camera motion
- error $f/50$ in image 2, orthogonal to epipolar plane

X_T – noiseless ground truth position
X_r – reprojection error minimizer
X_a – algebraic error minimizer
m – measurement (m_T with noise in v^2)





▶We Have Added to The ZOO

continuation from →68

problem	given	unknown	slide
camera resection	6 world–img correspondences $\left\{(X_i, m_i)\right\}_{i=1}^{6}$	$\mathbf{P}$	62
exterior orientation	$\mathbf{K}$, 3 world–img correspondences $\left\{(X_i, m_i)\right\}_{i=1}^{3}$	$\mathbf{R}$, $\mathbf{t}$	66
fundamental matrix	7 img–img correspondences $\left\{(m_i, m_i')\right\}_{i=1}^{7}$	$\mathbf{F}$	80
relative orientation	$\mathbf{K}$, 5 img–img correspondences $\left\{(m_i, m_i')\right\}_{i=1}^{5}$	$\mathbf{R}$, $\mathbf{t}$	83
triangulation	$\mathbf{P}_1$, $\mathbf{P}_2$, 1 img–img correspondence (m_i, m_i')	X	84

A bigger ZOO at `http://cmp.felk.cvut.cz/minimal/`

calibrated problems

- have fewer degenerate configurations
- can do with fewer points (good for geometry proposal generators →112)

- algebraic error optimization (with SVD) makes sense in camera resection and triangulation only
- but it is not the best method; we will now focus on 'optimizing optimally'

Part V

Optimization for 3D Vision

covered by

[1] [H&Z] Secs: 11.4, 11.6, 4.7

[2] Fischler, M.A. and Bolles, R.C . Random Sample Consensus: A Paradigm for Model Fitting with Applications to Image Analysis and Automated Cartography. *Communications of the ACM* 24(6):381–395, 1981

additional references

P. D. Sampson. Fitting conic sections to 'very scattered' data: An iterative refinement of the Bookstein algorithm. *Computer Vision, Graphics, and Image Processing*, 18:97–108, 1982.

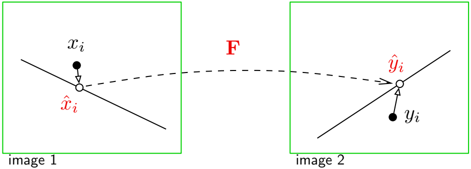
O. Chum, J. Matas, and J. Kittler. Locally optimized RANSAC. In *Proc DAGM*, LNCS 2781:236–243. Springer-Verlag, 2003.

O. Chum, T. Werner, and J. Matas. Epipolar geometry estimation via RANSAC benefits from the oriented epipolar constraint. In *Proc ICPR*, vol 1:112–115, 2004.

▶The Concept of Error for Epipolar Geometry

Problem: Given at least 8 matched points $x_i \leftrightarrow y_j$ in a general position, estimate the most likely (or most probable) fundamental matrix $\mathbf{F}$.

$$\mathbf{x}_i = (u_i^1,\, v_i^1), \quad \mathbf{y}_i = (u_i^2,\, v_i^2), \qquad i = 1, 2, \ldots, k, \quad k \geq 8$$



- detected points (measurements) x_i, y_i
- we introduce matches $\mathbf{Z}_i = (u_i^1, v_i^1, u_i^2, v_i^2) \in \mathbb{R}^4$; $\quad S = \{\mathbf{Z}_i\}_{i=1}^k$
- corrected points $\hat{x}_i$, $\hat{y}_i$; $\quad \hat{\mathbf{Z}}_i = (\hat{u}_i^1, \hat{v}_i^1, \hat{u}_i^2, \hat{v}_i^2)$; $\hat{S} = \{(\hat{\mathbf{Z}}_i\}_{i=1}^k$ are correspondences
- correspondences satisfy the epipolar geometry exactly $\underline{\hat{\mathbf{y}}}_i^\top \mathbf{F}\, \underline{\hat{\mathbf{x}}}_i = 0$, $i = 1, \ldots, k$
- small correction is more probable
- let $\mathbf{e}_i(\cdot)$ be the 'reprojection error' (vector) per match i,

$$\mathbf{e}_i(x_i, y_i \mid \hat{x}_i, \hat{y}_i, \mathbf{F}) = \begin{bmatrix} \mathbf{x}_i - \hat{\mathbf{x}}_i \\ \mathbf{y}_i - \hat{\mathbf{y}}_i \end{bmatrix} = \mathbf{e}_i(\mathbf{Z}_i \mid \hat{\mathbf{Z}}_i, \mathbf{F}) = \mathbf{Z}_i - \hat{\mathbf{Z}}_i(\mathbf{F}) \tag{15}$$

$$\|\mathbf{e}_i(\cdot)\|^2 \stackrel{\text{def}}{=} \mathbf{e}_i^2(\cdot) = \|\mathbf{x}_i - \hat{\mathbf{x}}_i\|^2 + \|\mathbf{y}_i - \hat{\mathbf{y}}_i\|^2 = \|\mathbf{Z}_i - \hat{\mathbf{Z}}_i\|^2$$

▶cont'd

- the total reprojection error (of all data) then is

$$L(S \mid \hat{S}, \mathbf{F}) = \sum_{i=1}^{k} \mathbf{e}_i^2(x_i, y_i \mid \hat{x}_i, \hat{y}_i, \mathbf{F}) \quad = \sum_{i=1}^{k} \mathbf{e}_i^2(\mathbf{Z}_i \mid \hat{\mathbf{Z}}_i, \mathbf{F})$$

- and the optimization problem is

$$(\hat{S}^*, \mathbf{F}^*) = \arg \min_{\substack{\mathbf{F} \\ \operatorname{rank} \mathbf{F} = 2}} \; \min_{\substack{\hat{S} \\ \underline{\hat{\mathbf{y}}}_i^\top \mathbf{F}\, \underline{\hat{\mathbf{x}}}_i = 0}} \sum_{i=1}^{k} \mathbf{e}_i^2(x_i, y_i \mid \hat{x}_i, \hat{y}_i, \mathbf{F}) \tag{16}$$

Three possible approaches

- they differ in how the correspondences $\hat{x}_i$, $\hat{y}_i$ are obtained:
  1. direct optimization of reprojection error over all variables $\hat{S}$, $\mathbf{F}$ →93
  2. Sampson optimal correction = partial correction of $\mathbf{Z}_i$ towards $\hat{\mathbf{Z}}_i$ used in an iterative minimization over $\mathbf{F}$ →94
  3. removing $\hat{x}_i$, $\hat{y}_i$ altogether = marginalization of $L(S, \hat{S} \mid \mathbf{F})$ over $\hat{S}$ followed by minimization over $\mathbf{F}$ — not covered, the marginalization is difficult

Method 1: Geometric Error Optimization

- we need to encode the constraints $\hat{\underline{\mathbf{y}}}_i \mathbf{F} \hat{\underline{\mathbf{x}}}_i = 0$, $\operatorname{rank} \mathbf{F} = 2$
- idea: reconstruct 3D point via equivalent projection matrices and use reprojection error
- equivalent projection matrices are see [H&Z,Sec. 9.5] for complete characterization

$$\mathbf{P}_1 = \begin{bmatrix} \mathbf{I} & \mathbf{0} \end{bmatrix}, \quad \mathbf{P}_2 = \begin{bmatrix} [\underline{\mathbf{e}}_2]_\times \mathbf{F} + \underline{\mathbf{e}}_2 \underline{\mathbf{e}}_1^\top & \underline{\mathbf{e}}_2 \end{bmatrix} \tag{17}$$

⊛ H3; 2pt: Verify that $\mathbf{F}$ is a f.m. of $\mathbf{P}_1$, $\mathbf{P}_2$, for instance that there is a regular $\mathbf{H}$ such that $\mathbf{F} \simeq \mathbf{H}^{-\top}[\mathbf{e}_1]_\times$

---

1. compute $\mathbf{F}^{(0)}$ by the 7-point algorithm →80; construct camera $\mathbf{P}_2^{(0)}$ from $\mathbf{F}^{(0)}$ using (17)
2. triangulate 3D points $\hat{\mathbf{X}}_i^{(0)}$ from matches (x_i, y_i) for all $i = 1, \ldots, k$ →84
3. starting from $\mathbf{P}_2^{(0)}$, $\hat{\mathbf{X}}^{(0)}$ minimize the reprojection error (15)

$$(\hat{\mathbf{X}}^*, \mathbf{P}_2^*) = \arg \min_{\mathbf{P}_2, \hat{\mathbf{X}}} \sum_{i=1}^{k} \mathbf{e}_i^2(\mathbf{Z}_i \mid \hat{\mathbf{Z}}_i(\hat{\mathbf{X}}_i, \mathbf{P}_2))$$

where

$$\hat{\mathbf{Z}}_i = (\hat{\mathbf{x}}_i, \hat{\mathbf{y}}_i) \ \text{(Cartesian)}, \quad \hat{\underline{\mathbf{x}}}_i \simeq \mathbf{P}_1 \hat{\underline{\mathbf{X}}}_i, \ \ \hat{\underline{\mathbf{y}}}_i \simeq \mathbf{P}_2 \hat{\underline{\mathbf{X}}}_i \ \text{(homogeneous)}$$

Non-linear, non-convex problem

4. compute $\mathbf{F}$ from $\mathbf{P}_1$, $\mathbf{P}_2^*$

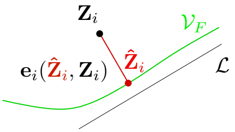
- $3k + 12$ parameters to be found: latent: $\hat{\mathbf{X}}_i$, for all i (correspondences!), non-latent: $\mathbf{P}_2$
- minimal representation: $3k + 7$ parameters, $\mathbf{P}_2 = \mathbf{P}_2(\mathbf{F})$ →139
- there are pitfalls; this is essentially bundle adjustment; we will return to this later →131

▶Method 2: First-Order Error Approximation

An elegant method for solving problems like (16):

- we will get rid of the latent parameters $\hat{X}$ needed for obtaining the correction
  [H&Z, p. 287], [Sampson 1982]
- we will recycle the algebraic error $\boldsymbol{\varepsilon} = \underline{\mathbf{y}}^\top \mathbf{F}\, \underline{\mathbf{x}}$ from →80

- consider matches $\mathbf{Z}_i$, correspondences $\hat{\mathbf{Z}}_i$, and reprojection error $\mathbf{e}_i = \|\mathbf{Z}_i - \hat{\mathbf{Z}}_i\|^2$
- correspondences satisfy $\underline{\hat{\mathbf{y}}}_i^\top \mathbf{F}\, \underline{\hat{\mathbf{x}}}_i = 0$, $\quad \underline{\hat{\mathbf{x}}}_i = (\hat{u}^1, \hat{v}^1, 1)$, $\underline{\hat{\mathbf{y}}}_i = (\hat{u}^2, \hat{v}^2, 1)$
- this is a manifold $\mathcal{V}_F \in \mathbb{R}^4$: a set of points $\hat{\mathbf{Z}} = (\hat{u}^1, \hat{v}^1, \hat{u}^2, \hat{v}^2)$ consistent with $\mathbf{F}$
- algebraic error vanishes for $\hat{\mathbf{Z}}_i$: $\boxed{\mathbf{0} = \boldsymbol{\varepsilon}_i(\hat{\mathbf{Z}}_i) = \underline{\hat{\mathbf{y}}}_i^\top \mathbf{F}\, \underline{\hat{\mathbf{x}}}_i}$



Sampson's idea: Linearize the algebraic error $\boldsymbol{\varepsilon}(\mathbf{Z})$ at $\mathbf{Z}_i$ (where it is non-zero) and evaluate the resulting linear function at $\hat{\mathbf{Z}}_i$ (where it is zero). The zero-crossing replaces $\mathcal{V}_F$ by a linear manifold $\mathcal{L}$. The point on $\mathcal{V}_F$ closest to $\mathbf{Z}_i$ is replaced by the closest point on $\mathcal{L}$.

$$\boldsymbol{\varepsilon}_i(\hat{\mathbf{Z}}_i) \;\approx\; \boldsymbol{\varepsilon}_i(\mathbf{Z}_i) + \frac{\partial \boldsymbol{\varepsilon}_i(\mathbf{Z}_i)}{\partial \mathbf{Z}_i}\,(\hat{\mathbf{Z}}_i - \mathbf{Z}_i)$$

▶Sampson's Approximation of Reprojection Error

- linearize $\varepsilon(\mathbf{Z})$ at match $\mathbf{Z}_i$, evaluate it at correspondence $\hat{\mathbf{Z}}_i$

$$0 = \varepsilon_i(\hat{\mathbf{Z}}_i) \approx \varepsilon_i(\mathbf{Z}_i) + \underbrace{\frac{\partial \varepsilon_i(\mathbf{Z}_i)}{\partial \mathbf{Z}_i}}_{\mathbf{J}_i(\mathbf{Z}_i)} \underbrace{(\hat{\mathbf{Z}}_i - \mathbf{Z}_i)}_{\mathbf{e}_i(\hat{\mathbf{Z}}_i, \mathbf{Z}_i)} \stackrel{\text{def}}{=} \varepsilon_i(\mathbf{Z}_i) + \mathbf{J}_i(\mathbf{Z}_i)\,\mathbf{e}_i(\hat{\mathbf{Z}}_i, \mathbf{Z}_i)$$

- this is a linear equation for $\hat{\mathbf{Z}}_i$
- $\mathbf{e}_i(\cdot)$ is the distance of $\hat{\mathbf{Z}}_i$ from $\mathbf{Z}_i$
- we compute the distance by least squares per match i

$$\mathbf{e}_i^* = \arg\min_{\mathbf{e}_i} \|\mathbf{e}_i\|^2 \quad \text{subject to} \quad \varepsilon_i + \mathbf{J}_i\,\mathbf{e}_i = 0$$

- which has a closed-form solution note that $\mathbf{J}_i$ is not invertible! ⊛ P1; 1pt: derive $\mathbf{e}_i^*$

$$\mathbf{e}_i^* = -\mathbf{J}_i^\top(\mathbf{J}_i\mathbf{J}_i^\top)^{-1}\varepsilon_i$$
$$\|\mathbf{e}_i^*\|^2 = \varepsilon_i^\top(\mathbf{J}_i\mathbf{J}_i^\top)^{-1}\varepsilon_i \tag{18}$$

- this mapping translates $\varepsilon(\cdot)$ to an estimate of $\mathbf{e}(\cdot)$
- we often do not need $\mathbf{e}_i$, just $\|\mathbf{e}_i\|^2$ exception: triangulation →100
- the unknown parameters $\mathbf{F}$ are inside: $\mathbf{e}_i = \mathbf{e}_i(\mathbf{F})$, $\varepsilon_i = \varepsilon_i(\mathbf{F})$, $\mathbf{J}_i = \mathbf{J}_i(\mathbf{F})$

▸Example: Fitting A Circle To Scattered Points

Problem: Fit a zero-centered circle $\mathcal{C}$ to a set of 2D points $\{x_i\}_{i=1}^k$, $\mathcal{C}\colon\ \|\mathbf{x}\|^2 - r^2 = 0$.

1. consider radial error as the 'algebraic error' $\boldsymbol{\varepsilon}(\mathbf{x}) = \|\mathbf{x}\|^2 - r^2$
2. linearize it at $\hat{\mathbf{x}}$ — we are dropping i in $\boldsymbol{\varepsilon}_i$, $\mathbf{e}_i$ etc for clarity

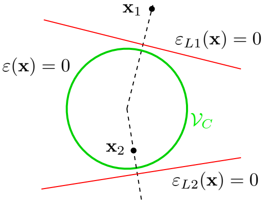
$$\boldsymbol{\varepsilon}(\hat{\mathbf{x}}) \approx \boldsymbol{\varepsilon}(\mathbf{x}) + \underbrace{\frac{\partial \boldsymbol{\varepsilon}(\mathbf{x})}{\partial \mathbf{x}}}_{\mathbf{J}(\mathbf{x})=2\mathbf{x}^\top} \underbrace{(\hat{\mathbf{x}} - \mathbf{x})}_{\mathbf{e}(\hat{\mathbf{x}},\mathbf{x})} = \cdots = 2\,\mathbf{x}^\top \hat{\mathbf{x}} - (r^2 + \|\mathbf{x}\|^2) \stackrel{\text{def}}{=} \boldsymbol{\varepsilon}_L(\hat{\mathbf{x}})$$

$\boldsymbol{\varepsilon}_L(\hat{\mathbf{x}}) = 0$ is a line with normal $\frac{\mathbf{x}}{\|\mathbf{x}\|}$ and intercept $\frac{r^2+\|\mathbf{x}\|^2}{2\|\mathbf{x}\|}$ — not tangent to $\mathcal{C}$, outside!

3. using (18), express error approximation $\mathbf{e}^*$ as

$$\|\mathbf{e}^*\|^2 = \boldsymbol{\varepsilon}^\top (\mathbf{J}\mathbf{J}^\top)^{-1} \boldsymbol{\varepsilon} = \frac{(\|\mathbf{x}\|^2 - r^2)^2}{4\|\mathbf{x}\|^2}$$

4. fit circle



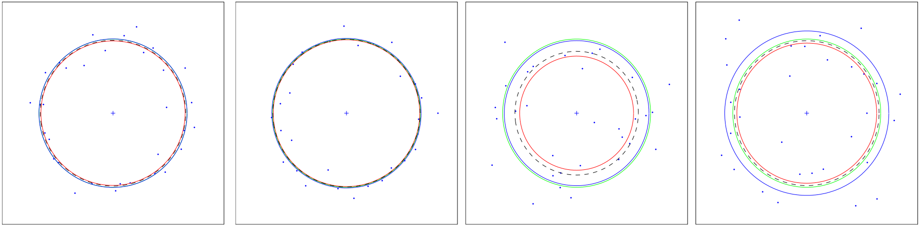
$$r^* = \arg\min_r \sum_{i=1}^k \frac{(\|\mathbf{x}_i\|^2 - r^2)^2}{4\|\mathbf{x}_i\|^2} = \cdots = \left(\frac{1}{k}\sum_{i=1}^k \frac{1}{\|\mathbf{x}_i\|^2}\right)^{-\frac{1}{2}}$$

- this example results in a convex quadratic optimization problem
- note that

$$\arg\min_r \sum_{i=1}^k (\|\mathbf{x}_i\|^2 - r^2)^2 = \left(\frac{1}{k}\sum_{i=1}^k \|\mathbf{x}_i\|^2\right)^{\frac{1}{2}}$$

Circle Fitting: Some Results

medium isotropic noise | medium radial noise | big isotropic noise | big radial noise



opt=1.8, dir=2.0, Smp=2.2 | 1.8, 1.9, 2.3 | 1.6, 2.0, 2.4 | 1.6, 1.8, 2.6

mean ranks over 10 000 random trials with $k = 32$ samples

green – ground truth
red – Sampson error minimizer
blue – direct radial error minimizer
black – optimal estimator for isotropic error

optimal estimator for isotropic error (black, dashed):

$$r \approx \frac{3}{4k}\sum_{i=1}^{k}\|\mathbf{x}_i\| + \sqrt{\left(\frac{3}{4k}\sum_{i=1}^{k}\|\mathbf{x}_i\|\right)^2 - \frac{1}{2k}\sum_{i=1}^{k}\|\mathbf{x}_i\|^2}$$

which method is better?

- error should model noise, radial noise and isotropic noise behave differently
- ground truth: Normally distributed isotropic error, Gamma-distributed radial error
- Sampson correction is closer to the radial distribution model
- no matter how corrected, the algebraic error minimizer is not an unbiased parameter estimator

Cramér-Rao bound tells us how close one can get with unbiased estimator and given k

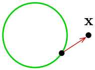
Discussion: On The Art of Probabilistic Model Design...

- a few models for fitting zero-centered circle C of radius r to points in $\mathbb{R}^2$

error model / **radial p.d.f.** / **random sample** / $p(\mathbf{x} \mid r)$

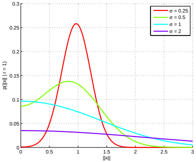
marginalized over C

error model:

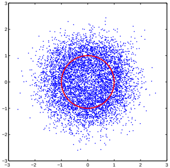


$N(\mathbf{0}, \sigma^2 \mathbf{I})$

radial p.d.f.:



random sample:

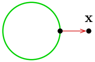


$$p(\mathbf{x} \mid r) \approx \frac{1}{\sigma\sqrt{(2\pi)^3 r\,\|\mathbf{x}\|}}\, e^{-\frac{(\|\mathbf{x}\|-r)^2}{2\sigma^2}}$$

- mode inside the circle
- models the inside well
- tends to normal distrib.

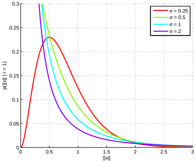
orthogonal deviation from C

error model:

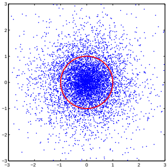


$\Gamma(\cdot,\cdot)$

radial p.d.f.:



random sample:

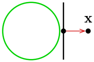


$$p(\mathbf{x} \mid r) = \frac{1}{2\pi\Gamma(\frac{r^2}{\sigma})}\, \frac{1}{\|\mathbf{x}\|^2} \left(\frac{r\|\mathbf{x}\|}{\sigma}\right)^{\frac{r^2}{\sigma}} e^{-\frac{r\|\mathbf{x}\|}{\sigma}}$$

- peak at the center
- unusable for small radii
- tends to Dirac distrib.

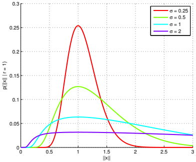
Sampson approximation

error model:

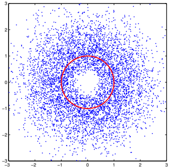


$N(\mathbf{0}, \sigma^2 \mathbf{I})$

radial p.d.f.:



random sample:



$$p(\mathbf{x} \mid r) = \frac{1}{r\sigma\sqrt{(2\pi)^3}}\, e^{-\frac{e^2(\mathbf{x};r)}{2\sigma^2}}$$

- mode at the circle
- hole at the center
- tends to normal distrib.

▶Sampson Error for Fundamental Matrix Estimation

The fundamental matrix estimation problem becomes

$\mathbf{e}_i$ is scalar, hence e_i

$$\mathbf{F}^* = \arg \min_{\mathbf{F}, \operatorname{rank} \mathbf{F}=2} \sum_{i=1}^{k} e_i^2(\mathbf{F})$$

Let $\mathbf{F} = \begin{bmatrix} \mathbf{F}_1 & \mathbf{F}_2 & \mathbf{F}_3 \end{bmatrix}$ (per columns) $= \begin{bmatrix} (\mathbf{F}^1)^\top \\ (\mathbf{F}^2)^\top \\ (\mathbf{F}^3)^\top \end{bmatrix}$ (per rows), $\mathbf{S} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, then

Sampson

$$\varepsilon_i(\mathbf{F}) = \underline{\mathbf{y}}_i^\top \mathbf{F}\, \underline{\mathbf{x}}_i$$

$\varepsilon_i \in \mathbb{R}$ — scalar algebraic error →80

$$\mathbf{J}_i(\mathbf{F}) = \left[\frac{\partial \varepsilon_i(\mathbf{F})}{\partial u_i^1}, \frac{\partial \varepsilon_i(\mathbf{F})}{\partial v_i^1}, \frac{\partial \varepsilon_i(\mathbf{F})}{\partial u_i^2}, \frac{\partial \varepsilon_i(\mathbf{F})}{\partial v_i^2} \right]$$

$\mathbf{J}_i \in \mathbb{R}^{1,4}$ — derivatives over point coords.

$$e_i(\mathbf{F}) = \frac{\varepsilon_i(\mathbf{F})}{\|\mathbf{J}_i(\mathbf{F})\|}$$

$e_i \in \mathbb{R}$ — Sampson error

$$\mathbf{J}_i(\mathbf{F}) = \left[(\mathbf{F}_1)^\top \underline{\mathbf{y}}_i, \; (\mathbf{F}_2)^\top \underline{\mathbf{y}}_i, \; (\mathbf{F}^1)^\top \underline{\mathbf{x}}_i, \; (\mathbf{F}^2)^\top \underline{\mathbf{x}}_i \right] \qquad e_i(\mathbf{F}) = \frac{\underline{\mathbf{y}}_i^\top \mathbf{F} \underline{\mathbf{x}}_i}{\sqrt{\|\mathbf{S}\mathbf{F}\underline{\mathbf{x}}_i\|^2 + \|\mathbf{S}\mathbf{F}^\top \underline{\mathbf{y}}_i\|^2}}$$

- Sampson correction 'normalizes' the algebraic error
- automatically copes with multiplicative factors $\mathbf{F} \mapsto \lambda \mathbf{F}$
- actual optimization not yet covered →103

▶Back to Triangulation: The Golden Standard Method

We are given $\mathbf{P}_1$, $\mathbf{P}_2$ and a single correspondence $x \leftrightarrow y$ and we look for 3D point $\mathbf{X}$ projecting to x and y. →84

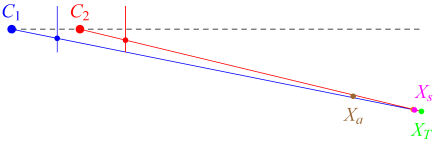
Idea:

1. compute $\mathbf{F}$ from $\mathbf{P}_1$, $\mathbf{P}_2$, e.g. $\mathbf{F} = (\mathbf{Q}_1\mathbf{Q}_2^{-1})^\top[\mathbf{q}_1 - (\mathbf{Q}_1\mathbf{Q}_2^{-1})\mathbf{q}_2]_\times$
2. correct measurement by the linear estimate of the correction vector →95

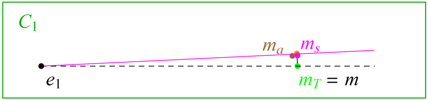
$$\begin{bmatrix}\hat{u}^1\\\hat{v}^1\\\hat{u}^2\\\hat{v}^2\end{bmatrix} \approx \begin{bmatrix}u^1\\v^1\\u^2\\v^2\end{bmatrix} - \frac{\varepsilon}{\|\mathbf{J}\|^2}\mathbf{J}^\top = \begin{bmatrix}u^1\\v^1\\u^2\\v^2\end{bmatrix} - \frac{\underline{\mathbf{y}}^\top\mathbf{F}\underline{\mathbf{x}}}{\|\mathbf{S}\mathbf{F}\underline{\mathbf{x}}\|^2 + \|\mathbf{S}\mathbf{F}^\top\underline{\mathbf{y}}\|^2}\begin{bmatrix}(\mathbf{F}_1)^\top\mathbf{y}\\(\mathbf{F}_2)^\top\mathbf{y}\\(\mathbf{F}^1)^\top\mathbf{x}\\(\mathbf{F}^2)^\top\mathbf{x}\end{bmatrix}$$

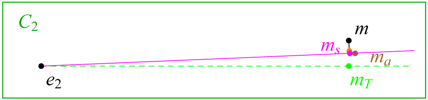
3. use the SVD algorithm with numerical conditioning →85

Ex (cont'd from →88):



X_T – noiseless ground truth position
● – reprojection error minimizer
X_s – Sampson-corrected algebraic error minimizer
X_a – algebraic error minimizer
m – measurement (m_T with noise in v^2)





Levenberg-Marquardt (LM) Iterative Estimation in a Nutshell

Consider error function $\mathbf{e}_i(\boldsymbol{\theta}) = f(\mathbf{x}_i, \mathbf{y}_i, \boldsymbol{\theta}) \in \mathbb{R}^m$, with $\mathbf{x}_i, \mathbf{y}_i$ given, $\boldsymbol{\theta} \in \mathbb{R}^q$ unknown

$\theta = \mathbf{F}$, $q = 9$, $m = 1$ for f.m. estimation

Our goal: $\boldsymbol{\theta}^* = \arg\min_{\boldsymbol{\theta}} \sum_{i=1}^{k} \|\mathbf{e}_i(\boldsymbol{\theta})\|^2$

Idea 1 (Gauss-Newton approximation): proceed iteratively for $s = 0, 1, 2, \ldots$

$$\boldsymbol{\theta}^{s+1} := \boldsymbol{\theta}^s + \mathbf{d}_s\,, \qquad \text{where} \qquad \mathbf{d}_s = \arg\min_{\mathbf{d}} \sum_{i=1}^{k} \|\mathbf{e}_i(\boldsymbol{\theta}^s + \mathbf{d})\|^2 \tag{19}$$

$$\mathbf{e}_i(\boldsymbol{\theta}^s + \mathbf{d}) \approx \mathbf{e}_i(\boldsymbol{\theta}^s) + \mathbf{L}_i\,\mathbf{d},$$

$$(\mathbf{L}_i)_{jl} = \frac{\partial\big(\mathbf{e}_i(\boldsymbol{\theta})\big)_j}{\partial(\boldsymbol{\theta})_l}, \qquad \mathbf{L}_i \in \mathbb{R}^{m,q}$$

typically a long matrix

Then the solution to Problem (19) is a set of normal eqs

$$-\underbrace{\sum_{i=1}^{k} \mathbf{L}_i^\top \mathbf{e}_i(\boldsymbol{\theta}^s)}_{\mathbf{e} \in \mathbb{R}^{q,1}} = \underbrace{\left(\sum_{i=1}^{k} \mathbf{L}_i^\top \mathbf{L}_i\right)}_{\mathbf{L} \in \mathbb{R}^{q,q}} \mathbf{d}_s, \tag{20}$$

- $\mathbf{d}_s$ can be solved for by Gaussian elimination using Choleski decomposition of $\mathbf{L}$
  $\mathbf{L}$ symmetric $\Rightarrow$ use Choleski, almost $2\times$ faster than Gauss-Seidel, see bundle adjustment $\rightarrow$134
- such updates do not lead to stable convergence $\longrightarrow$ ideas of Levenberg and Marquardt

LM (cont'd)

Idea 2 (Levenberg): replace $\sum_i \mathbf{L}_i^\top \mathbf{L}_i$ with $\sum_i \mathbf{L}_i^\top \mathbf{L}_i + \lambda\,\mathbf{I}$ for some damping factor $\lambda \geq 0$

Idea 3 (Marquardt): replace $\lambda\,\mathbf{I}$ with $\lambda \sum_i \operatorname{diag}(\mathbf{L}_i^\top \mathbf{L}_i)$ to adapt to local curvature:

$$-\sum_{i=1}^{k} \mathbf{L}_i^\top \mathbf{e}_i(\boldsymbol{\theta}^s) = \left(\sum_{i=1}^{k} \left(\mathbf{L}_i^\top \mathbf{L}_i + \lambda \operatorname{diag} \mathbf{L}_i^\top \mathbf{L}_i \right) \right) \mathbf{d}_s$$

Idea 4 (Marquardt): adaptive λ small $\lambda \to$ Gauss-Newton, large $\lambda \to$ gradient descend

1. choose $\lambda \approx 10^{-3}$ and compute $\mathbf{d}_s$
2. if $\sum_i \|\mathbf{e}_i(\boldsymbol{\theta}^s + \mathbf{d}_s)\|^2 < \sum_i \|\mathbf{e}_i(\boldsymbol{\theta}^s)\|^2$ then accept $\mathbf{d}_s$ and set $\lambda := \lambda/10$, $s := s + 1$
3. otherwise set $\lambda := 10\lambda$ and recompute $\mathbf{d}_s$

- sometimes different constants are needed for the 10 and 10^{-3}
- note that $\mathbf{L}_i \in \mathbb{R}^{m,q}$ (long matrix) but each contribution $\mathbf{L}_i^\top \mathbf{L}_i$ is a square singular $q \times q$ matrix (always singular for $k < q$)
- error can be made robust to outliers, see the trick →106
- we have approximated the least squares Hessian by ignoring second derivatives of the error function (Gauss-Newton approximation) See [Triggs et al. 1999, Sec. 4.3]
- λ helps avoid the consequences of gauge freedom →136

LM with Sampson Error for Fundamental Matrix Estimation

Sampson (derived by linearization over point coordinates u^1, v^1, u^2, v^2)

$$e_i(\mathbf{F}) = \frac{\varepsilon_i}{\|\mathbf{J}_i\|} = \frac{\underline{\mathbf{y}}_i^\top \mathbf{F} \underline{\mathbf{x}}_i}{\sqrt{\|\mathbf{S}\mathbf{F}\underline{\mathbf{x}}_i\|^2 + \|\mathbf{S}\mathbf{F}^\top \underline{\mathbf{y}}_i\|^2}} \quad \text{where} \quad \mathbf{S} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

LM (by linearization over parameters $\mathbf{F}$)

$$\mathbf{L}_i = \frac{\partial e_i(\mathbf{F})}{\partial \mathbf{F}} = \cdots = \frac{1}{2\|\mathbf{J}_i\|} \left[\left(\underline{\mathbf{y}}_i - \frac{2e_i}{\|\mathbf{J}_i\|} \mathbf{S}\mathbf{F}\underline{\mathbf{x}}_i \right) \underline{\mathbf{x}}_i^\top + \underline{\mathbf{y}}_i \left(\underline{\mathbf{x}}_i - \frac{2e_i}{\|\mathbf{J}_i\|} \mathbf{S}\mathbf{F}^\top \underline{\mathbf{y}}_i \right)^\top \right] \tag{21}$$

- $\mathbf{L}_i$ is a 3×3 matrix, must be reshaped to dimension-9 vector
- $\underline{\mathbf{x}}_i$ and $\underline{\mathbf{y}}_i$ in Sampson error are normalized to unit homogeneous coordinate (21) relies on this
- reinforce $\operatorname{rank} \mathbf{F} = 2$ after each LM update to stay in the fundamental matrix manifold and $\|\mathbf{F}\| = 1$ to avoid gauge freedom by SVD →104
- LM linearization could be done by numerical differentiation (small dimension)

▶Local Optimization for Fundamental Matrix Estimation

Given a set $\{(x_i, y_i)\}_{i=1}^{k}$ of $k > 7$ inlier correspondences, compute a (reasonably) efficient estimate for fundamental matrix $\mathbf{F}$.

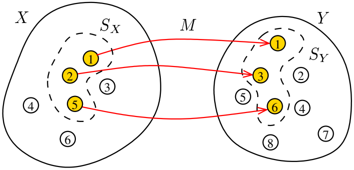
1. Find the conditioned (→87) 7-point $\mathbf{F}_0$ (→80) from a suitable 7-tuple
2. Improve the $\mathbf{F}_0^*$ using the LM optimization (→101–102) and the Sampson error (→103) on <u>all inliers</u>, reinforce rank-2, unit-norm $\mathbf{F}_k^*$ after each LM iteration using SVD

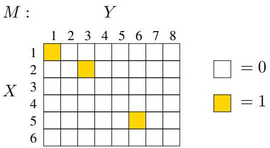
- if there are no wrong matches (outliers), this gives a local optimum
- contamination of (inlier) correspondences by outliers may wreak havoc with this algorithm
- the full problem involves finding the inliers!
- in addition, we need a mechanism for jumping out of local minima (and exploring the space of all fundamental matrices)

▸The Full Problem of Matching and Fundamental Matrix Estimation

Problem: Given image point sets $X = \{x_i\}_{i=1}^m$ and $Y = \{y_j\}_{j=1}^n$ and their descriptors D, find the most probable

1. inliers $S_X \subseteq X$, $S_Y \subseteq Y$
2. one-to-one perfect matching $M \colon S_X \to S_Y$ — perfect matching: 1-factor of the bipartite graph
3. fundamental matrix $\mathbf{F}$ such that $\operatorname{rank} \mathbf{F} = 2$
4. such that for each $x_i \in S_X$ and $y_j = M(x_i)$ it is probable that
   a) the image descriptor $D(x_i)$ is similar to $D(y_j)$, and
   b) the total geometric error $E = \sum_{ij} e_{ij}^2(\mathbf{F})$ is small — note a slight change in notation: e_{ij}
5. inlier-outlier and outlier-outlier matches are improbable





$$(M^*, \mathbf{F}^*) = \arg\max_{M,\mathbf{F}} p(E, D, \mathbf{F} \mid M)\, P(M) \tag{22}$$

- probabilistic model: an efficient language for task formulation — it also unifies 4.a and 4.b
- the (22) is a Bayesian probabilistic model — there is a constant number of random variables!
- binary matching table $M_{ij} \in \{0, 1\}$ of fixed size $m \times n$
  - each row/column contains at most one unity
  - zero rows/columns correspond to unmatched point x_i/y_j

For algorithmic efficiency, instead of $(M^*, \mathbf{F}^*) = \arg\max_{M,\mathbf{F}} p(E, D, \mathbf{F} \mid M)\, P(M)$ solve

$$\boxed{\mathbf{F}^* = \arg\max_{\mathbf{F}} p(E, D, \mathbf{F})} \tag{23}$$

by marginalization of $p(E, D, \mathbf{F} \mid M)\, P(M)$ over M — this changes the problem!

ignoring that M are 1:1 matchings and assuming correspondence-wise independence:

$$p(E, D, \mathbf{F} \mid M)P(M) = \prod_{i=1}^{m}\prod_{j=1}^{n} p_e(e_{ij}, d_{ij}, \mathbf{F} \mid m_{ij})P(m_{ij})$$

- e_{ij} represents geometric error for match $x_i \leftrightarrow y_i$: $e_{ij}(x_i, y_i, \mathbf{F})$
- d_{ij} represents descriptor similarity for match $x_i \leftrightarrow y_i$: $d_{ij} = \|\mathbf{d}(x_i) - \mathbf{d}(y_j)\|$

Marginalization:

$$p(E, D, \mathbf{F}) \approx \sum_{m_{11}\in\{0,1\}} \sum_{m_{12}} \cdots \sum_{m_{mn}} p(E, D, \mathbf{F} \mid M)P(M) =$$

$$= \sum_{m_{11}} \cdots \sum_{m_{mn}} \prod_{i=1}^{m}\prod_{j=1}^{n} p_e(e_{ij}, d_{ij}, \mathbf{F} \mid m_{ij})P(m_{ij}) \overset{\circledast 1}{=} \cdots =$$

$$= \prod_{i=1}^{m}\prod_{j=1}^{n} \underbrace{\sum_{m_{ij}\in\{0,1\}} p_e(e_{ij}, d_{ij}, \mathbf{F} \mid m_{ij})P(m_{ij})}_{\text{we will continue with this term}}$$

Robust Matching Model (cont'd)

$$\sum_{m_{ij} \in \{0,1\}} p_e(e_{ij}, d_{ij}, \mathbf{F} \mid m_{ij}) P(m_{ij}) =$$

$$= \underbrace{p_e(e_{ij}, d_{ij}, \mathbf{F} \mid m_{ij} = 1)}_{p_1(e_{ij}, d_{ij}, \mathbf{F})} \underbrace{P(m_{ij} = 1)}_{1-P_0} + \underbrace{p_e(e_{ij}, d_{ij}, \mathbf{F} \mid m_{ij} = 0)}_{p_0(e_{ij}, d_{ij}, \mathbf{F})} \underbrace{P(m_{ij} = 0)}_{P_0} =$$

$$= (1 - P_0)\, p_1(e_{ij}, d_{ij}, \mathbf{F}) + P_0\, p_0(e_{ij}, d_{ij}, \mathbf{F}) \tag{24}$$

- the $p_0(e_{ij}, d_{ij}, \mathbf{F})$ is a penalty for 'missing a correspondence' but it should be a p.d.f. (cannot be a constant) (→108 for a simplification)

choose $P_0 \to 1$, $p_0(\cdot) \to 0$ so that $\dfrac{P_0}{1 - P_0}\, p_0(\cdot) \approx \text{const}$

- the $p_1(e_{ij}, d_{ij}, \mathbf{F})$ is typically an easy-to-design component: assuming independence of geometric error and descriptor similarity:

$$p_1(e_{ij}, d_{ij}, \mathbf{F}) = p_1(e_{ij} \mid \mathbf{F})\, p_F(\mathbf{F})\, p_1(d_{ij})$$

- we choose, eg.

$$p_1(e_{ij} \mid \mathbf{F}) = \frac{1}{T_e(\sigma_1)}\, e^{-\frac{e_{ij}^2(\mathbf{F})}{2\sigma_1^2}}, \quad p_1(d_{ij}) = \frac{1}{T_d(\sigma_d, \dim \mathbf{d})}\, e^{-\frac{\|\mathbf{d}(x_i) - \mathbf{d}(y_j)\|^2}{2\sigma_d^2}} \tag{25}$$

- $\mathbf{F}$ is a random variable and σ_1, σ_d, P_0 are parameters
- the form of $T(\sigma_1)$ depends on error definition, it may depend on x_i, y_j but not on $\mathbf{F}$
- we will continue with the result from (24)

▶Simplified Robust Energy (Error) Function

- assuming the choice of p_1 as in (25), we are simplifying

$$p(E, D, \mathbf{F}) = p(E, D \mid \mathbf{F})\, p_F(\mathbf{F}) =$$

$$= p_F(\mathbf{F}) \prod_{i=1}^{m} \prod_{j=1}^{n} \Big[(1 - P_0)\, p_1(e_{ij}, d_{ij} \mid \mathbf{F}) + P_0\, p_0(e_{ij}, d_{ij} \mid \mathbf{F}) \Big]$$

- we choose $\sigma_0 \gg \sigma_1$ and omit d_{ij} for simplicity; then the square-bracket term is

$$\frac{1 - P_0}{T_e(\sigma_1)}\, e^{-\frac{e_{ij}^2(\mathbf{F})}{2\sigma_1^2}} + \frac{P_0}{T_e(\sigma_0)}\, e^{-\frac{e_{ij}^2(\mathbf{F})}{2\sigma_0^2}}$$

- we define the 'potential function' as: $V(x) = -\log p(x)$, then

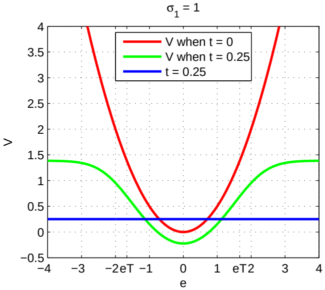
$$V(E, D \mid \mathbf{F}) = \sum_{i=1}^{m} \sum_{j=1}^{n} \left[\underbrace{-\log \frac{1 - P_0}{T_e(\sigma_1)}}_{\Delta\, =\, \text{const}} - \log\Big(e^{-\frac{e_{ij}^2(\mathbf{F})}{2\sigma_1^2}} + \underbrace{\frac{P_0}{1 - P_0} \frac{T_e(\sigma_1)}{T_e(\sigma_0)}\, e^{-\frac{e_{ij}^2(\mathbf{F})}{2\sigma_0^2}}}_{t\, \approx\, \text{const}} \Big) \right] =$$

$$= \boxed{m\, n\, \Delta + \sum_{i=1}^{m} \sum_{j=1}^{n} \underbrace{-\log\Big(e^{-\frac{e_{ij}^2(\mathbf{F})}{2\sigma_1^2}} + t \Big)}_{\hat{V}(e_{ij})}} \qquad (26)$$

- note we are summing over all $m\, n$ matches (m, n are constant!)

►The Action of the Robust Matching Model on Data

Example for $\hat{V}(e)$ from (26):



red – the usual (non-robust) error when $t = 0$

blue – the rejected correspondence penalty t

green – 'robust energy' (26)

- if the error of a correspondence exceeds a limit, it is ignored
- then $\hat{V}(e) = \text{const}$ and we essentially count outliers in (26)
- t controls the 'turn-off' point
- the inlier/outlier threshold is e_T – the error for which $(1 - P_0)\, p_1(e_T) = P_0\, p_0(e_T)$: note that $t \approx 0$

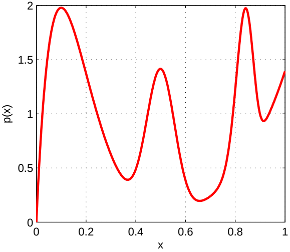
$$e_T = \sigma_1 \sqrt{-\log t^2} \qquad (27)$$

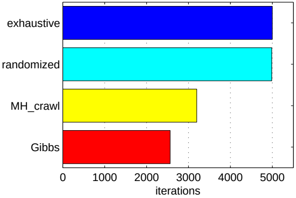
The full optimization problem (23) uses (26):

$$\mathbf{F}^* = \arg\max_{\mathbf{F}} \frac{\overbrace{p(E, D \mid \mathbf{F})}^{\text{likelihood}} \cdot \overbrace{p(\mathbf{F})}^{\text{prior}}}{\underbrace{p(E, D)}_{\text{evidence}}} \approx \arg\min_{\mathbf{F}} \left[V(\mathbf{F}) + \sum_{i=1}^{m} \sum_{j=1}^{n} \log\left(e^{-\frac{e_{ij}^2(\mathbf{F})}{2\sigma_1^2}} + t \right) \right]$$

- typically we take $V(\mathbf{F}) = -\log p(\mathbf{F}) = 0$ unless we need to stabilize a computation, e.g. when video camera moves smoothly (on a high-mass vehicle) and we have a prediction for $\mathbf{F}$
- evidence is not needed unless we want to compare different models (eg. homography vs. epipolar geometry)

How To Find the Global Maxima (Modes) of a PDF?





- averaged over 10^4 trials
- number of proposals before $|x - x_{\text{true}}| \leq \text{step}$

- given the function $p(x)$ at left

p.d.f. on $[0, 1]$, mode at 0.1

consider several methods:

1. exhaustive search

```
step = 1/(iterations-1);
for x = 0:step:1
 if p(x) > bestp
  bestx = x; bestp = p(x);
 end
end
```

- slow algorithm (definite quantization)
- fast to implement

2. randomized search with uniform sampling

```
while t < iterations
 x = rand(1);
 if p(x) > bestp
  bestx = x; bestp = p(x);
 end
 t = t+1;  % time
end
```

- equally slow algorithm
- fast to implement
- how to stop it?

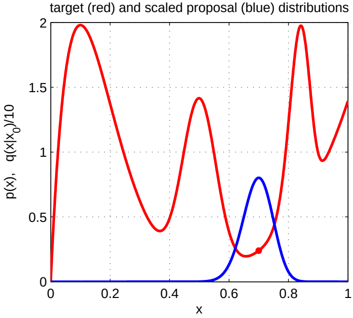
3. random sampling from $p(x)$ (Gibbs sampler)

• faster algorithm • fast to implement but often infeasible (e.g. when $p(x)$ is data dependent (our case in correspondence prob.))

4. Metropolis-Hastings sampling

• almost as fast (with care) • not so fast to implement
• rarely infeasible • RANSAC belongs here

How To Generate Random Samples from a Complex Distribution?



- red: probability density function $p(x)$ of the toy distribution on the unit interval target distribution

$$p(x) = \sum_{i=1}^{4} \gamma_i \, \mathrm{Be}(x; \alpha_i, \beta_i), \quad \sum_{i=1}^{4} \gamma_i = 1, \ \gamma_i \geq 0$$

$$\mathrm{Be}(x; \alpha, \beta) = \frac{1}{\mathrm{B}(\alpha, \beta)} \cdot x^{\alpha-1} (1-x)^{\beta-1}$$

- note we can generate samples from this $p(x)$ how?

- suppose we cannot sample from $p(x)$ but we can sample from some 'simple' distribution $q(x \mid x_0)$, given the last sample x_0 (blue) proposal distribution

$$q(x \mid x_0) = \begin{cases} \mathrm{U}_{0,1}(x) & \text{(independent) uniform sampling} \\ \mathrm{Be}(x; \frac{x_0}{T} + 1, \frac{1-x_0}{T} + 1) & \text{'beta' diffusion (crawler)} \quad T \text{ – temperature} \\ p(x) & \text{(independent) Gibbs sampler} \end{cases}$$

- note we have unified all the random sampling methods from the previous slide
- how to transform proposal samples $q(x \mid x_0)$ to target distribution $p(x)$ samples?

▶Metropolis-Hastings (MH) Sampling

C – configuration (of all variable values) eg. $C = \mathbf{F}$ and $p(C) = p(\mathbf{F} \mid E, D)$

Goal: Generate a sequence of random samples $\{C_t\}$ from $p(C)$

- setup a Markov chain with a suitable transition probability to generate the sequence

Sampling procedure

1. given C_t, draw a random sample S from $q(S \mid C_t)$

   q may use some information from C_t (Hastings)
2. compute acceptance probability — the evidence term drops out

$$a = \min\left\{1,\ \frac{p(S)}{p(C_t)} \cdot \frac{q(C_t \mid S)}{q(S \mid C_t)}\right\}$$

3. draw a random number u from unit-interval uniform distribution $\mathrm{U}_{0,1}$
4. if $u \leq a$ then $C_{t+1} := S$ else $C_{t+1} := C_t$

'Programming' an MH sampler

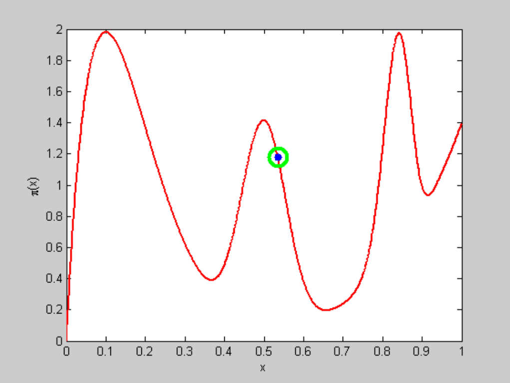
1. design a proposal distribution (mixture) q and a sampler from q
2. write functions $q(C_t \mid S)$ and $q(S \mid C_t)$ that are proper distributions — not always simple

Finding the mode

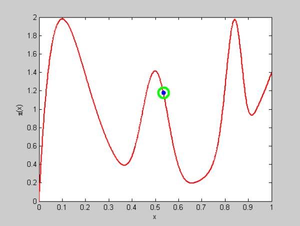
- remember the best sample — fast implementation but must wait long to hit the mode
- use simulated annealing — very slow
- start local optimization from the best sample — good trade-off between speed and accuracy

an optimal algorithm does not use just the best sample: a Stochastic EM Algorithm (e.g. SAEM)

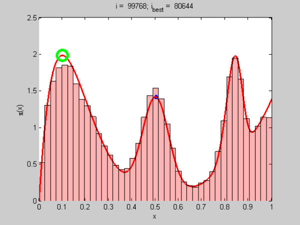
MH Sampling Demo



sampling process (video, 7:33, 100k samples)



initial sample



final distribution
of visited states

- blue point: current sample
- green circle: best sample so far

quality $= p(x)$

- histogram: current distribution of visited states
- the vicinity of modes are the most often visited states

Demo Source Code (Matlab)

```
function x = proposal_gen(x0)
% proposal generator q(x | x0)

 T = 0.01; % temperature
 x = betarnd(x0/T+1,(1-x0)/T+1);
end


function p = proposal_q(x, x0)
% proposal distribution q(x | x0)

 T = 0.01;
 p = betapdf(x, x0/T+1, (1-x0)/T+1);
end


function p = target_p(x)
% target distribution p(x)

 % shape parameters:
 a = [2   40   100 6];
 b = [10  40   20   1];

 % mixing coefficients:
 w = [1 0.4 0.253 0.50]; w = w/sum(w);
 p = 0;
 for i = 1:length(a)
  p = p + w(i)*betapdf(x,a(i),b(i));
 end
end
```

```
%% DEMO script

k = 10000;    % number of samples
X = NaN(1,k); % list of samples

x0 = proposal_gen(0.5);
for i = 1:k
 x1 = proposal_gen(x0);
 a = target_p(x1)/target_p(x0) * ...
     proposal_q(x0,x1)/proposal_q(x1,x0);
 if rand(1) < a
  X(i) = x1;  x0 = x1;
 else
  X(i) = x0;
 end
end

figure(1)
x = 0:0.001:1;
plot(x, target_p(x), 'r', 'linewidth',2);
hold on
binw = 0.025; % histogram bin width
n = histc(X, 0:binw:1);
h = bar(0:binw:1, n/sum(n)/binw, 'histc');
set(h, 'facecolor', 'r', 'facealpha', 0.3)
xlim([0 1]); ylim([0 2.5])
xlabel 'x'
ylabel 'p(x)'
title 'MH demo'
hold off
```

▶Stripping MH Down

- when we are interested in the best sample only... and we need fast data exploration...

Simplified sampling procedure

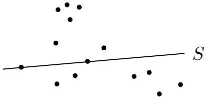
1. ~~given C_t,~~ draw a random sample S from $\cancel{q(S \mid C_t)}$ $q(S)$ — independent sampling; no use of information from C_t
2. ~~compute acceptance probability~~
$$a = \min\left\{1,\ \frac{p(S)}{p(C_t)} \cdot \frac{q(C_t \mid S)}{q(S \mid C_t)}\right\}$$
3. ~~draw a random number u from unit-interval uniform distribution $\mathrm{U}_{0,1}$~~
4. ~~if $u \le a$ then $C_{t+1} := S$ else $C_{t+1} := C_t$~~
5. if $p(S) > p(C_{\text{best}})$ then remember $C_{\text{best}} := S$

- ... but getting a good accuracy sample might take very long this way
- good overall exploration but slow convergence in the vicinity of a mode where C_t could serve as an attractor
- cannot use the past generated samples to estimate any parameters
- we will fix these problems by (possibly robust) 'local optimization'

▶Putting Some Clothes Back: RANSAC with Local Optimization

1. initialize the best sample as empty $C_{\text{best}} := \emptyset$ and time $t := 0$
2. estimate the number of needed iterations as $N := \binom{mn}{s}$
3. while $t \leq N$:

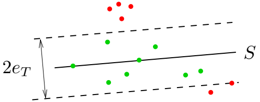
   a) draw a minimal random sample S of size s from $q(S)$

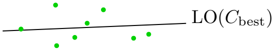
   b) if $p(S) > p(C_{\text{best}})$ then

      i) update the best sample $C_{\text{best}} := S$ — $p(S)$ marginalized as in (26); $p(S)$ includes a prior $\Rightarrow$ MAP

      ii) threshold-out inliers

      iii) start local optimization from the inliers of C_{best} — LM optimization with robustified (→108) Sampson error possibly weighted by posterior $p(m_{ij})$ [Chum et al. 2003]

      iv) update C_{best}, update inliers using (27), re-estimate N from inlier counts →117 for derivation

$$N = \frac{\log(1-P)}{\log(1-\varepsilon^s)}, \quad \varepsilon = \frac{|\operatorname{inliers}(C_{\text{best}})|}{m\,n},$$

   c) $t := t + 1$

4. output C_{best}

- see MPV course for RANSAC details — see also [Fischler & Bolles 1981], [25 years of RANSAC]

▶Stopping RANSAC

Principle: what is the number of proposals N that are needed to hit an all-inlier sample?

this will tell us nothing about the accuracy of the result

P ...probability that at least one sample is all-inlier $\quad 1-P$ ...all previous N samples were bad

ε ...the fraction of inliers among tentative correspondences, $\varepsilon \leq 1$

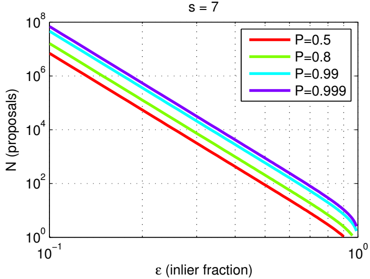
s ...sample size (7 in 7-point algorithm)

$$N \geq \frac{\log(1-P)}{\log(1-\varepsilon^s)}$$

- ε^s ...proposal does not contain an outlier
- $1-\varepsilon^s$ ...proposal contains at least one outlier
- $(1-\varepsilon^s)^N$ ...N previous proposals contained an outlier $= 1-P$

N for $s=7$

	P	
ε	0.8	0.99
0.5	205	590
0.2	$1.3 \cdot 10^5$	$3.5 \cdot 10^5$
0.1	$1.6 \cdot 10^7$	$4.6 \cdot 10^7$



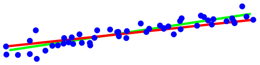
- N can be re-estimated using the current estimate for ε (if there is LO, then after LO)

  the quasi-posterior estimate for ε is the average over all samples generated so far

- for $\varepsilon \to 0$ we gain nothing over the standard MH-sampler stopping criterion

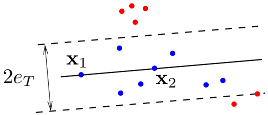
The Core Ideas in RANSAC [Fischler & Bolles 1981]

1. configuration $=$ s-tuple of inlier correspondences
   - the minimization will be over a discrete set of epipolar geometries proposable from 7-tuples
2. proposal distribution $q(\cdot)$ is given by the empirical distribution of data sample:
   a) select s-tuple from data independently $q(S \mid C_t) = q(S)$
      i) q uniform $q(S) = \binom{mn}{s}^{-1}$ MAPSAC ($p(S)$ includes the prior)
      ii) q dependent on descriptor similarity PROSAC (similar pairs are proposed more often)
   b) solve the minimal geometric problem $\mapsto$ geometry proposal e.g. $\mathbf{F}$ from $s = 7$



- pairs of points define line distribution from $p(\mathbf{n} \mid X)$ (left)
- random correspondence tuples drawn uniformly propose samples of $\mathbf{F}$ from a data-driven distribution $q(\mathbf{F} \mid E)$

3. independent sampling & looking for the best sample $\Rightarrow$ no need to filter proposals by a
4. standard RANSAC replaces probability maximization with consensus maximization



the e_T is the inlier/outlier threshold from (27)

5. stopping based on the probability of mode-hitting $\rightarrow$117

Example Matching Results for the 7-point Algorithm with RANSAC

input images | interest points (ca. 3600) | tentative corresp. (416) | matching (340)

- notice some wrong matches (they have wrong depth, even negative)
- they cannot be rejected without additional constraints or scene knowledge
- without local optimization the minimization is over a discrete set of epipolar geometries proposable from 7-tuples

Beyond RANSAC

By marginalization in (23) we have lost constraints on M (eg. uniqueness). One can choose a better model when not marginalizing:

$$p(M, \mathbf{F}, E, D) = \underbrace{p(E \mid M, \mathbf{F})}_{\text{geometric error}} \cdot \underbrace{p(D \mid M)}_{\text{similarity}} \cdot \underbrace{p(M)}_{\text{constraints}} \cdot \underbrace{p(\mathbf{F})}_{\text{prior}}$$

this is a global model: decisions on m_{ij} are no longer independent!

In the MH scheme

- one can work with full $p(M, \mathbf{F} \mid E, D)$, then $S = (M, \mathbf{F})$
  - explicit labeling m_{ij} can be done by, e.g. sampling from

    $$q(m_{ij} \mid \mathbf{F}) \sim \big((1 - P_0)\, p_1(e_{ij} \mid \mathbf{F}),\ P_0\, p_0(e_{ij} \mid \mathbf{F})\big)$$

    when $p(M)$ uniform then always accepted, $a = 1$ ⊛ derive
  - we can compute the posterior probability of each match $p(m_{ij})$ by histogramming m_{ij} over $\{S_i\}$
  - local optimization can then use explicit inliers and $p(m_{ij})$
- error can be estimated for elements of $\mathbf{F}$ from $\{S_i\}$ does not work in RANSAC!
- large error indicates problem degeneracy this is not directly available in RANSAC
- good conditioning is not a requirement we work with the entire distribution $p(\mathbf{F})$
- one can find the most probable number of epipolar geometries (homographies or other models) by reversible jump MCMC and model selection

if there are multiple models explaning data, RANSAC will return one of them randomly

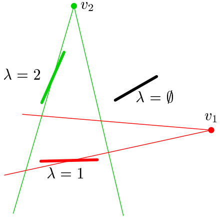
Example: MH Sampling for a More Complex Problem

Task: Find two vanishing points from line segments detected in input image.

video

Model

- principal point known, square pixel
- latent variables
  1. each line has a vanishing point label $\lambda_i \in \{\emptyset, 1, 2\}$, $\emptyset$ represents an outlier
- explicit variables
  1. two unknown vanishing points v_1, v_2
  2. 'mother line' parameters θ_L (they pass through their vanishing points)



$$\arg \min_{v_1, v_2, \Lambda, \theta_L} V(v_1, v_2, \Lambda, L \mid S)$$

simplifications

- vanishing points restricted to the set of all pairwise segment intersections
- mother lines fixed by segment centroid (then θ_L uniquely given by λ_i)

Part VI

3D Structure and Camera Motion

covered by

[1] [H&Z] Secs: 9.5.3, 10.1, 10.2, 10.3, 12.1, 12.2, 12.4, 12.5, 18.1

[2] Triggs, B. et al. Bundle Adjustment—A Modern Synthesis. In *Proc ICCV Workshop on Vision Algorithms.* Springer-Verlag. pp. 298–372, 1999.

additional references

D. Martinec and T. Pajdla. Robust Rotation and Translation Estimation in Multiview Reconstruction. In *Proc CVPR*, 2007

M. I. A. Lourakis and A. A. Argyros. SBA: A Software Package for Generic Sparse Bundle Adjustment. ACM Trans Math Software 36(1):1–30, 2009.

▶Constructing Cameras from the Fundamental Matrix

Given $\mathbf{F}$, construct some cameras $\mathbf{P}_1$, $\mathbf{P}_2$ such that $\mathbf{F}$ is their fundamental matrix.

Solution

See [H&Z, p. 256]

$$\mathbf{P}_1 = \begin{bmatrix} \mathbf{I} & \mathbf{0} \end{bmatrix}$$
$$\mathbf{P}_2 = \begin{bmatrix} [\underline{\mathbf{e}}_2]_\times \mathbf{F} + \underline{\mathbf{e}}_2 \, \underline{\mathbf{v}}^\top & \lambda \, \underline{\mathbf{e}}_2 \end{bmatrix}$$

where

- $\underline{\mathbf{v}}$ is any 3-vector, e.g. $\underline{\mathbf{v}} = \underline{\mathbf{e}}_1$ to make the camera finite
- $\lambda \neq 0$ is a scalar,
- $\underline{\mathbf{e}}_2 = \mathrm{null}(\mathbf{F}^\top)$, i.e. $\underline{\mathbf{e}}_2^\top \mathbf{F} = 0$

Proof

1. $\mathbf{S}$ is skew-symmetric iff $\mathbf{x}^\top \mathbf{S} \mathbf{x} = 0$ for all $\mathbf{x}$ — look-up the proof!
2. we have $\underline{\mathbf{x}} \simeq \mathbf{P} \underline{\mathbf{X}}$
3. a non-zero $\mathbf{F}$ is a f.m. iff $\mathbf{P}_2^\top \mathbf{F} \mathbf{P}_1$ is skew-symmetric
4. if $\mathbf{P}_1 = \begin{bmatrix} \mathbf{I} & \mathbf{0} \end{bmatrix}$ and $\mathbf{P}_2 = \begin{bmatrix} \mathbf{S}\mathbf{F} & \underline{\mathbf{e}}_2 \end{bmatrix}$ then $\mathbf{F}$ corresponds to $(\mathbf{P}_1, \mathbf{P}_2)$ by Step 3
5. we can write $\mathbf{S} = [\mathbf{s}]_\times$
6. a suitable choice is $\mathbf{s} = \underline{\mathbf{e}}_2$ — [Luong96]
7. for the full the class including $\underline{\mathbf{v}}$, see [H&Z, Sec. 9.5]

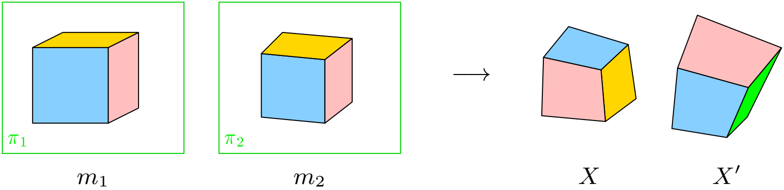
▶The Projective Reconstruction Theorem

Observation: Unless $\mathbf{P}_i$ are constrained, then for any number of cameras $i = 1, \ldots, k$

$$\underline{\mathbf{m}}_i \simeq \mathbf{P}_i \underline{\mathbf{X}} = \underbrace{\mathbf{P}_i \mathbf{H}^{-1}}_{\mathbf{P}'_i} \underbrace{\mathbf{H}\underline{\mathbf{X}}}_{\underline{\mathbf{X}}'} = \mathbf{P}'_i \underline{\mathbf{X}}'$$

- when $\mathbf{P}_i$ and $\underline{\mathbf{X}}$ are both determined from correspondences (including calibrations $\mathbf{K}_i$), they are given up to a common 3D <u>homography</u> $\mathbf{H}$

  (translation, rotation, scale, shear, pure perspectivity)

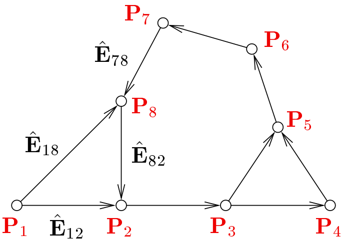


- when cameras are internally calibrated ($\mathbf{K}_i$ known) then $\mathbf{H}$ is restricted to a <u>similarity</u> since it must preserve the calibrations $\mathbf{K}_i$ [H&Z, Secs. 10.2, 10.3], [Longuet-Higgins 1981] (translation, rotation, scale)

►Reconstructing Camera Systems

Problem: Given a set of p decomposed pairwise essential matrices $\hat{\mathbf{E}}_{ij} = [\hat{\mathbf{t}}_{ij}]_\times \hat{\mathbf{R}}_{ij}$ and calibration matrices $\mathbf{K}_i$ reconstruct the camera system $\mathbf{P}_i$, $i = 1, \ldots, k$

→77 and →139 on representing $\mathbf{E}$



We construct calibrated camera pairs $\hat{\mathbf{P}}_{ij} \in \mathbb{R}^{6,4}$ →123

$$\hat{\mathbf{P}}_{ij} = \begin{bmatrix} \mathbf{K}_i^{-1}\hat{\mathbf{P}}_i \\ \mathbf{K}_j^{-1}\hat{\mathbf{P}}_j \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \hat{\mathbf{R}}_{ij} & \hat{\mathbf{t}}_{ij} \end{bmatrix} \in \mathbb{R}^{6,4}$$

- singletons i, j correspond to graph nodes — k nodes
- pairs ij correspond to graph edges — p edges

$\hat{\mathbf{P}}_{ij}$ are in different coordinate systems but these are related by similarities $\hat{\mathbf{P}}_{ij}\mathbf{H}_{ij} = \mathbf{P}_{ij}$

$$\underbrace{\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \hat{\mathbf{R}}_{ij} & \hat{\mathbf{t}}_{ij} \end{bmatrix}}_{\mathbb{R}^{6,4}} \underbrace{\begin{bmatrix} \mathbf{R}_{ij} & \mathbf{t}_{ij} \\ \mathbf{0}^\top & s_{ij} \end{bmatrix}}_{\mathbf{H}_{ij} \in \mathbb{R}^{4,4}} \overset{!}{=} \underbrace{\begin{bmatrix} \mathbf{R}_i & \mathbf{t}_i \\ \mathbf{R}_j & \mathbf{t}_j \end{bmatrix}}_{\mathbb{R}^{6,4}} \tag{28}$$

- (28) is a linear system of $24p$ eqs. in $7p + 6k$ unknowns — $7p \sim (\mathbf{t}_{ij}, \mathbf{R}_{ij}, s_{ij})$, $6k \sim (\mathbf{R}_i, \mathbf{t}_i)$
- each $\mathbf{P}_i$ appears on the right side as many times as is the degree of node $\mathbf{P}_i$ — eg. P_5 3-times

►cont'd

Eq. (28) implies
$$\begin{bmatrix}\mathbf{R}_{ij}\\ \hat{\mathbf{R}}_{ij}\mathbf{R}_{ij}\end{bmatrix}=\begin{bmatrix}\mathbf{R}_i\\ \mathbf{R}_j\end{bmatrix}\qquad\begin{bmatrix}\mathbf{t}_{ij}\\ \hat{\mathbf{R}}_{ij}\mathbf{t}_{ij}+s_{ij}\hat{\mathbf{t}}_{ij}\end{bmatrix}=\begin{bmatrix}\mathbf{t}_i\\ \mathbf{t}_j\end{bmatrix}$$

- $\mathbf{R}_{ij}$ and $\mathbf{t}_{ij}$ can be eliminated:

$$\hat{\mathbf{R}}_{ij}\mathbf{R}_i=\mathbf{R}_j,\qquad \hat{\mathbf{R}}_{ij}\mathbf{t}_i+s_{ij}\hat{\mathbf{t}}_{ij}=\mathbf{t}_j,\qquad s_{ij}>0 \tag{29}$$

- note transformations that do not change these equations — assuming no error in $\hat{\mathbf{R}}_{ij}$

  1. $\mathbf{R}_i\mapsto\mathbf{R}_i\mathbf{R}$, 2. $\mathbf{t}_i\mapsto\sigma\,\mathbf{t}_i$ and $s_{ij}\mapsto\sigma s_{ij}$, 3. $\mathbf{t}_i\mapsto\mathbf{t}_i+\mathbf{R}_i\mathbf{t}$

- the global frame is fixed, e.g. by selecting

$$\mathbf{R}_1=\mathbf{I},\qquad\sum_{i=1}^{k}\mathbf{t}_i=\mathbf{0},\qquad\frac{1}{p}\sum_{i,j}s_{ij}=1 \tag{30}$$

- rotation equations are decoupled from translation equations
- in principle, s_{ij} could correct the sign of $\hat{\mathbf{t}}_{ij}$ from essential matrix decomposition →77
  but $\mathbf{R}_i$ cannot correct the α sign in $\hat{\mathbf{R}}_{ij}$
  ⇒ therefore make sure all points are in front of cameras and constrain $s_{ij}>0$; →79

\+ pairwise correspondences are sufficient

− suitable for well-located cameras only (dome-like configurations)

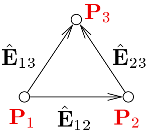
otherwise intractable or numerically unstable

Task: Solve $\hat{\mathbf{R}}_{ij}\mathbf{R}_i = \mathbf{R}_j$, $i, j \in V$, $(i,j) \in E$ where $\mathbf{R}$ are a 3×3 rotation matrix each.
Per columns $c = 1, 2, 3$ of $\mathbf{R}_j$:

$$\hat{\mathbf{R}}_{ij}\mathbf{r}_i^c - \mathbf{r}_j^c = \mathbf{0}, \qquad \text{for all } i,\ j \tag{31}$$

- fix c and denote $\mathbf{r}^c = \left[\mathbf{r}_1^c, \mathbf{r}_2^c, \dots, \mathbf{r}_k^c\right]^\top$ c-th columns of all rotation matrices stacked; $\mathbf{r}^c \in \mathbb{R}^{3k}$
- then (31) becomes $\mathbf{D}\,\mathbf{r}^c = \mathbf{0}$ $\qquad \mathbf{D} \in \mathbb{R}^{3p,3k}$
- $3p$ equations for $3k$ unknowns $\to p \geq k$ $\qquad$ in a 1-connected graph we have to fix $\mathbf{r}_1^c = [1, 0, 0]$

Ex: $(k = p = 3)$



$$\rightarrow \quad \begin{aligned} \hat{\mathbf{R}}_{12}\mathbf{r}_1^c - \mathbf{r}_2^c &= \mathbf{0} \\ \hat{\mathbf{R}}_{23}\mathbf{r}_2^c - \mathbf{r}_3^c &= \mathbf{0} \\ \hat{\mathbf{R}}_{13}\mathbf{r}_1^c - \mathbf{r}_3^c &= \mathbf{0} \end{aligned} \quad \rightarrow \quad \mathbf{D}\,\mathbf{r}^c = \begin{bmatrix} \hat{\mathbf{R}}_{12} & -\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{R}}_{23} & -\mathbf{I} \\ \hat{\mathbf{R}}_{13} & \mathbf{0} & -\mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{r}_1^c \\ \mathbf{r}_2^c \\ \mathbf{r}_3^c \end{bmatrix} = \mathbf{0}$$

- must hold for any c

Idea: $\qquad$ [Martinec & Pajdla CVPR 2007]

1. find the space of all $\mathbf{r}^c \in \mathbb{R}^{3k}$ that solve (31) $\quad \mathbf{D}$ is sparse, use `[V,E] = eigs(D'*D,3,0);` (Matlab)
2. choose 3 unit orthogonal vectors in this space $\qquad$ 3 smallest eigenvectors
3. find closest rotation matrices per cam. using SVD $\qquad$ because $\|\mathbf{r}^c\| = 1$ is necessary but insufficient
$\mathbf{R}_i^* = \mathbf{U}\mathbf{V}^\top$, where $\mathbf{R}_i = \mathbf{U}\mathbf{D}\mathbf{V}^\top$

- global world rotation is arbitrary

Finding The Translation Component in Eq. (29)

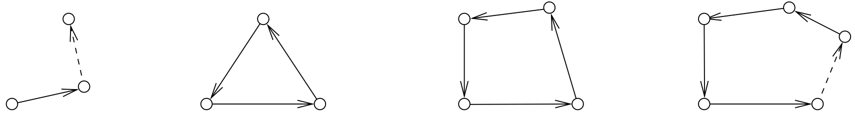
From (29) and (30): $d \le 3$ – rank of camera center set, p – #pairs, k – #cameras

$$\hat{\mathbf{R}}_{ij}\mathbf{t}_i + s_{ij}\hat{\mathbf{t}}_{ij} - \mathbf{t}_j = \mathbf{0}, \qquad \sum_{i=1}^{k} \mathbf{t}_i = \mathbf{0}, \qquad \sum_{i,j} s_{ij} = p, \qquad s_{ij} > 0, \qquad \mathbf{t}_i \in \mathbb{R}^d$$

- in rank d: $d \cdot p + d + 1$ equations for $d \cdot k + p$ unknowns $\to p \ge \frac{d(k-1)-1}{d-1}$

Ex: Chains and circuits construction from sticks of known orientation and unknown length?

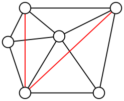
$p = k - 1$ | $k = p = 3$ | $k = p = 4$ | $k = p > 4$



$k \le 2$ for any d | $3 \ge d \ge 2$: non-collinear ok | $3 \ge d \ge 3$: non-planar ok | $3 \ge d \ge k - 1$: impossible

- equations insufficient for chains, trees, or when $d = 1$ collinear cameras
- 3-connectivity implies sufficient equations for $d = 3$ cams. in general pos. in 3D
  - s-connected graph has $p \ge \lceil \frac{sk}{2} \rceil$ edges for $s \ge 2$, hence $p \ge \lceil \frac{3k}{2} \rceil \ge \frac{3k}{2} - 2$
- 4-connectivity implies sufficient eqns. for any k when $d = 2$ coplanar cams
  - since $p \ge \lceil 2k \rceil \ge 2k - 3$
  - maximal planar tringulated graphs have $p = 3k - 6$ and give a solution for $k \ge 3$

maximal planar triangulated graph example:



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Linear equations in (29) and (30) can be rewritten to

$$\mathbf{D}\mathbf{t} = \mathbf{0}, \qquad \mathbf{t} = \left[\mathbf{t}_1^\top, \mathbf{t}_2^\top, \ldots, \mathbf{t}_k^\top, s_{12}, \ldots, s_{ij}, \ldots\right]^\top$$

for $d = 3$: $\mathbf{t} \in \mathbb{R}^{3k+p}$, $\quad \mathbf{D} \in \mathbb{R}^{3p,3k+p}$ is sparse

$$\mathbf{t}^* = \arg\min_{\mathbf{t},\, s_{ij}>0} \mathbf{t}^\top \mathbf{D}^\top \mathbf{D}\, \mathbf{t}$$

- this is a quadratic programming problem (mind the constraints!)

```
z = zeros(3*k+p,1);
t = quadprog(D.'*D, z, diag([zeros(3*k,1); -ones(p,1)]), z);
```

- but check the rank first!

▶Solving Eq. (29) by Stepwise Gluing

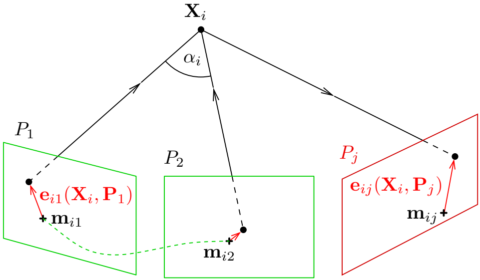
Given: Calibration matrices $\mathbf{K}_j$ and tentative correspondences per camera triples.

Initialization

1. initialize camera cluster $\mathcal{C}$ with P_1, P_2,
2. find essential matrix $\mathbf{E}_{12}$ and matches M_{12} by the 5-point algorithm →83
3. construct camera pair

$$\mathbf{P}_1 = \mathbf{K}_1 \begin{bmatrix} \mathbf{I} & \mathbf{0} \end{bmatrix}, \ \mathbf{P}_2 = \mathbf{K}_2 \begin{bmatrix} \mathbf{R} & \mathbf{t} \end{bmatrix}$$

4. compute 3D reconstruction $\{X_i\}$ per match from M_{12} →100
5. initialize point cloud $\mathcal{X}$ with $\{X_i\}$ satisfying chirality constraint $z_i > 0$ and apical angle constraint $|\alpha_i| > \alpha_T$



Attaching camera $P_j \notin \mathcal{C}$

1. select points $\mathcal{X}_j$ from $\mathcal{X}$ that have matches to P_j
2. estimate $\mathbf{P}_j$ using $\mathcal{X}_j$, RANSAC with the 3-pt alg. (P3P), projection errors $\mathbf{e}_{ij}$ in $\mathcal{X}_j$ →66
3. reconstruct 3D points from all tentative matches from P_j to all P_l, $l \neq k$ that are not in $\mathcal{X}$
4. filter them by the chirality and apical angle constraints and add them to $\mathcal{X}$
5. add P_j to $\mathcal{C}$
6. perform bundle adjustment on $\mathcal{X}$ and $\mathcal{C}$ coming next

▶Bundle Adjustment

Given:

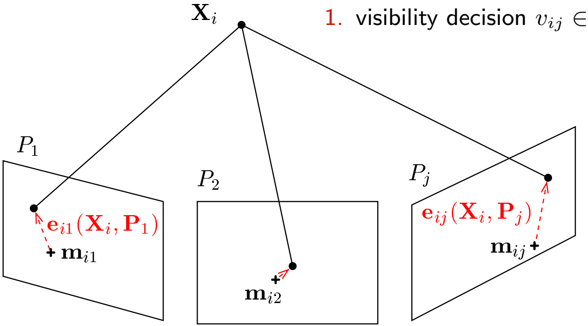
1. set of 3D points $\{\mathbf{X}_i\}_{i=1}^{p}$
2. set of cameras $\{\mathbf{P}_j\}_{j=1}^{c}$
3. fixed tentative projections $\mathbf{m}_{ij}$

Required:

1. corrected 3D points $\{\mathbf{X}'_i\}_{i=1}^{p}$
2. corrected cameras $\{\mathbf{P}'_j\}_{j=1}^{c}$

Latent:

1. visibility decision $v_{ij} \in \{0, 1\}$ per $\mathbf{m}_{ij}$



- for simplicity, $\mathbf{X}$, $\mathbf{m}$ are considered Cartesian (not homogeneous)
- we have projection error $\mathbf{e}_{ij}(\mathbf{X}_i, \mathbf{P}_j) = \mathbf{x}_i - \mathbf{m}_i$ per image feature, where $\underline{\mathbf{x}}_i = \mathbf{P}_j \underline{\mathbf{X}}_i$
- for simplicity, we will work with scalar error $e_{ij} = \|\mathbf{e}_{ij}\|$

Robust Objective Function for Bundle Adjustment

Likelihood is constructed by marginalization, as in Robust Matching Model →107

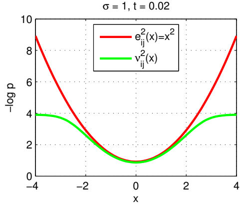
$$p(\{\mathbf{e}\} \mid \{\mathbf{P}, \mathbf{X}\}) = \prod_{\text{pts}:i=1}^{p} \prod_{\text{cams}:j=1}^{c} \Big((1-P_0) p_1(e_{ij} \mid \mathbf{X}_i, \mathbf{P}_j) + P_0\, p_0(e_{ij} \mid \mathbf{X}_i, \mathbf{P}_j) \Big)$$

marginalized negative log-likelihood is (→108)

$$-\log p(\{\mathbf{e}\} \mid \{\mathbf{P}, \mathbf{X}\}) = \sum_i \sum_j \underbrace{-\log\Big(e^{-\frac{e_{ij}^2(\mathbf{X}_i,\mathbf{P}_j)}{2\sigma_1^2}} + t\Big)}_{\rho(e_{ij}^2(\mathbf{X}_i,\mathbf{P}_j)) = \nu_{ij}^2(\mathbf{X}_i,\mathbf{P}_j)} \stackrel{\text{def}}{=} \sum_i \sum_j \nu_{ij}^2(\mathbf{X}_i, \mathbf{P}_j)$$

- e_{ij} is the projection error (not Sampson error)
- ν_{ij} is a 'robust' error fcn.; it is non-robust ($\nu_{ij} = e_{ij}$) when $t = 0$
- $\rho(\cdot)$ is a 'robustification function' we often find in M-estimation
- the $\mathbf{L}_{ij}$ in Levenberg-Marquardt changes to vector

$$(\mathbf{L}_{ij})_l = \frac{\partial \nu_{ij}}{\partial \theta_l} = \underbrace{\frac{1}{1 + t\, e^{e_{ij}^2(\theta)/(2\sigma_1^2)}}}_{\text{small for big } e_{ij}} \cdot \frac{1}{\nu_{ij}(\theta)} \cdot \frac{1}{4\sigma_1^2} \cdot \frac{\partial e_{ij}^2(\theta)}{\partial \theta_l} \qquad (32)$$



but the LM method stays the same as on →101–102

- outliers: almost no impact on $\mathbf{d}_s$ in normal equations because the red term in (32) scales contributions to both sums down for the particular ij

$$-\sum_{i,j}^{k} \mathbf{L}_{ij}^\top \nu_{ij}(\theta^s) = \Big(\sum_{i,j} \mathbf{L}_{ij}^\top \mathbf{L}_{ij}\Big) \mathbf{d}_s$$

▶Sparsity in Bundle Adjustment

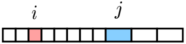
We have $q = 3p + 11k$ parameters: $\theta = (\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_p; \mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_k)$ points, cameras

We will use a running index $r = 1, \ldots, z$, $z = p \cdot k$. Then each r corresponds to some i, j

$$\theta^* = \arg\min_{\theta} \sum_{r=1}^{z} \nu_r^2(\theta), \quad \boldsymbol{\theta}^{s+1} := \boldsymbol{\theta}^s + \mathbf{d}_s, \quad -\sum_{r=1}^{z} \mathbf{L}_r^\top \nu_r(\theta^s) = \left(\sum_{r=1}^{z} \mathbf{L}_r^\top \mathbf{L}_r + \lambda \,\mathrm{diag}\, \mathbf{L}_r^\top \mathbf{L}_r \right) \mathbf{d}_s$$

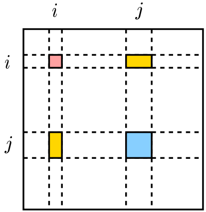
The block form of $\mathbf{L}_r$ in Levenberg-Marquardt (→101) is zero except in columns i and j:
r-th error term is $\nu_r^2 = \rho(e_{ij}^2(\mathbf{X}_i, \mathbf{P}_j))$

$\mathbf{L}_r =$



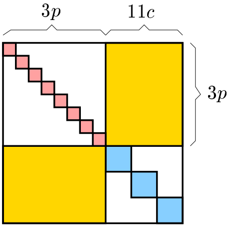
$r = (i, j)$ blocks:
- ■: $\mathbf{X}_i, 1 \times 3$
- ■: $\mathbf{P}_j, 1 \times 11$

$\mathbf{L}_r^\top \mathbf{L}_r =$



blocks:
- ■: $\mathbf{X}_i - \mathbf{X}_i$, 3×3
- ■: $\mathbf{X}_i - \mathbf{P}_j$, 3×11
- ■: $\mathbf{P}_j - \mathbf{P}_j$, 11×11

$\sum_{r=1}^{z} \mathbf{L}_r^\top \mathbf{L}_r =$



- "points first, then cameras" scheme
- standard bundle adjustment eliminates points and solves cameras, then back-substitutes

▶Choleski Decomposition for B. A.

The most expensive computation in B. A. is solving the normal eqs:

$$\text{find } \mathbf{d}_s \text{ such that} \quad -\sum_{r=1}^{z} \mathbf{L}_r^\top \nu_r(\theta^s) = \Big(\sum_{r=1}^{z} \mathbf{L}_r^\top \mathbf{L}_r + \lambda \operatorname{diag} \mathbf{L}_r^\top \mathbf{L}_r\Big) \mathbf{d}_s$$

This is a linear set of equations $\mathbf{A}\mathbf{x} = \mathbf{b}$, where

- $\mathbf{A}$ is very large — approx. $3 \cdot 10^4 \times 3 \cdot 10^4$ for a small problem of 10000 points and 5 cameras
- $\mathbf{A}$ is sparse and symmetric, $\mathbf{A}^{-1}$ is dense — direct matrix inversion is prohibitive

Choleski: Every symmetric positive definite matrix $\mathbf{A}$ can be decomposed to $\mathbf{A} = \mathbf{L}\mathbf{L}^\top$, where $\mathbf{L}$ is lower triangular. If $\mathbf{A}$ is sparse then $\mathbf{L}$ is sparse, too.

1. decompose $\mathbf{A} = \mathbf{L}\mathbf{L}^\top$ — transforms the problem to solving $\mathbf{L}\underbrace{\mathbf{L}^\top\mathbf{x}}_{\mathbf{c}} = \mathbf{b}$
2. solve for $\mathbf{x}$ in two passes:

$$\mathbf{L}\,\mathbf{c} = \mathbf{b} \qquad \mathbf{c}_i := \mathbf{L}_{ii}^{-1}\Big(\mathbf{b}_i - \sum_{j<i} \mathbf{L}_{ij}\mathbf{c}_j\Big)$$

forward substitution, $i = 1, \ldots, q$

$$\mathbf{L}^\top\mathbf{x} = \mathbf{c} \qquad \mathbf{x}_i := \mathbf{L}_{ii}^{-1}\Big(\mathbf{c}_i - \sum_{j>i} \mathbf{L}_{ji}\mathbf{x}_j\Big)$$

back-substitution

- Choleski decomposition is fast (does not touch zero blocks)

  non-zero elements are $9p + 121k + 66pk \approx 3.4 \cdot 10^6$; ca. $250\times$ fewer than all elements
- it can be computed on single elements or on entire blocks
- use profile Choleski for sparse $\mathbf{A}$ and diagonal pivoting for semi-definite $\mathbf{A}$ [Triggs et al. 1999]
- λ controls the definiteness

Profile Choleski Decomposition is Simple

```
function L = pchol(A)
%
% PCHOL profile Choleski factorization,
%    L = PCHOL(A) returns lower-triangular sparse L such that A = L*L'
%    for sparse square symmetric positive definite matrix A,
%    especially useful for arrowhead sparse matrices.

 [p,q] = size(A);
 if p ~= q, error 'Matrix must be square'; end

 L = sparse(q,q);
 F = ones(q,1);
 for i=1:q
  F(i) = find(A(i,:),1); % 1st non-zero on row i; we are building F gradually
  for j = F(i):i-1
   k = max(F(i),F(j));
   a = A(i,j) - L(i,k:(j-1))*L(j,k:(j-1))';
   L(i,j) = a/L(j,j);
  end
  a = A(i,i) - sum(full(L(i,F(i):(i-1))).^2);
  if a < 0, error 'Matrix must be positive definite'; end
  L(i,i) = sqrt(a);
 end
end
```

▶Gauge Freedom

1. The external frame is not fixed: See Projective Reconstruction Theorem →124

$$\underline{\mathbf{m}}_i \simeq \mathbf{P}_j \underline{\mathbf{X}}_i = \mathbf{P}_j \mathbf{H}^{-1} \mathbf{H} \underline{\mathbf{X}}_i = \mathbf{P}'_j \underline{\mathbf{X}}'_i$$

2. Some representations are not minimal, e.g.
   - $\mathbf{P}$ is 12 numbers for 11 parameters
   - we may represent $\mathbf{P}$ in decomposed form $\mathbf{K}$, $\mathbf{R}$, $\mathbf{t}$
   - but $\mathbf{R}$ is 9 numbers representing the 3 parameters of rotation

As a result

- there is no unique solution
- matrix $\sum_r \mathbf{L}_r^\top \mathbf{L}_r$ is singular

Solutions

1. fixing the external frame (e.g. a selected camera frame) explicitly or by constraints

2a. either imposing constraints on projective entities
   - cameras, e.g. $\mathbf{P}_{3,4} = 1$ — this excludes affine cameras
   - points, e.g. $\|\underline{\mathbf{X}}_i\|^2 = 1$ — this way we can represent points at infinity

2b. or using minimal representations
   - points in their Euclidean representation $\mathbf{X}_i$ — but finite points may be an unrealistic model
   - rotation matrix can be represented by Cayley transform see next

▶Implementing Simple Constraints

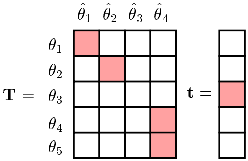
What for?

1. fixing external frame as in $\theta_i = \mathbf{t}_i$ ‘trivial gauge’
2. representing additional knowledge as in $\theta_i = \theta_j$ e.g. cameras share calibration matrix $\mathbf{K}$

Introduce reduced parameters $\hat{\theta}$ and replication matrix $\mathbf{T}$:

$$\theta = \mathbf{T}\,\hat{\theta} + \mathbf{t}, \quad \mathbf{T} \in \mathbb{R}^{p,\hat{p}}, \quad \hat{p} \le p$$

then $\mathbf{L}_r$ in LM changes to $\mathbf{L}_r\,\mathbf{T}$ and everything else stays the same →101



these $\mathbf{T}$, $\mathbf{t}$ represent	
$\theta_1 = \hat{\theta}_1$	no change
$\theta_2 = \hat{\theta}_2$	no change
$\theta_3 = t_3$	constancy
$\theta_4 = \theta_5 = \hat{\theta}_4$	equality

- $\mathbf{T}$ deletes columns of $\mathbf{L}_r$ that correspond to fixed parameters it reduces the problem size
- consistent initialisation: $\theta^0 = \mathbf{T}\,\hat{\theta}^0 + \mathbf{t}$ or filter the init by pseudoinverse $\theta^0 \mapsto \mathbf{T}^\dagger\theta^0$
- no need for computing derivatives for θ_j corresponding to all-zero rows of $\mathbf{T}$ fixed θ

- constraining projective entities →138–139
- more complex constraints tend to make normal equations dense
- implementing constraints is safer than explicit renaming of the parameters, gives a flexibility to experiment
- other methods are much more involved, see [Triggs et al. 1999]
- **BA resource:** `http://www.ics.forth.gr/~lourakis/sba/` [Lourakis 2009]

▶Minimal Representations for Rotation

- $\mathbf{o}$ – rotation axis, $\|\mathbf{o}\| = 1$, φ – rotation angle
- **wanted**: simple mapping to/from rotation matrices

1. Rodrigues' representation

$$\mathbf{R} = \mathbf{I} + \sin\varphi[\mathbf{o}]_\times + (1 - \cos\varphi)[\mathbf{o}]_\times^2$$

$$\sin\varphi\,[\mathbf{o}]_\times = \frac{1}{2}(\mathbf{R} - \mathbf{R}^\top), \quad \cos\varphi = \frac{1}{2}(\operatorname{tr}\mathbf{R} - 1)$$

   - hiding φ in the vector $\mathbf{o}$ as in $[\sin\varphi\,\mathbf{o}]_\times$ is not so easy
   - Cayley tried:

2. Cayley's representation; let $\mathbf{a} = \mathbf{o}\tan\frac{\varphi}{2}$, then

$$\mathbf{R} = (\mathbf{I} + [\mathbf{a}]_\times)(\mathbf{I} - [\mathbf{a}]_\times)^{-1}$$

$$[\mathbf{a}]_\times = (\mathbf{R} + \mathbf{I})^{-1}(\mathbf{R} - \mathbf{I})$$

$$\mathbf{a}_1 \circ \mathbf{a}_2 = \frac{\mathbf{a}_1 + \mathbf{a}_2 - \mathbf{a}_1 \times \mathbf{a}_2}{1 - \mathbf{a}_1^\top \mathbf{a}_2}$$

   composition of rotations $\mathbf{R} = \mathbf{R}_1\mathbf{R}_2$

   - no trigonometric functions
   - cannot represent rotation by 180°
   - explicit composition formula

3. exponential map $\mathbf{R} = \exp\,[\varphi\,\mathbf{o}]_\times$, inverse by Rodrigues' formula

Minimal Representations for Other Entities

1. with the help of rotation we can minimally represent
   - fundamental matrix

$$\mathbf{F} = \mathbf{U}\mathbf{D}\mathbf{V}^\top, \quad \mathbf{D} = \operatorname{diag}(1, d^2, 0), \quad \mathbf{U}, \mathbf{V} \text{ are rotations}, \quad 3 + 1 + 3 = 7 \text{ DOF}$$

   - essential matrix

$$\mathbf{E} = [-\mathbf{t}]_\times \mathbf{R}, \quad \mathbf{R} \text{ is rotation}, \quad \|\mathbf{t}\| = 1, \qquad 3 + 2 = 5 \text{ DOF}$$

   - camera

$$\mathbf{P} = \mathbf{K}\begin{bmatrix}\mathbf{R} & \mathbf{t}\end{bmatrix}, \qquad 5 + 3 + 3 = 11 \text{ DOF}$$

2. homography can be represented via exponential map

$$\exp \mathbf{A} = \sum_{k=0}^{\infty} \frac{1}{k!} \mathbf{A}^k \qquad \text{note: } \mathbf{A}^0 = \mathbf{I}$$

some properties

$$\exp \mathbf{0} = \mathbf{I}, \quad \exp(-\mathbf{A}) = \left(\exp \mathbf{A}\right)^{-1}, \quad \exp(\mathbf{A} + \mathbf{B}) \neq \exp(\mathbf{A}) \exp(\mathbf{B})$$

$\exp(\mathbf{A}^\top) = (\exp \mathbf{A})^\top$ hence if $\mathbf{A}$ skew symmetric then $\exp \mathbf{A}$ orthogonal

$$\left(\exp(\mathbf{A})\right)^\top = \exp(\mathbf{A}^\top) = \exp(-\mathbf{A}) = \left(\exp(\mathbf{A})\right)^{-1}$$

$\det \exp \mathbf{A} = \exp(\operatorname{tr} \mathbf{A})$ … a key to homography representation:

$$\mathbf{H} = \exp \mathbf{Z} \text{ such that } \operatorname{tr} \mathbf{Z} = 0, \text{ eg. } \mathbf{Z} = \begin{bmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & -(z_{11} + z_{22}) \end{bmatrix}, \quad 8 \text{ DOF}$$

Part VII

Stereovision

mostly covered by

Šára, R. How To Teach Stereoscopic Vision. Proc. ELMAR 2010 referenced as [SP]

additional references

C. Geyer and K. Daniilidis. Conformal rectification of omnidirectional stereo pairs. In *Proc Computer Vision and Pattern Recognition Workshop*, p. 73, 2003.

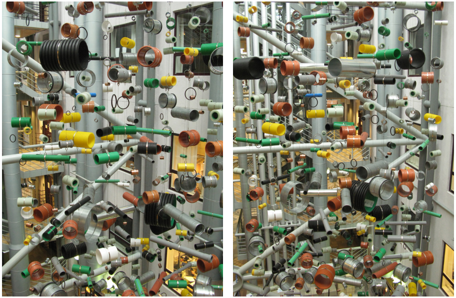
J. Gluckman and S. K. Nayar. Rectifying transformations that minimize resampling effects. In *Proc IEEE CS Conf on Computer Vision and Pattern Recognition*, vol. 1:111–117. 2001.

M. Pollefeys, R. Koch, and L. V. Gool. A simple and efficient rectification method for general motion. In *Proc Int Conf on Computer Vision*, vol. 1:496–501, 1999.

What Are The Relative Distances?

- monocular vision already gives a rough 3D sketch because we understand the scene

What Are The Relative Distances?

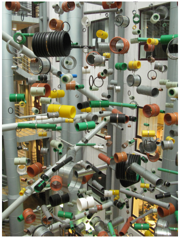


Centrum för teknikstudier at Malmö Högskola, Sweden

The Vyšehrad Fortress, Prague

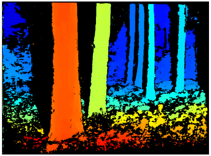
- left: we have no help from image interpretation
- right: ambiguous interpretation due to a combination of lack of texture and occlusion

▶How Difficult Is Stereo?

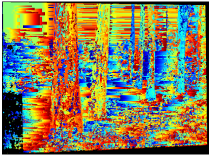


- when we do not recognize the scene and cannot use high-level constraints the problem seems difficult (right, less so in the center)
- most stereo matching algorithms do not require scene understanding prior to matching
- the success of a model-free stereo matching algorithm is unlikely:

left image



a good disparity map



disparity map from WTA

WTA Matching:

for every left-image pixel
find the most similar right-image pixel along the corresponding epipolar line [Marroquin 83]

A Summary of Our Observations and an Outlook

1. simple matching algorithms do not work
2. in sufficiently complex scenes stereopsis requires image interpretation

   or another-modality measurement

we have a tradeoff: model strength $\leftrightarrow$ universality

Outlook:

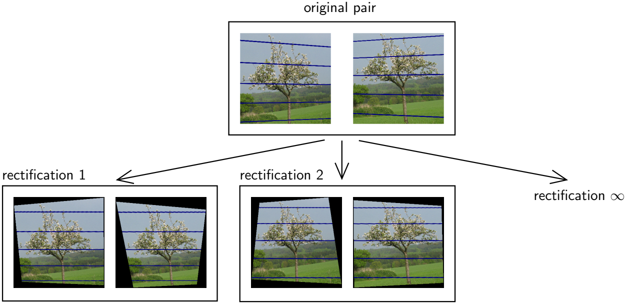
1. represent the occlusion constraint: correspondences are not independent due to occlusions
   - epipolar rectification
   - disparity
   - uniqueness as an occlusion constraint
2. represent piecewise continuity the weakest of interpretations; piecewise: object boundaries
   - ordering as a weak continuity model
3. use a consistent framework
   - looking for the most probable solution (MAP)

▶Linear Epipolar Rectification for Easier Correspondence Search

Problem: Given fundamental matrix $\mathbf{F}$ or camera matrices $\mathbf{P}_1$, $\mathbf{P}_2$, transform images by a pair of homographies so that epipolar lines become horizontal with the same row coordinate. The result is a standard stereo pair.

Procedure:

1. find a pair of rectification homographies $\mathbf{H}_1$ and $\mathbf{H}_2$.
2. warp images using $\mathbf{H}_1$ and $\mathbf{H}_2$ and modify fundamental matrix $\mathbf{F} \mapsto \mathbf{H}_2^{-\top} \mathbf{F} \mathbf{H}_1^{-1}$ or cameras $\mathbf{P}_1 \mapsto \mathbf{H}_1 \mathbf{P}_1$, $\mathbf{P}_2 \mapsto \mathbf{H}_2 \mathbf{P}_2$.



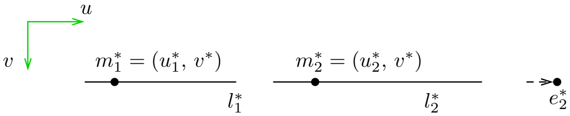
- binocular rectification: there is a 9-parameter family of rectification homographies, see next
- trinocular rectification: has 9 or 6 free parameters (depending on additional constrains)
- in general, linear rectification is not possible for more than three cameras

▶Rectification Homographies

Assumption: Cameras $(\mathbf{P}_1, \mathbf{P}_2)$ are rectified by a homography pair $(\mathbf{H}_1, \mathbf{H}_2)$:

$$\mathbf{P}_i^* = \mathbf{H}_i \mathbf{P}_i = \mathbf{H}_i \mathbf{K}_i \mathbf{R}_i \begin{bmatrix} \mathbf{I} & -\mathbf{C}_i \end{bmatrix}, \quad i = 1, 2$$

rectified entities: $\mathbf{F}^*$, $\underline{\mathbf{l}}_2^*$, $\underline{\mathbf{l}}_1^*$, etc:



corresponding epipolar lines must be:

1. parallel to image rows $\Rightarrow$ epipoles become $e_1^* = e_2^* = (1, 0, 0)$
2. equivalent $l_2^* = l_1^*$ $\Rightarrow$ $\underline{\mathbf{l}}_2^* \simeq \underline{\mathbf{l}}_1^* \simeq \underline{\mathbf{e}}_1^* \times \underline{\mathbf{m}}_1 = [\underline{\mathbf{e}}_1^*]_\times \underline{\mathbf{m}}_1 = \mathbf{F}^* \underline{\mathbf{m}}_1$

- both conditions together give the rectified fundamental matrix

$$\mathbf{F}^* \simeq \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

- the rectified location difference $d = u_1^* - u_2^*$ is called disparity

A two-step rectification procedure

1. find some pair of primitive rectification homographies $\hat{\mathbf{H}}_1$, $\hat{\mathbf{H}}_2$
2. upgrade to a pair of optimal rectification homographies while preserving $\mathbf{F}^*$

▶Geometric Interpretation of Linear Rectification

What pair of physical cameras is compatible with $\mathbf{F}^*$?

- we know that $\mathbf{F} = (\mathbf{Q}_1 \mathbf{Q}_2^{-1})^\top [\underline{\mathbf{e}}_1]_\times$ →75
- we choose $\mathbf{Q}_1^* = \mathbf{K}_1^*$, $\mathbf{Q}_2^* = \mathbf{K}_2^* \mathbf{R}^*$; then

$$(\mathbf{Q}_1^* \mathbf{Q}_2^{*\,-1})^\top [\underline{\mathbf{e}}_1^*]_\times = (\mathbf{K}_1^* \mathbf{R}^{*\top} \mathbf{K}_2^{*\,-1})^\top \mathbf{F}^*$$

- we look for $\mathbf{R}^*$, $\mathbf{K}_1^*$, $\mathbf{K}_2^*$ compatible with

$$(\mathbf{K}_1^* \mathbf{R}^{*\top} \mathbf{K}_2^{*\,-1})^\top \mathbf{F}^* = \lambda \mathbf{F}^*, \qquad \mathbf{R}^* \mathbf{R}^{*\top} = \mathbf{I}, \qquad \mathbf{K}_1^*, \mathbf{K}_2^* \text{ upper triangular}$$

- we also want $\mathbf{b}^*$ from $\underline{\mathbf{e}}_1^* \simeq \mathbf{P}_1^* \underline{\mathbf{C}}_2^* = \mathbf{K}_1^* \mathbf{b}^*$ $\quad$ $\mathbf{b}^*$ in cam. 1 frame
- result:

$$\mathbf{R}^* = \mathbf{I}, \quad \mathbf{b}^* = \begin{bmatrix} b \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{K}_1^* = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ 0 & f & v_0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{K}_2^* = \begin{bmatrix} k_{21} & k_{22} & k_{23} \\ 0 & f & v_0 \\ 0 & 0 & 1 \end{bmatrix} \tag{33}$$

- rectified cameras are in canonical position with respect to each other

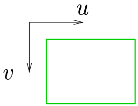
  not rotated, canonical baseline

- rectified calibration matrices can differ in the first row only
- when $\mathbf{K}_1^* = \mathbf{K}_2^*$ then the rectified pair is called the <u>standard stereo pair</u> and the homographies <u>standard rectification homographies</u>

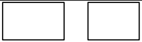
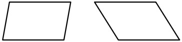
▶The Degrees of Freedom in Epipolar Rectification

Proposition 1 Homographies $\mathbf{A}_1$ and $\mathbf{A}_2$ are <u>rectification-preserving</u> if the images stay rectified, i.e. if ${\mathbf{A}_2}^{-\top}\,\mathbf{F}^*\,{\mathbf{A}_1}^{-1} \simeq \mathbf{F}^*$, which gives

$$\mathbf{A}_1 = \begin{bmatrix} l_1 & l_2 & l_3 \\ 0 & s_v & t_v \\ 0 & q & 1 \end{bmatrix}, \qquad \mathbf{A}_2 = \begin{bmatrix} r_1 & r_2 & r_3 \\ 0 & s_v & t_v \\ 0 & q & 1 \end{bmatrix},$$



where $s \neq 0,\ u_0,\ l_1,\ l_2 \neq 0,\ l_3,\ r_1,\ r_2 \neq 0,\ r_3,\ q$ are <u>9 free parameters.</u>

general	transformation		canonical	type
$l_1,\ r_1$	horizontal scales		$l_1 = r_1$	algebraic
$l_2,\ r_2$	horizontal shears		$l_2 = r_2$	algebraic
$l_3,\ r_3$	horizontal shifts		$l_3 = r_3$	algebraic
q	common special projective			geometric
s_v	common vertical scale			geometric
t_v	common vertical shift			algebraic
9 DoF			$9 - 3 = 6$ DoF	

- q is rotation about the baseline
- s_v changes the focal length

proof: find a rotation $\mathbf{G}$ that brings $\mathbf{K}$ to upper triangular form via RQ decomposition: $\mathbf{A}_1\mathbf{K}_1^* = \hat{\mathbf{K}}_1\mathbf{G}$ and $\mathbf{A}_2\mathbf{K}_2^* = \hat{\mathbf{K}}_2\mathbf{G}$

The Rectification Group

Corollary for Proposition 1 Let $\bar{\mathbf{H}}_1$ and $\bar{\mathbf{H}}_2$ be (primitive or other) rectification homographies. Then $\mathbf{H}_1 = \mathbf{A}_1 \bar{\mathbf{H}}_1, \quad \mathbf{H}_2 = \mathbf{A}_2 \bar{\mathbf{H}}_2$ are also rectification homographies.

Proposition 2 Pairs of rectification-preserving homographies $(\mathbf{A}_1, \mathbf{A}_2)$ form a group with group operation $(\mathbf{A}'_1, \mathbf{A}'_2) \circ (\mathbf{A}_1, \mathbf{A}_2) = (\mathbf{A}'_1 \mathbf{A}_1, \mathbf{A}'_2 \mathbf{A}_2)$.

Proof:

- closure by Proposition 1
- associativity by matrix multiplication
- identity belongs to the set
- inverse element belongs to the set by $\mathbf{A}_2^\top \mathbf{F}^* \mathbf{A}_1 \simeq \mathbf{F}^* \Leftrightarrow \mathbf{F}^* \simeq \mathbf{A}_2^{-\top} \mathbf{F}^* \mathbf{A}_1^{-1}$

▶Primitive Rectification

Goal: Given fundamental matrix $\mathbf{F}$, derive some simple rectification homographies $\mathbf{H}_1$, $\mathbf{H}_2$

1. Let the SVD of $\mathbf{F}$ be $\mathbf{UDV}^\top = \mathbf{F}$, where $\mathbf{D} = \mathrm{diag}(1, d^2, 0)$, $\quad 1 \geq d^2 > 0$
2. Write $\mathbf{D}$ as $\mathbf{D} = \mathbf{A}^\top \mathbf{F}^* \mathbf{B}$. For instance ($\mathbf{F}^*$ is given →146)

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -d & 0 \\ 1 & 0 & 0 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & d & 0 \end{bmatrix}$$

3. Then

$$\mathbf{F} = \mathbf{UDV}^\top = \underbrace{\mathbf{UA}^\top}_{\hat{\mathbf{H}}_2^\top} \mathbf{F}^* \underbrace{\mathbf{BV}^\top}_{\hat{\mathbf{H}}_1}$$

and the primitive rectification homographies are

$$\hat{\mathbf{H}}_2 = \mathbf{AU}^\top, \qquad \hat{\mathbf{H}}_1 = \mathbf{BV}^\top$$

⊛ P1; 1pt: derive some $\mathbf{A}$, $\mathbf{B}$ from the admissible class

- rectification homographies do exist →146
- there are other primitive rectification homographies, these suggested are just simple to obtain

►Primitive Rectification Suffices for Calibrated Cameras

Obs: calibrated cameras: $d = 1 \Rightarrow \hat{\mathbf{H}}_1,\ \hat{\mathbf{H}}_2$ are orthogonal

1. determine primitive rectification homographies $(\hat{\mathbf{H}}_1, \hat{\mathbf{H}}_2)$ from the <u>essential</u> matrix
2. choose a suitable common calibration matrix $\mathbf{K}$, e.g.

$$\mathbf{K} = \begin{bmatrix} f & 0 & u_0 \\ 0 & f & v_0 \\ 0 & 0 & 1 \end{bmatrix}, \quad f = \frac{1}{2}(f^1 + f^2), \quad u_0 = \frac{1}{2}(u_0^1 + u_0^2), \quad \text{etc.}$$

3. the final rectification homographies applied as $\mathbf{P}_i \mapsto \mathbf{H}_i\,\mathbf{P}_i$ are

$$\mathbf{H}_1 = \mathbf{K}\hat{\mathbf{H}}_1\mathbf{K}_1^{-1}, \quad \mathbf{H}_2 = \mathbf{K}\hat{\mathbf{H}}_2\mathbf{K}_2^{-1}$$

- we got a standard camera pair and non-negative disparity

let $\mathbf{K}_i^{-1}\mathbf{P}_i = \mathbf{R}_i\,\begin{bmatrix}\mathbf{I} & -\mathbf{C}_i\end{bmatrix}, \quad i = 1, 2$ note we started from $\mathbf{E}$, not $\mathbf{F}$

$$\mathbf{H}_1\mathbf{P}_1 = \mathbf{K}\hat{\mathbf{H}}_1\mathbf{K}_1^{-1}\mathbf{P}_1 = \mathbf{K}\,\underbrace{\mathbf{B}\mathbf{V}^\top\mathbf{R}_1}_{\mathbf{R}^*}\begin{bmatrix}\mathbf{I} & -\mathbf{C}_1\end{bmatrix} = \mathbf{K}\mathbf{R}^*\begin{bmatrix}\mathbf{I} & -\mathbf{C}_1\end{bmatrix}$$

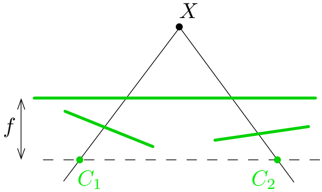
$$\mathbf{H}_2\mathbf{P}_2 = \mathbf{K}\hat{\mathbf{H}}_2\mathbf{K}_2^{-1}\mathbf{P}_2 = \mathbf{K}\,\underbrace{\mathbf{A}\mathbf{U}^\top\mathbf{R}_2}_{\mathbf{R}^*}\begin{bmatrix}\mathbf{I} & -\mathbf{C}_2\end{bmatrix} = \mathbf{K}\mathbf{R}^*\begin{bmatrix}\mathbf{I} & -\mathbf{C}_2\end{bmatrix}$$

- one can prove that $\mathbf{B}\mathbf{V}^\top\mathbf{R}_1 = \mathbf{A}\mathbf{U}^\top\mathbf{R}_2$ with the help of (13)
- points at infinity project to $\mathbf{K}\mathbf{R}^*$ in both images $\Rightarrow$ they have zero disparity →154

▶Summary

- rectification is a homography (per image)
  - ⇒ rectified camera centers are equal to the original ones
- standard rectified cameras are in canonical orientation
  - ⇒ rectified image projection planes are coplanar
- standard rectification guarantees equal rectified calibration matrices
  - ⇒ rectified image projection planes are equal

standard rectification homographies reproject onto a common image plane parallel to the baseline



Corollary

- standard rectified pair: disparity vanishes when corresponding 3D points are at infinity
  - known $\mathbf{F}$ used alone gives no constraints on standard rectification homographies
  - for that we need either of these:
    1. projection matrices, or
    2. calibrated cameras, or
    3. a few points at infinity calibrating k_{1i}, k_{2i}, $i = 1, 2, 3$ in (33)

Optimal and Non-linear Rectification

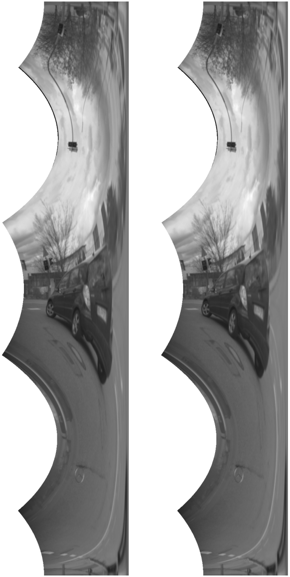
Optimal choice for the free parameters

- by minimization of residual image distortion, eg. [Gluckman & Nayar 2001]

$$\mathbf{A}_1^* = \arg\min_{\mathbf{A}_1} \iint_\Omega \left(\det J(\mathbf{A}_1 \hat{\mathbf{H}}_1 \underline{\mathbf{x}}) - 1\right)^2 d\mathbf{x}$$

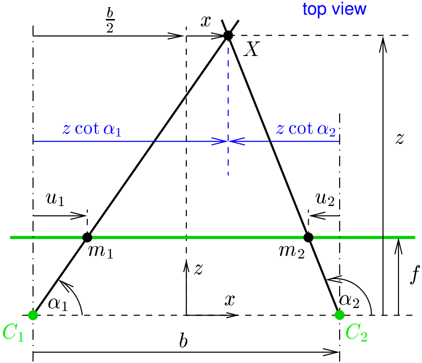
- by minimization of image information loss [Matoušek, ICIG 2004]
- non-linear rectification   suitable for forward motion [Pollefeys et al. 1999], [Geyer & Daniilidis 2003]

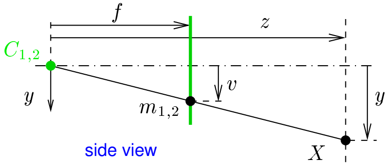
forward egomotion



rectified images, Pollefeys' method

▶Binocular Disparity in Standard Stereo Pair





- Assumptions: single image line, standard camera pair

$$b = z\cot\alpha_1 - z\cot\alpha_2$$

$$u_1 = f\cot\alpha_1 \qquad\qquad u_2 = f\cot\alpha_2$$

$$b = \frac{b}{2} + x - z\cot\alpha_2$$

$X = (x, z)$ from **disparity** $d = u_1 - u_2$:

$$\boxed{z = \frac{b\,f}{d}}, \quad x = \frac{b}{d}\,\frac{u_1 + u_2}{2}, \quad y = \frac{b\,v}{d}$$

f, d, u, v in pixels, b, x, y, z in meters

Observations

- constant disparity surface is a frontoparallel plane
- distant points have small disparity
- relative error in z is large for small disparity

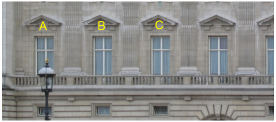
$$\frac{1}{z}\,\frac{dz}{dd} = -\frac{1}{d}$$

- increasing baseline increases disparity and reduces the error

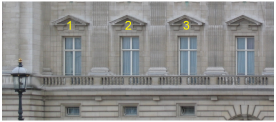
Understanding Basic Occlusion Constraints

- we can recognize matches but have no scene model
- lack of an occlusion model
- lack of a continuity model

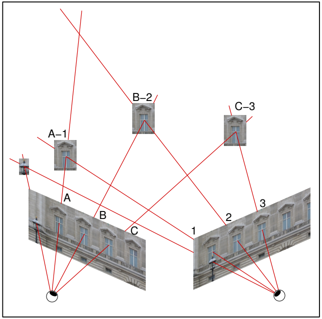
$\Rightarrow$ structural ambiguity in the presence of repetitions (or lack of texture)



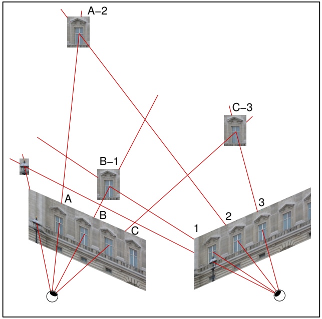
left image



right image

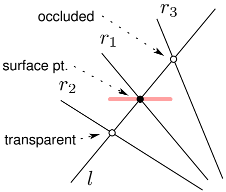


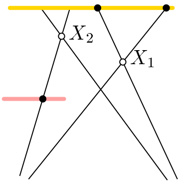
interpretation 1



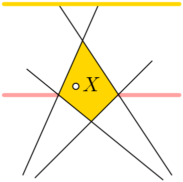
interpretation 2

▶Understanding Basic Occlusion Types More Deeply





half occlusion

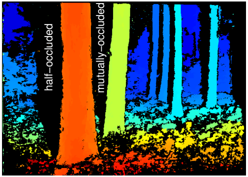


mutual occlusion

- surface point at the intersection of rays l and r_1 occludes a world point at the intersection (l, r_3) and implies the world point (l, r_2) is transparent, therefore

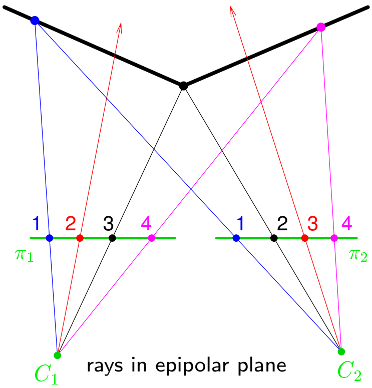
(l, r_3) and (l, r_2) are excluded by (l, r_1)

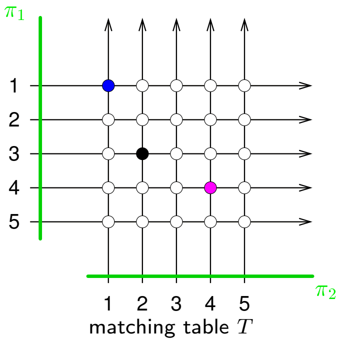
- in half-occlusion, every world point such as X_1 or X_2 is excluded by a binocularly visible surface point $\Rightarrow$ decisions on correspondences are not independent
- in mutual occlusion this is no longer the case: any X in the yellow zone is not excluded $\Rightarrow$ decisions in the zone are independent on the rest



▶Matching Table

Based on the observation on mutual exclusion we expect each pixel to match at most once.





matching table

- rows and columns represent optical rays
- nodes: possible correspondence pairs
- full nodes: matches
- numerical values associated with nodes: descriptor similarities

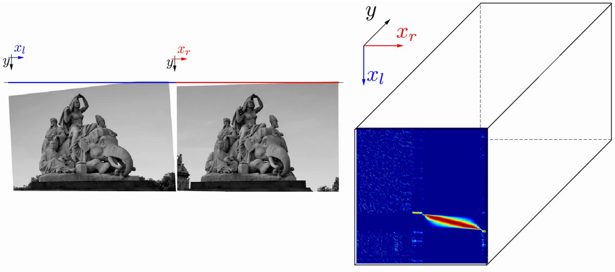
see next

Image Point Descriptors And Their Similarity

Descriptors: Tag image points by their (viewpoint-invariant) physical properties:

- texture window [Moravec 77]
- a descriptor like DAISY [Tola et al. 2010]
- reflectance profile under a moving illuminant
- photometric ratios [Wolff & Angelopoulou 93-94]
- dual photometric stereo [Ikeuchi 87]
- polarization signature
- ...

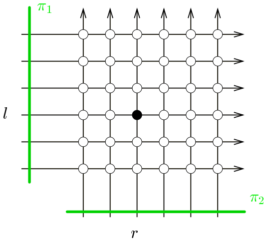
- similar points are more likely to match
- we will compute image similarity for all 'match candidates' and get the matching table



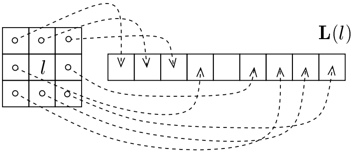
▶Constructing A Suitable Image Similarity

- let $p_i = (l, r)$ and $\mathbf{L}(l)$, $\mathbf{R}(r)$ be (left, right) image descriptors (vectors) constructed from local image neighborhood windows

in matching table T:



in the left image:



- a natural descriptor similarity is $\quad \mathrm{sim}(l, r) = \dfrac{\|\mathbf{L}(l) - \mathbf{R}(r)\|^2}{\sigma_I^2(l, r)}$
- σ_I^2 – the difference scale; a suitable (plug-in) estimate is $\frac{1}{2}\left[s^2\big(\mathbf{L}(l)\big) + s^2\big(\mathbf{R}(r)\big)\right]$, giving

$$\mathrm{sim}(l, r) = 1 - \underbrace{\frac{2\, s\big(\mathbf{L}(l), \mathbf{R}(r)\big)}{s^2\big(\mathbf{L}(l)\big) + s^2\big(\mathbf{R}(r)\big)}}_{\rho\left(\mathbf{L}(l), \mathbf{R}(r)\right)} \qquad (34)$$

$s^2(\cdot)$ is sample (co-)variance

- ρ – MNCC – Moravec's Normalized Cross-Correlation [Moravec 1977]

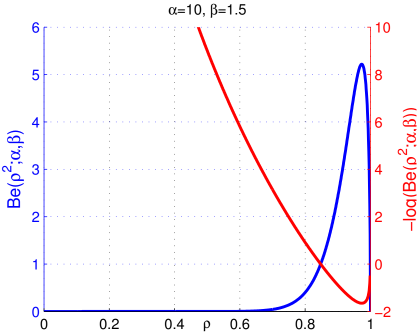
$$\rho^2 \in [0, 1], \qquad \mathrm{sign}\,\rho \sim \text{'phase'}$$

Similarity vs. Match Likelihood

- ρ can be considered a similarity feature
- we choose some probability distribution on $[0, 1]$, e.g. Beta distribution

$$p_1\big(\rho(l,r)\big) = \frac{1}{B(\alpha,\beta)}\,\rho^{2(\alpha-1)}(1-\rho^2)^{\beta-1}$$

- note that uniform distribution is obtained for $\alpha = \beta = 1$
- when $\alpha = 3/2$ and $\beta = 1$ then $p_1(\cdot) = \frac{2}{3}|\rho|$



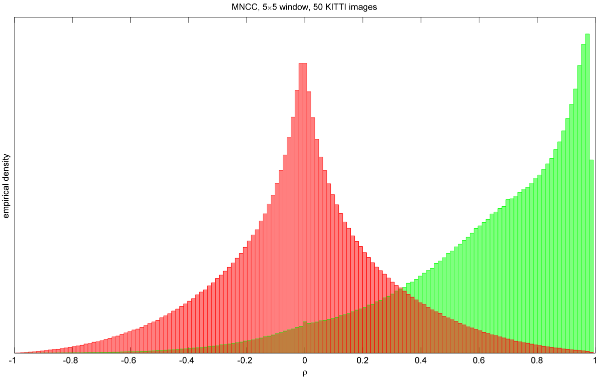
- the mode is at $\sqrt{\frac{\alpha-1}{\alpha+\beta-2}} \approx 0.9733$ for $\alpha = 10,\ \beta = 1.5$
- if we chose $\beta = 1$ then the mode was at $\rho = 1$
- perfect similarity is 'suspicious' (depends on expected camera noise level)
- from now on we will work with negative log-likelihood

$$V_1\big(\rho(l,r)\big) = -\log p_1\big(\rho(l,r)\big) \tag{35}$$

smaller is better

- we may also define similarity (and negative log-likelihood $V_0(\rho(l,r))$) for non-matches

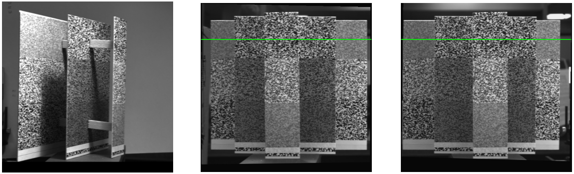
Example: Empirical Distribution for Matches and Non-Matches



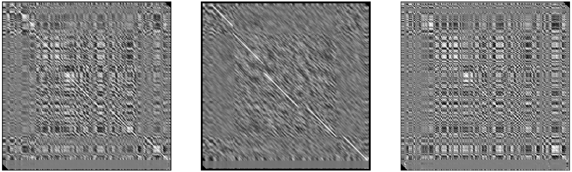
- KITTI dataset
  - $4.2 \cdot 10^6$ ground-truth (LiDAR) matches for $p_1(\rho)$ (green),
  - $4.2 \cdot 10^6$ random non-matches for $p_0(\rho)$ (red)
- histograms of ρ computed over 5×5 correlation window

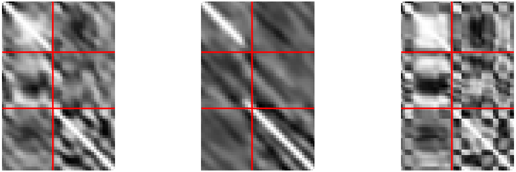
How A Scene Looks in The Filled-In Matching Table

scene | left image | right image



5×5 window | 11×11 window | 3×3 window





a good tradeoff | occlusion artefacts | undiscrimiable

- MNCC ρ used ($\alpha = 1.5,\ \beta = 1$)
- high-correlation structures correspond to scene objects

constant disparity

- a diagonal in matching table
- zero disparity is the main diagonal

depth discontinuity

- horizontal or vertical jump in matching table

large image window

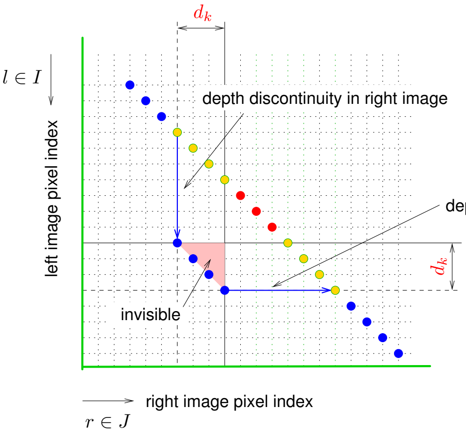
- better correlation
- worse occlusion localization

see next

repeated texture

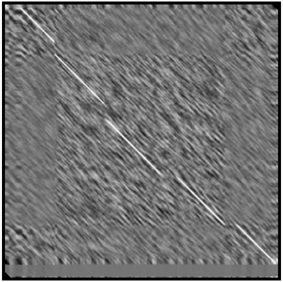
- horizontal and vertical block repetition

Understanding Matching Tables

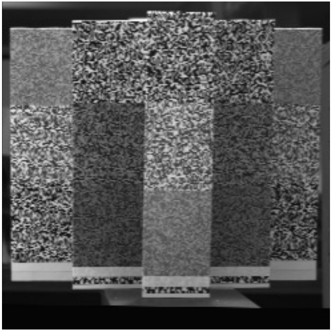


- binocularly visible foreground points
- binocularly visible background pts violating ordering
- monocularly visible points

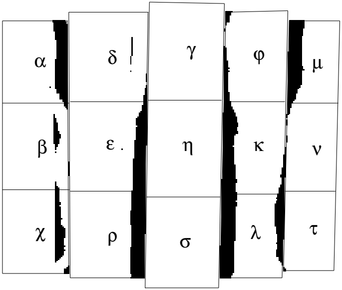
d_k critical disparity



Note: Errors at Occlusion Boundaries for Large Windows



NCC, Disparity Error



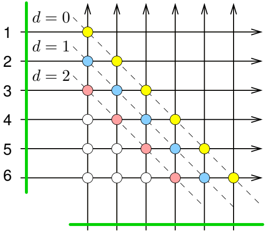
- this used really large window of 25×25 px
- errors depend on the relative contrast across the occlusion boundary
- the direction of 'overflow' depends on the combination of texture and edge contrasts
- solutions:
  1. small windows (5×5 typically suffices)
  2. eg. 'guided filtering' methods for computing image similarity [Hosni 2011]

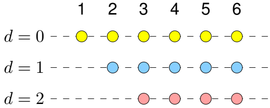
▶Marroquin's Winner Take All (WTA) Matching Algorithm

1. per left-image pixel: find the most similar right-image pixel

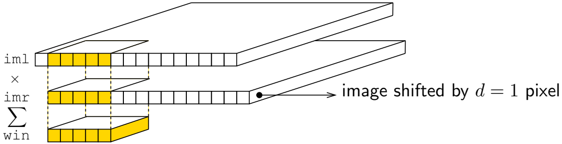
$$\mathrm{SAD}(l,r) = \|\mathbf{L}(l) - \mathbf{R}(r)\|_1 \qquad L_1 \text{ norm instead of the } L_2 \text{ norm in (34); unnormalized}$$

2. represent the matching table diagonals in a compact form





3. use the 'image sliding aggregation algorithm'



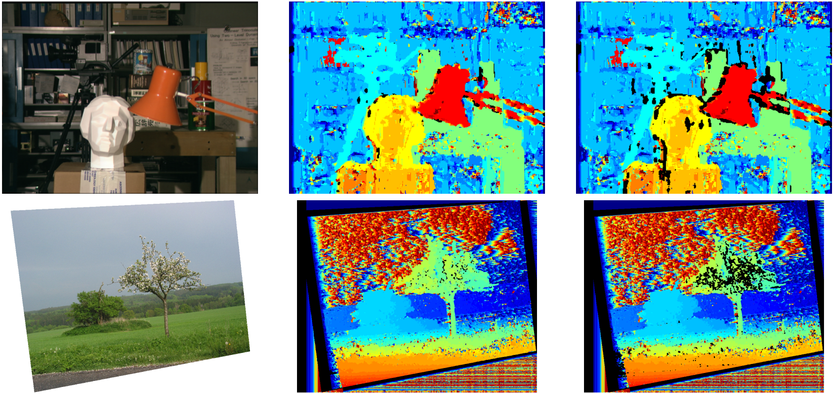
4. threshold results by maximal allowed dissimilarity

The Matlab Code for WTA

```
function dmap = marroquin(iml,imr,disparityRange)
%       iml, imr - rectified gray-scale images
% disparityRange - non-negative disparity range

% (c) Radim Sara (sara@cmp.felk.cvut.cz) FEE CTU Prague, 10 Dec 12

 thr = 20;             % bad match rejection threshold
 r = 2;
 winsize = 2*r+[1 1];  % 5x5 window (neighborhood)

 % the size of each local patch; it is N=(2r+1)^2 except for boundary pixels
 N = boxing(ones(size(iml)), winsize);

 % computing dissimilarity per pixel (unscaled SAD)
 for d = 0:disparityRange                       % cycle over all disparities
  slice = abs(imr(:,1:end-d) - iml(:,d+1:end)); % pixelwise dissimilarity
  V(:,d+1:end,d+1) = boxing(slice, winsize)./N; % window aggregation
 end

 % collect winners, threshold, and output disparity map
 [cmap,dmap] = min(V,[],3);
 dmap(cmap > thr) = NaN;    % mask-out high dissimilarity pixels
end

function c = boxing(im, wsz)
 % if the mex is not found, run this slow version:
 c = conv2(ones(1,wsz(1)), ones(wsz(2),1), im, 'same');
end
```

WTA: Some Results

thr $=$ 20 thr $=$ 10



- results are bad
- false matches in textureless image regions and on repetitive structures (book shelf)
- a more restrictive threshold (thr $= 10$) does not work as expected
- we searched the true disparity range, results get worse if the range is set wider
- chief failure reasons:
  - unnormalized image dissimilarity does not work well

▶A Principled Approach: (1) Symmetric Matching

- given matching M what is the likelihood of observed data D?
- data – all pairwise costs in matching table T
- matches – pairs $p_i = (l_i, r_i), \quad i = 1, \dots, n$
- matching: partitioning matching table T to matched M and excluded E pairs

$$T = M \cup E, \quad M \cap E = \emptyset$$

- matching cost (negative log-likelihood, smaller is better)

$$V(D \mid M) = \sum_{p \in M} V_1(D \mid p) + \sum_{p \in E} V_0(D \mid p)$$

$V_1(D \mid p)$ – negative log-probability of data D at matched pixel p (35)
$V_0(D \mid p)$ – ditto at unmatched pixel p (e.g. uniform)

- matching problem

$$M^* = \arg \min_{M \in \mathcal{M}(T)} V(D \mid M)$$

$\mathcal{M}(T)$ – the set of all matchings in table T

- symmetric: formulated over pairs, invariant to left $\leftrightarrow$ right image swap

▶A Principled Approach: (2) Log-Likelihood Ratio

- we need to reduce the matching to a standard polynomial-complexity problem
- we convert the matching cost to an 'easier' sum

$$V(D \mid M) = \sum_{p \in M} V_1(D \mid p) + \sum_{p \in E} V_0(D \mid p) \quad + \overbrace{\sum_{p \in M} V_0(D \mid p) - \sum_{p \in M} V_0(D \mid p)}^{0}$$
$$= \sum_{p \in M} \underbrace{\Big(V_1(D \mid p) - V_0(D \mid p)\Big)}_{-L(D \mid p)} + \underbrace{\sum_{p \in E} V_0(D \mid p) + \sum_{p \in M} V_0(D \mid p)}_{\sum_{p \in T} V_0(D \mid p) = \text{const}}$$

- hence

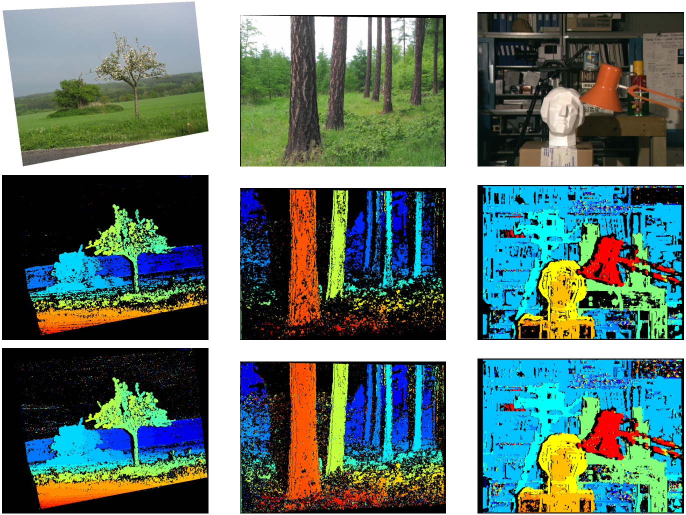
$$\arg \min_{M \in \mathcal{M}(T)} V(D \mid M) = \arg \max_{M \in \mathcal{M}(T)} \sum_{p \in M} L(D \mid p) \tag{36}$$

$L(D \mid p)$ – logarithm of matched-to-unmatched likelihood ratio (bigger is better)

why this way: we want to use maximum-likelihood but our measurement is all data D

- (36) is max-cost matching (maximum assignment) for the maximum-likelihood (ML) matching problem
  - it must contain no pairs p with $L(D \mid p) < 0$
  - use Hungarian (Munkres) algorithm and threshold the result based on $L(D \mid p)$
  - or step back: sacrifice symmetry to speed and use dynamic programming

Some Results for the Maximum-Likelihood (ML) Matching



- unlike the WTA we can efficiently control the density/accuracy tradeoff
- middle row: V_{e} set to error rate of 3% (and 61% density is achieved)
- bottom row: V_{e} set to density of 76% (and 4.3% error rate is achieved)

holes are black

▶Basic Stereoscopic Matching Models

- notice many small isolated errors in the ML matching
- we need a stronger model

Potential models for M

1. Uniqueness: Every image point matches at most once
   - excludes semi-transparent objects
   - used by the ML matching algorithm (but not by the WTA algorithm)

2. Monotonicity: Matched pixel ordering is preserved
   - For all $(i,j) \in M, (k,l) \in M, \quad k > i \Rightarrow l > j$

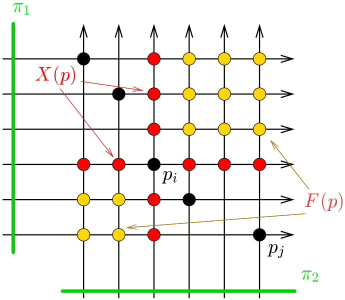
     Notation: $(i,j) \in M$ or $j = M(i)$ – left-image pixel i matches right-image pixel j
   - excludes thin objects close to the cameras

3. Coherence: Objects occupy well defined 3D volumes
   - concept by [Prazdny 85]
   - algorithms are based on image/disparity map segmentation
   - currently the most popular model (segment-based, bilateral filtering and their successors)

4. Continuity: There are no occlusions or self-occlusions
   - too strong, except in some applications

▶Uniqueness and Ordering in Matching Table T

X-zone and F-zone



$$p_j \notin X(p_i), \quad p_j \notin F(p_i)$$

- **Uniqueness Constraint:**

  A set of pairs $M = \{p_i\}_{i=1}^{n}$, $p_i \in T$ is a matching iff
  $$\forall p_i, p_j \in M, i \neq j : \; p_j \notin X(p_i).$$

  X-zone

- **Ordering Constraint:**

  Matching M is monotonic iff
  $$\forall p_i, p_j \in M : \; p_j \notin F(p_i).$$

  F-zone

  - ordering constraint: matched points form a monotonic set in both images
  - ordering is a powerful constraint: monotonic matchings $O(4^n) \ll O(n!)$ all matchings in $n \times n$ table

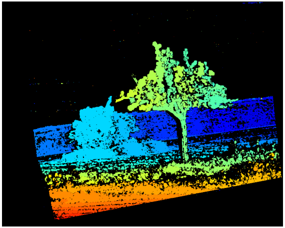
  ⊛ 2: how many are there maximal monotonic matchings?

- uniqueness constraint is a basic occlusion model
- ordering constraint is a weak continuity model — and partly also an occlusion model
- ordered matching can be found by dynamic programming

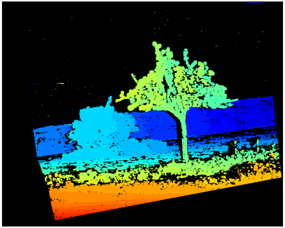
Some Results: AppleTree

left image

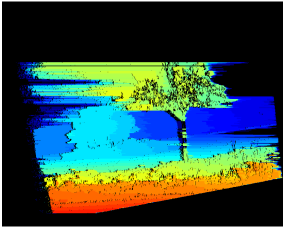
right image



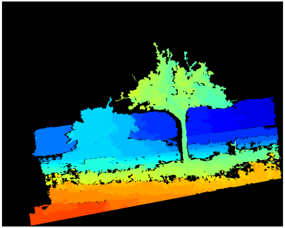
ML →169



3LDP w/ordering [SP]



naïve DP [Cox et al. 1992]



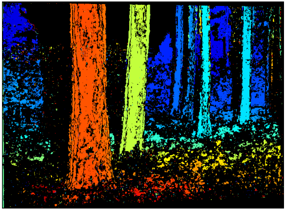
stable segmented 3LDP

- 3LDP parameters α_i, V_e learned on Middlebury stereo data http://vision.middlebury.edu/stereo/

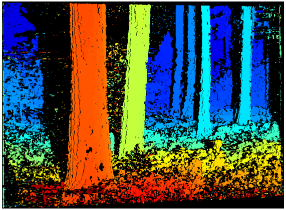
Some Results: Larch

left image

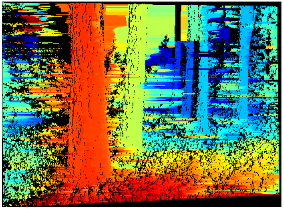
right image



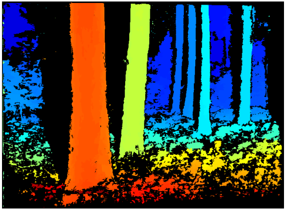
ML →169



3LDP w/ordering [SP]



naïve DP



stable segmented 3LDP

- naïve DP does not model mutual occlusion
- but even 3LDP has errors in mutually occluded region
- stable segmented 3LDP has few errors in mutually occluded region since it uses a coherence model

Algorithm Comparison

Winner-Take-All (WTA →165)

- the ur-algorithm very weak model
- dense disparity map
- $O(N^3)$ algorithm, simple but it rarely works

Maximum Likelihood Matching (ML →169)

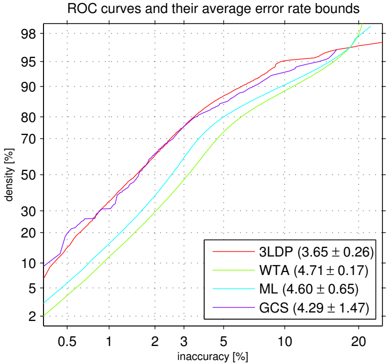
- semi-dense disparity map
- many small isolated errors
- models basic occlusion
- $O(N^3 \log(NV))$ algorithm max-flow by cost scaling

MAP with Min-Cost Labeled Path (3LDP)

- semi-dense disparity map
- models occlusion in flat, piecewise continuos scenes
- has 'illusions' if ordering does not hold
- $O(N^3)$ algorithm

Stable Segmented 3LDP

- better (fewer errors at any given density)
- $O(N^3 \log N)$ algorithm
- requires image segmentation itself a difficult task



- ROC-like curve captures the density/accuracy tradeoff
- GCS is the one used in the exercises
- more algorithms at `http://vision.middlebury.edu/stereo/` (good luck!)

Part VIII

Shape from Reflectance

mostly covered by

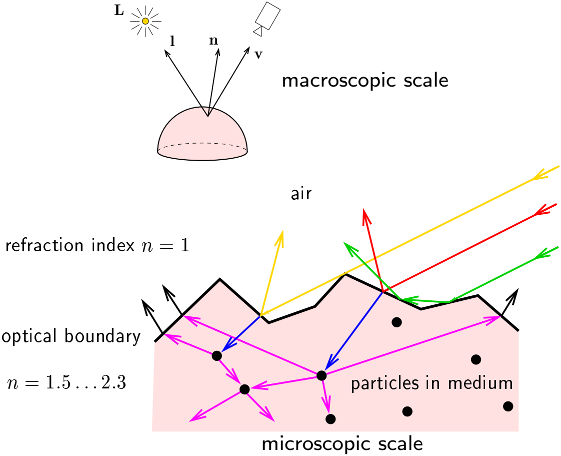
Forsyth, David A. and Ponce, Jean. *Computer Vision: A Modern Approach*. Prentice Hall 2003. Chap. 5

additional references

R. T. Frankot and R. Chellappa. A method for enforcing integrability in shape from shading algorithms. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 10(4):439–451, July 1988.

P. N. Belhumeur, D. J. Kriegman, and A. L. Yuille. The bas-relief ambiguity. In *Proc Conf Computer Vision and Pattern Recognition*, pp. 1060–1066, 1997.

Basic Surface Reflectance Mechanisms



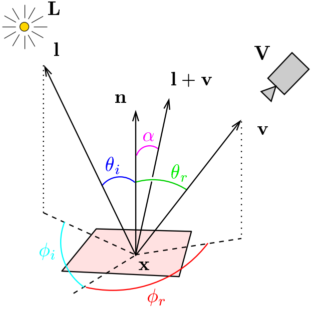
- reflection on (rough) optical boundary
- masking and shadowing
- interreflection
- refraction into the body
- subsurface scattering
- refraction into the air

Parametric Reflectance Models

Image intensity (measurement) at pixel m given by surface reflectance function R

$$J(m) = \eta\, f_{i,r}(\theta_i, \phi_i; \theta_r, \phi_r) \cdot \underbrace{\frac{\Phi_e}{4\pi\|\mathbf{L}-\mathbf{x}\|^2}}_{\sigma}\, \mathbf{n}^\top \mathbf{l} = R(\mathbf{n}), \qquad \mathbf{l} = \frac{\mathbf{L}-\mathbf{x}}{\|\mathbf{L}-\mathbf{x}\|}$$

- η – sensor sensitivity   for simplicity, we select $\eta = 2\pi$
- $f_{i,r}()$ – bidirectional reflectance distribution function (BRDF)
  $[f_{i,r}()] = \mathrm{sr}^{-1}$ how much of irradiance in $\mathrm{W m}^{-2}$ is redistributed per solid angle element
- $\mathbf{L}$ – point light source position in 3D
- $\mathbf{x}$ – surface patch position in 3D
- Φ_e – radiant power of the light source, $[\Phi_e] = \mathrm{W}$
- $\mathbf{n}$ – surface normal
- σ – irradiance of a surface element orthogonal to incident light direction



pixel projected onto surface

Isotropic (Lambertian) reflection   [Lambert 1760]

no optical boundary

$$f_{i,r}(\theta_i, \phi_i; \theta_r, \phi_r) = \frac{\rho}{2\pi}, \qquad \rho \text{ – albedo}$$

$$J(m) = \sigma\rho\cos\theta_i = \sigma\rho\,\mathbf{n}^\top\mathbf{l}$$

▶Photometric Stereo

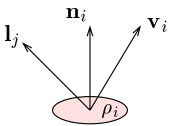
Lambertian model (light $j \in \{1,2,3\}$, pixel $i \in \{1,\ldots,n\}$)

$$J_{ji} = (\sigma_j \, \mathbf{l}_j)^\top (\rho_i \, \mathbf{n}_i) = \mathbf{s}_j^\top \, \mathbf{b}_i$$

$\mathbf{b}_i$ – scaled normals, $\mathbf{s}_j$ – scaled lights

3 independent scaled lights and n scaled normals (per pixel); stacked:

$$\begin{bmatrix} J_{11} & \cdots & J_{1n} \\ J_{21} & \cdots & J_{2n} \\ J_{31} & & J_{3n} \end{bmatrix} = \begin{bmatrix} \mathbf{s}_1^\top \mathbf{b}_1 & & \mathbf{s}_1^\top \mathbf{b}_n \\ \mathbf{s}_2^\top \mathbf{b}_1 & \cdots & \mathbf{s}_2^\top \mathbf{b}_n \\ \mathbf{s}_3^\top \mathbf{b}_1 & & \mathbf{s}_3^\top \mathbf{b}_n \end{bmatrix} = \begin{bmatrix} \mathbf{s}_1^\top \\ \mathbf{s}_2^\top \\ \mathbf{s}_3^\top \end{bmatrix} \begin{bmatrix} \mathbf{b}_1 \cdots \mathbf{b}_n \end{bmatrix} = \mathbf{S}^\top \mathbf{B}$$



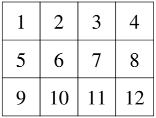
Solution to Photometric Stereo

$$\mathbf{J} = \mathbf{S}^\top \mathbf{B} \quad \Rightarrow \quad \mathbf{B} = \mathbf{S}^{-\top} \mathbf{J}$$

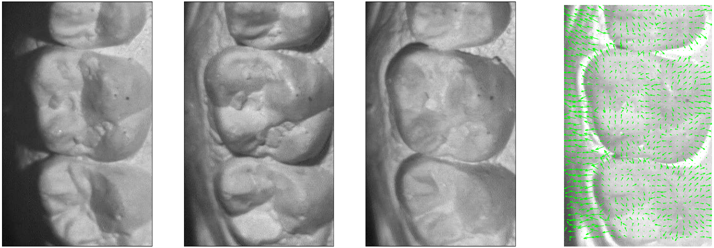
$\mathbf{S} \in \mathbb{R}^{3,3}$, $\mathbf{B} \in \mathbb{R}^{3,n}$, $\mathbf{J} \in \mathbb{R}^{3,n}$

$$\rho_i = \|\mathbf{b}_i\| \quad \text{albedo map}, \qquad \mathbf{n}_i = \frac{1}{\rho_i} \mathbf{b}_i \quad \text{needle map}$$

pixel indexing i:



Photometric Stereo: Plaster Cast Example



input images (known lights) needle & albedo maps

We have: 1. shape (surface normals), 2. intrinsic texture (albedo)

- depth map $\big(u,\,v,\,z(u,v)\big)$, u, v – image coordinates, z – depth Monge patch
- represented as unit normal vectors $\mathbf{n}$ or as a gradient field $\big(p(u,v), q(u,v)\big)$:

$$\mathbf{n}(u,v) = \big(n_1(u,v),\, n_2(u,v),\, n_3(u,v)\big) \;\simeq\; \big(p(u,v),\, q(u,v),\, 1\big)$$

see a book on differential geometry of surfaces

$$\frac{\partial z(u,v)}{\partial u} \stackrel{\text{def}}{=} z_u(u,v) = p(u,v) = \frac{n_1(u,v)}{n_3(u,v)}$$

$$\frac{\partial z(u,v)}{\partial v} \stackrel{\text{def}}{=} z_v(u,v) = q(u,v) = \frac{n_2(u,v)}{n_3(u,v)}$$

The Integration Algorithm of Frankot and Chellappa (FC)

Task: Given gradient fields $p(u,v)$, $q(u,v)$, find height function $z(u,v)$ such that z_u is close to p and z_v is close to q in the sense of a functional norm.

$$z^* = \arg\min_z Q(z), \qquad Q(z) = \iint |z_u(u,v) - p(u,v)|^2 + |z_v(u,v) - q(u,v)|^2 \, du\, dv$$

In the Fourier domain this can be written as $\mathcal{F}(z;\boldsymbol{\omega}) = \frac{1}{2\pi} \iint z(u,v) e^{-j(u\omega_u + v\omega_v)} \, du\, dv$

$$Q(z) = \iint \underbrace{|j\omega_u \, \mathcal{F}(z;\boldsymbol{\omega}) - \mathcal{F}(p;\boldsymbol{\omega})|^2 + |j\omega_v \, \mathcal{F}(z;\boldsymbol{\omega}) - \mathcal{F}(q;\boldsymbol{\omega})|^2}_{A(\mathcal{F}(z;\boldsymbol{\omega}))} \, d\boldsymbol{\omega}, \qquad \boldsymbol{\omega} = (\omega_u, \omega_v)$$

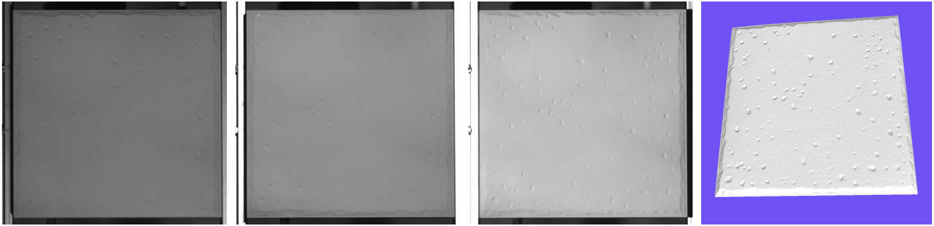
and its minimiser is

from vanishing formal derivative of $A(\mathcal{F}(z;\boldsymbol{\omega}))$ wrt $\mathcal{F}(z;\boldsymbol{\omega})$
[Frankot & Chellappa 1988]

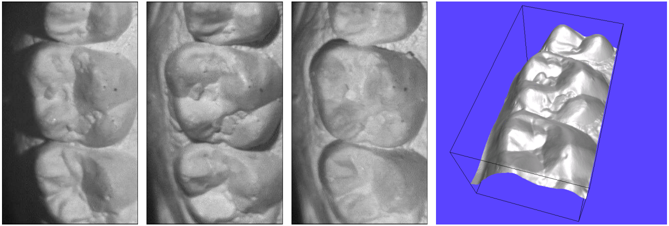
$$\mathcal{F}(z;\boldsymbol{\omega}) = -\frac{j\omega_u}{|\boldsymbol{\omega}|^2} \, \mathcal{F}(p;\boldsymbol{\omega}) - \frac{j\omega_v}{|\boldsymbol{\omega}|^2} \, \mathcal{F}(q;\boldsymbol{\omega})$$

```
[m,n] = size(p);
Wu = fft2(fftshift([-1,0,1]/2),m,n); % discrete differential operator
Wv = fft2(fftshift([-1;0;1]/2),m,n);
Z = -(Wu.*fft2(p) + Wv.*fft2(q))./(abs(Wu).^2 + abs(Wv).^2 + eps);
z = real(ifft2(Z));
```

Photometric Stereo: Examples



3 input images surface



3 input images surface

- integrated by the FC algorithm →181
- bias due to interreflections can be removed

[Drew & Funt, JOSA-A 1992]

Optimal Light Configurations

For n lights $\mathbf{S}$ the error $\Delta\mathbf{b} = \mathbf{S}^{-\top}\Delta\mathbf{J}$ in normal $\mathbf{b}$ due to error $\Delta\mathbf{J}$ in image is

$$\epsilon(\mathbf{S}) = E\big[\Delta\mathbf{b}^\top\Delta\mathbf{b}\big] = E\big[\Delta\mathbf{J}^\top(\mathbf{S}^\top\mathbf{S})^{-1}\Delta\mathbf{J}\big] = \sigma^2\,\mathrm{tr}\big[(\mathbf{S}\mathbf{S}^\top)^{-1}\big] \geq \frac{9\sigma^2}{n}.$$

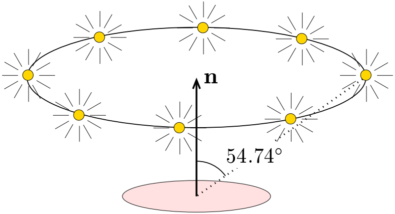
assuming pixel-independent normal camera noise $\Delta J_i \sim N(0,\sigma)$

The error ϵ is minimum if [Drbohlav & Chantler 2005]

$$\mathbf{S}\mathbf{S}^\top = \frac{n}{3}\,\mathbf{I}, \qquad \text{where} \qquad \mathbf{S} = [\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_n]$$

- either $n \geq 3$ equidistant and equiradiant lights on a circle of uniform slant of $\arctan\sqrt{2} \approx 54.74°$
- $n-1$ lights in this configuration plus a light parallel to the sum $\sum_{i=1}^{n-1}\mathbf{s}_i$
- or light matrix $\mathbf{S}$ is a concatenation of optimal solutions (each of ≥ 3 lights)

eg. 3 optimally placed $(\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3)$ + 3 lights $(\mathbf{s}_4, \mathbf{s}_5, \mathbf{s}_6) = (\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3) + \alpha$ rotated by angle α around $\mathbf{n}$



Uncalibrated Photometric Stereo

Factorization $\mathbf{J} = \mathbf{S}^\top \mathbf{B}, \qquad \mathbf{J} \in \mathbb{R}^{3,n}$ [Hayakawa94]

LS solution by SVD decomposition of $\mathbf{J} = \mathbf{U}\mathbf{D}\mathbf{V}^\top$, $\qquad \mathbf{U} \in \mathbb{R}^{3,3}$, $\mathbf{D} \in \mathbf{R}^{3,n}$, $\mathbf{V} \in \mathbb{R}^{n,n}$

$\mathbf{S} = \mathbf{D}_{1:3}\mathbf{U}^\top$ scaled pseudo-lights

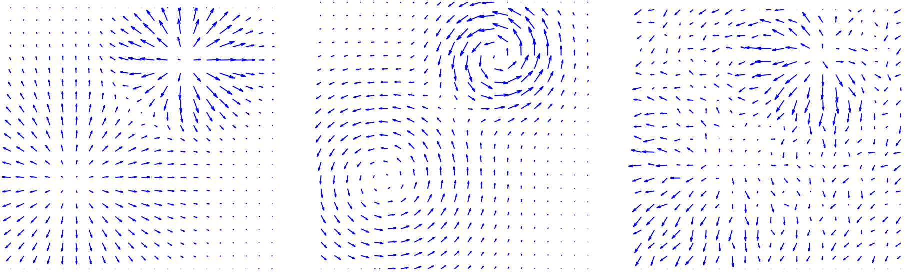
$\mathbf{B} = (\mathbf{V}_{1:3})^\top$ scaled pseudo-normals $\quad$ $\mathbf{V}_{1:3}$ are columns 1–3

Ambiguity $$\mathbf{J} = \mathbf{S}^\top \mathbf{B} = \underbrace{\mathbf{S}^\top \mathbf{A}^{-1}}_{\bar{\mathbf{S}}^\top} \underbrace{\mathbf{A}\mathbf{B}}_{\bar{\mathbf{B}}}, \qquad \mathbf{A} \in GL(3)$$ [Koenderink94]

known	remaninig ambiguity	algorithm
$n \geq 3$ normals $\bar{\mathbf{B}}$	$\lambda\mathbf{I}$	$\bar{\mathbf{B}} = \mathbf{A}\mathbf{B} \;\Rightarrow\; \mathbf{A}$ $\quad$ $\mathbf{B}$ is measured
uniform albedo in $n \geq 6$ points	$\lambda\mathbf{R}$	$\mathbf{b}_i^\top \mathbf{A}^\top \mathbf{A}\mathbf{b}_i = 1$: linear in $\mathbf{A}^\top\mathbf{A} \Rightarrow \mathbf{A}$ up to rotation $\mathbf{R}$ by Choleski [Drew92]
equal light intensity	$\lambda\mathbf{R}$	$\lVert\mathbf{s}_j \mathbf{A}^{-1}\rVert = 1 \Rightarrow \mathbf{A}$ up to rotation $\mathbf{R}$ [Hayakawa94]
integrability $p_v = q_u$	GBR	[Yuille99, Fan97, Belhumeur99]
uniform albedo and integrability	$\lambda\mathbf{I}$	
integrability and $n \geq 2$ specular points	$\lambda\mathbf{I}$	[Drbohlav & Chantler, ICCV 2005]

Integrability of a Vector Field

- not every vector field $p(u,v)$, $q(u,v)$ is integrable (born by a surface $z(u,v)$)
- integrability constraint
$$p_v(u,v) = q_u(u,v)$$
- this is because a regular surface has $\operatorname{rot} \nabla z(u,v) = 0$ irrotational gradient field
$$z_{uv}(u,v) = z_{vu}(u,v)$$
- noise causes non-integrability
- the FC algorithm finds the closest integrable surface



integrable non-integrable non-integrable (noisy)

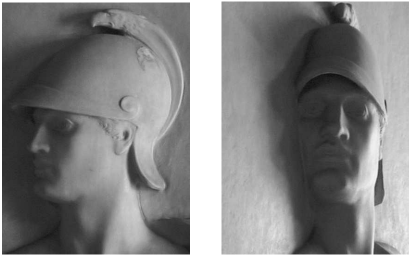
Generalized Bas Relief Ambiguity (GBR)

GBR maps surface $z'(u,v) = \lambda z(u,v) + \mu\, u + \nu\, v$, i.e. it maps normals to $\mathbf{n}' = \mathbf{G}\mathbf{n}$, where

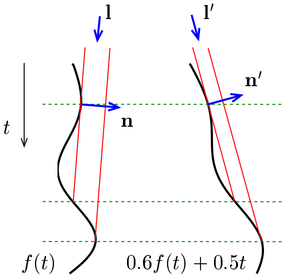
$$\mathbf{G} = \begin{bmatrix} \lambda & 0 & -\mu \\ 0 & \lambda & -\nu \\ 0 & 0 & 1 \end{bmatrix}$$

Obs: If normals change $\mathbf{n}' = \mathbf{G}\,\mathbf{n}$ and lights change $\mathbf{l}' = \mathbf{G}^{-\top}\,\mathbf{l}$ then Lambertian shading does not change:

$$\mathbf{n}'^{\top}\mathbf{l}' = (\mathbf{n}^{\top}\mathbf{G}^{\top})(\mathbf{G}^{-\top}\mathbf{l}) = \mathbf{n}^{\top}\mathbf{l}$$

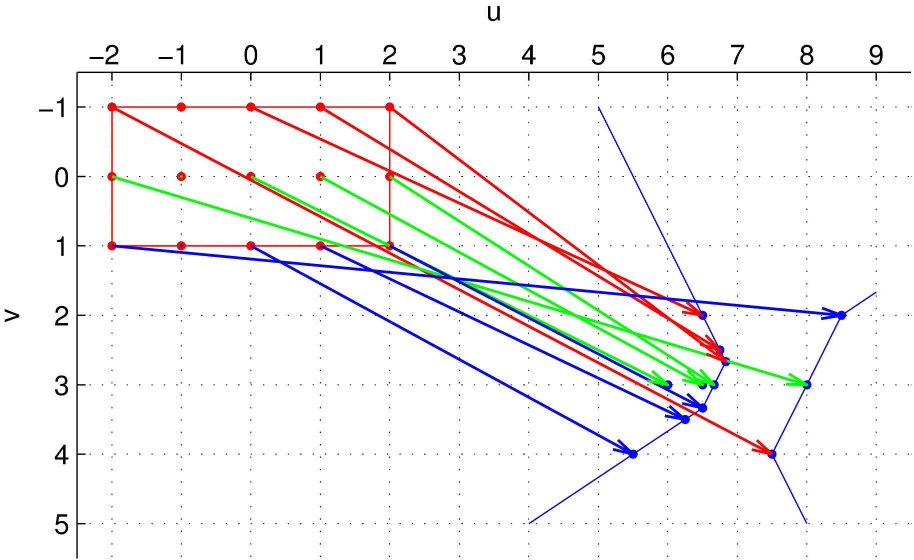


Reproduced from [Belhumeur et al. 1997]



Obs: Shadow boundaries of surface $\mathcal{S}$ illuminated by light $\mathbf{l}$ are identical to those of surface $\mathcal{S}'$ transformed by GBR $\mathbf{G}$ and illuminated by light $\mathbf{l}' = \mathbf{G}^{-\top}\,\mathbf{l}$ weak assumptions [Belhumeur et al. 1997]

Thank You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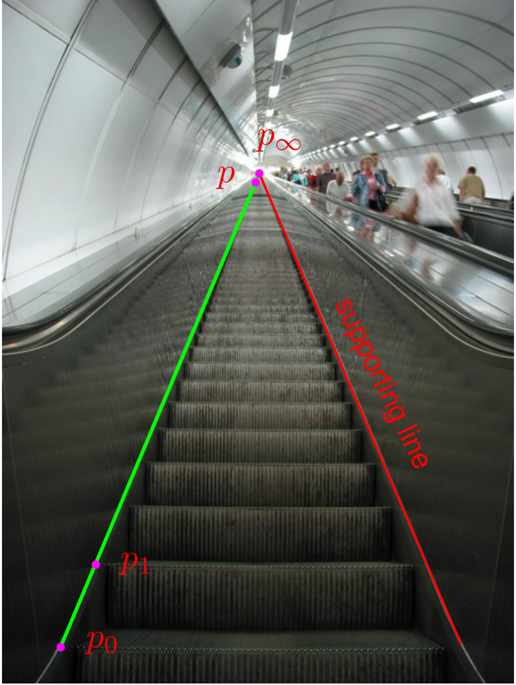
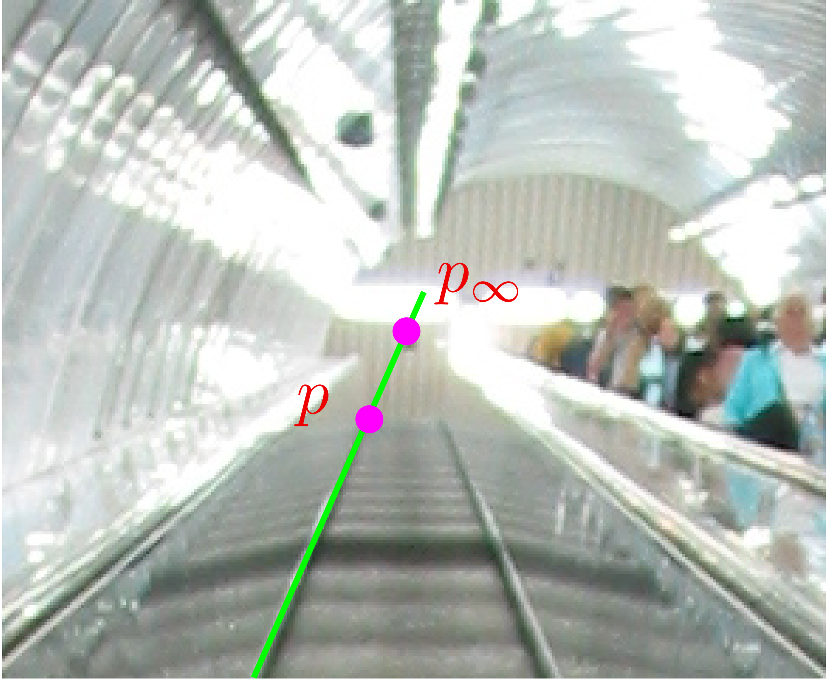
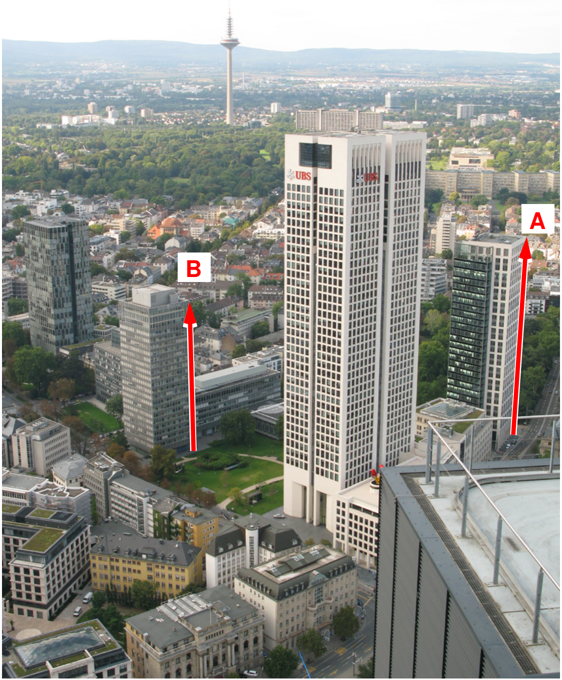
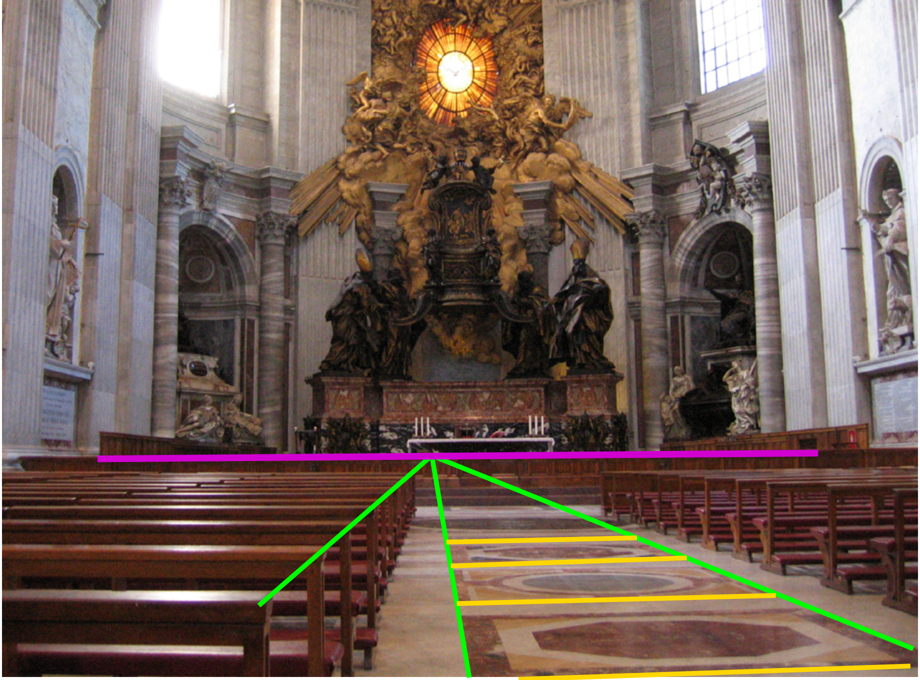

p_∞
p
supporting line
p_1
p_0

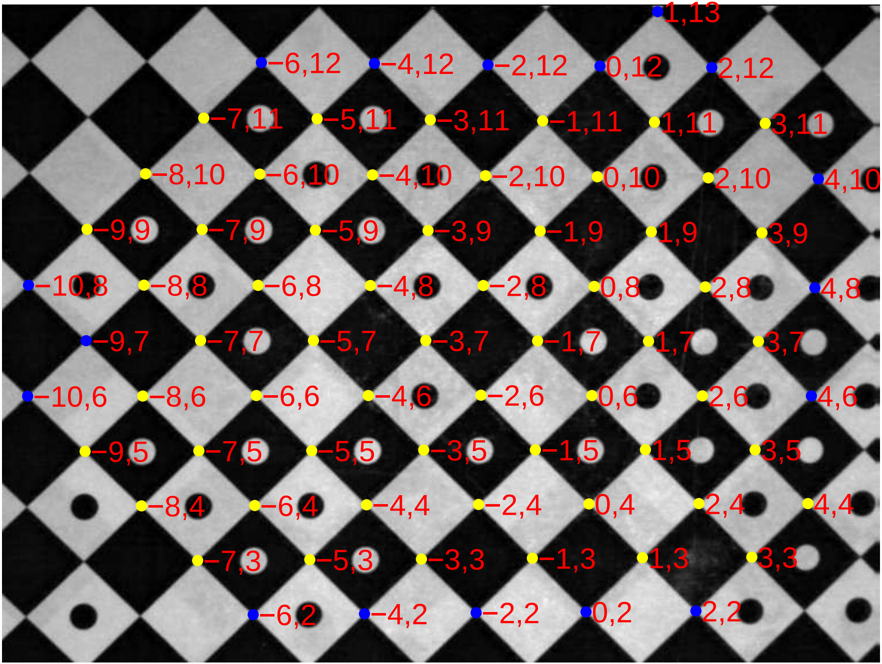

p_∞
p


UBS
UBS
A
B




CLEMENS·XII·PONT·MAX·ANN·SAC·MDCCXXXVII·PONT·VIII·


BOOKS
FILMS
TICKETS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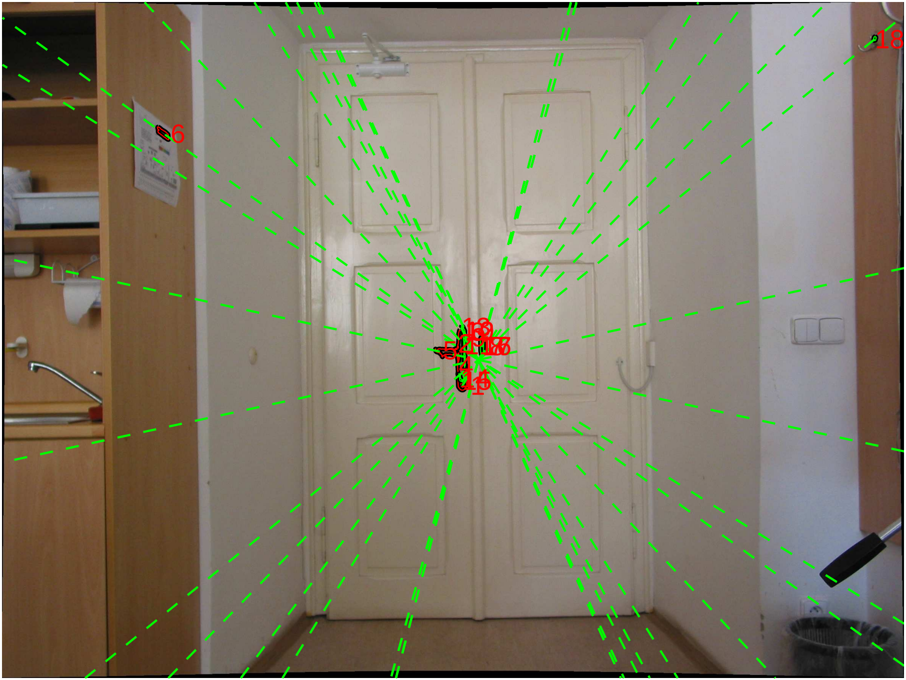
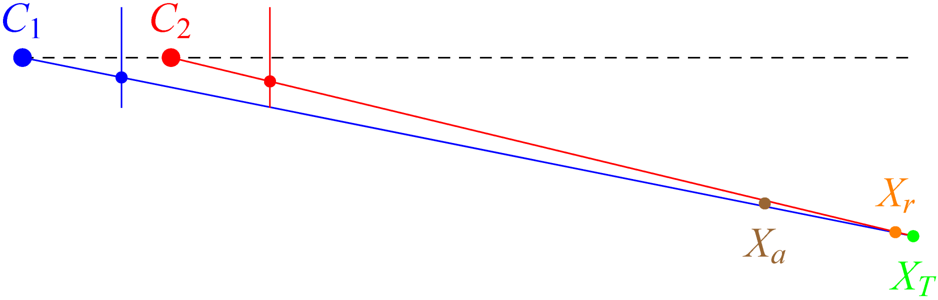
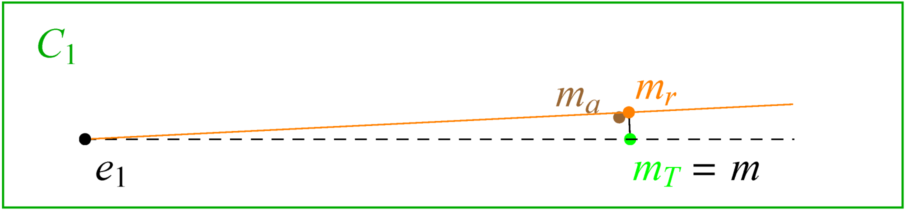
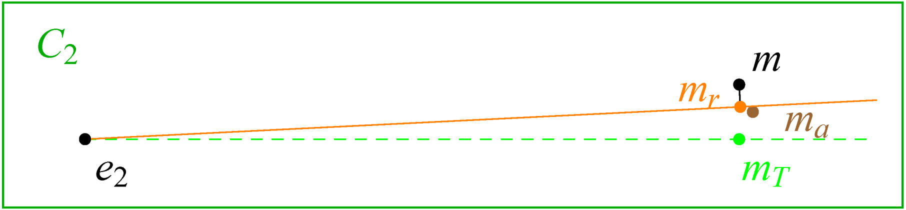
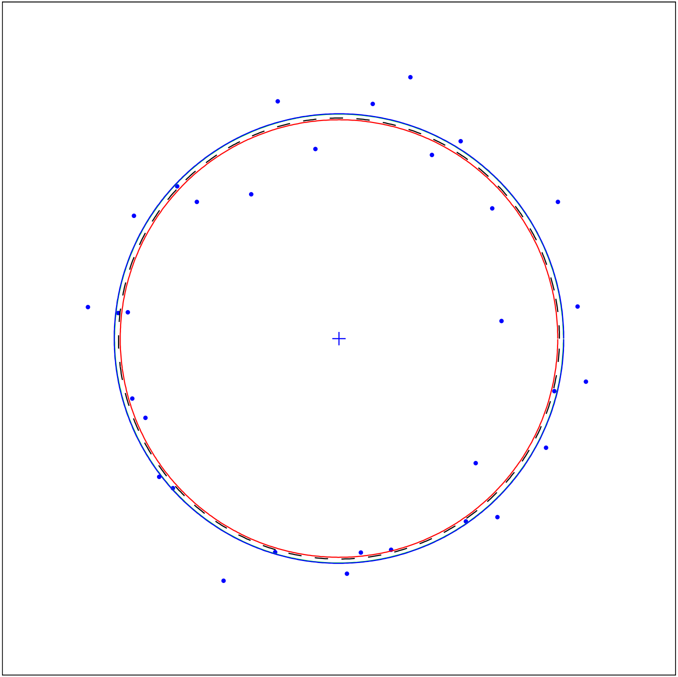
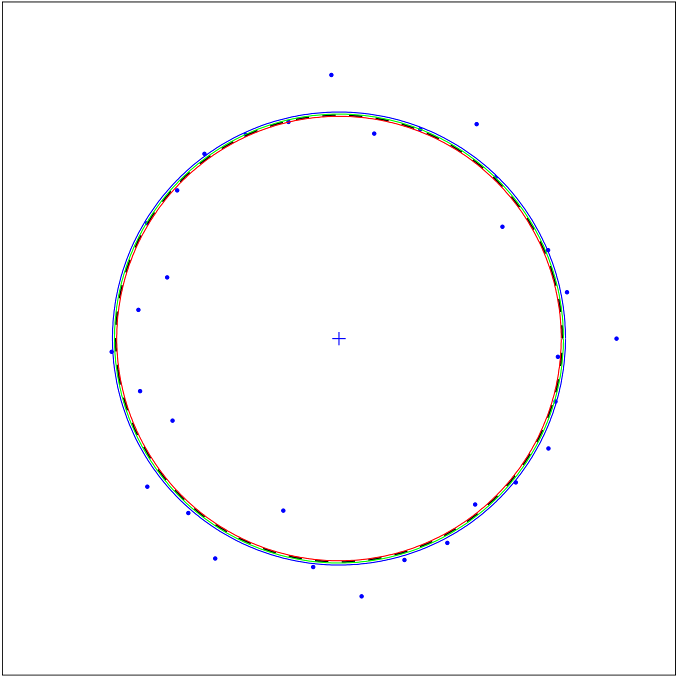

6
18

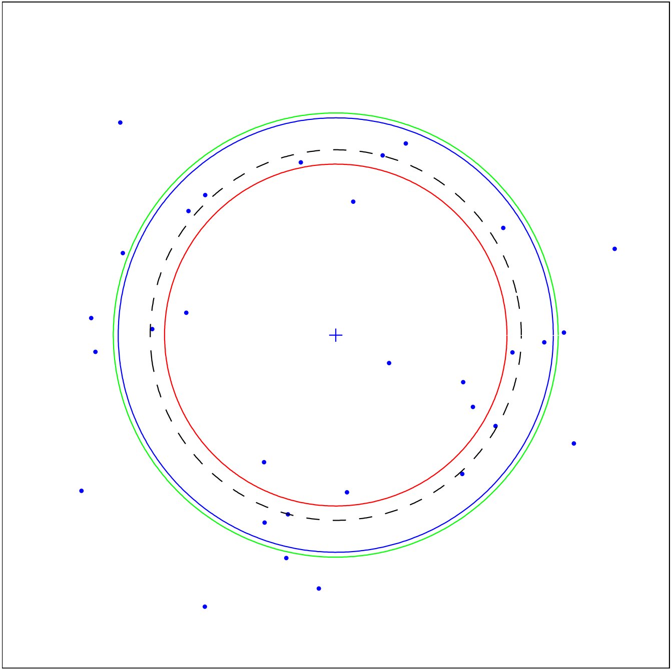

C_1
C_2
X_r
X_a
X_T

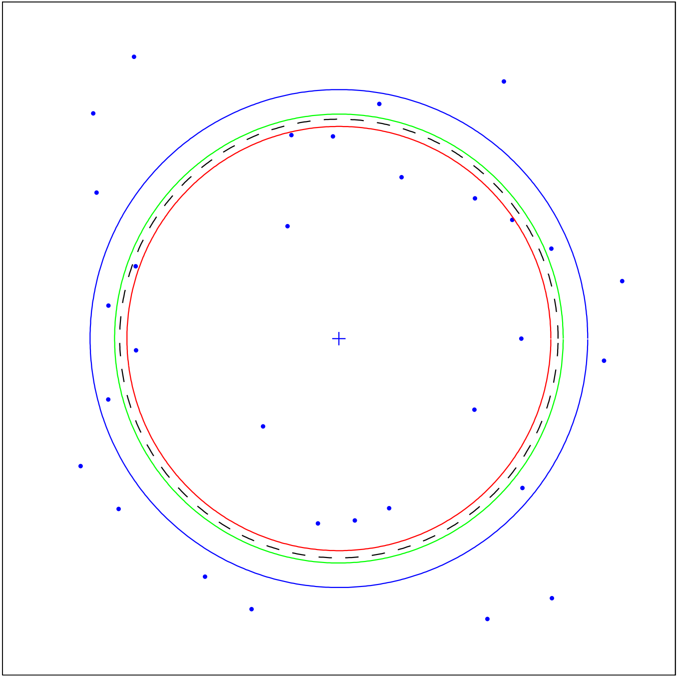

C_1
m_a
m_r
e_1
$m_T = m$

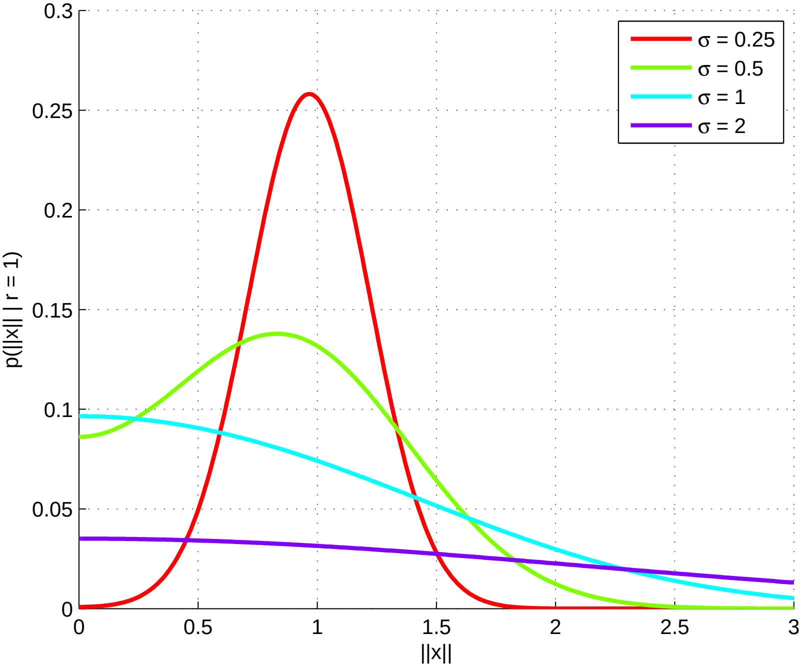

C_2
m
m_r
m_a
e_2
m_T

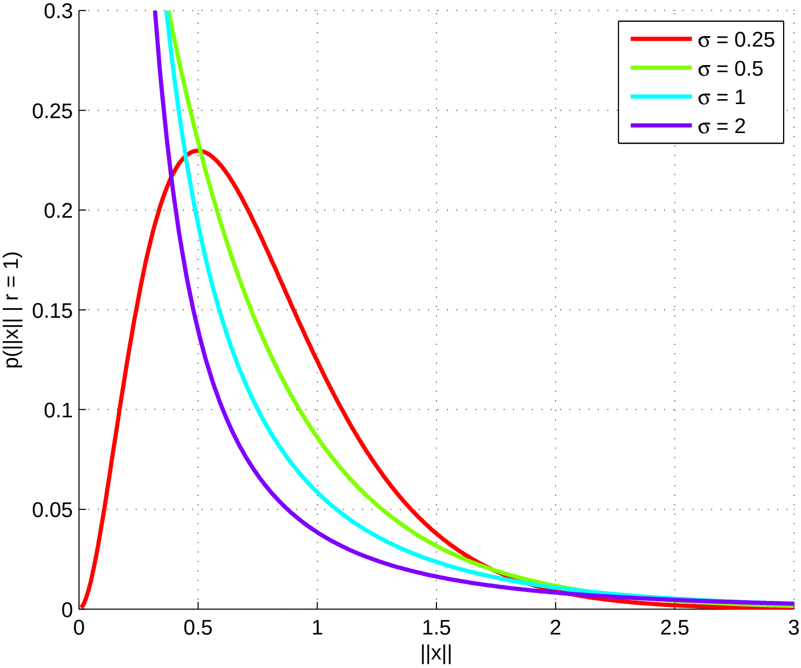




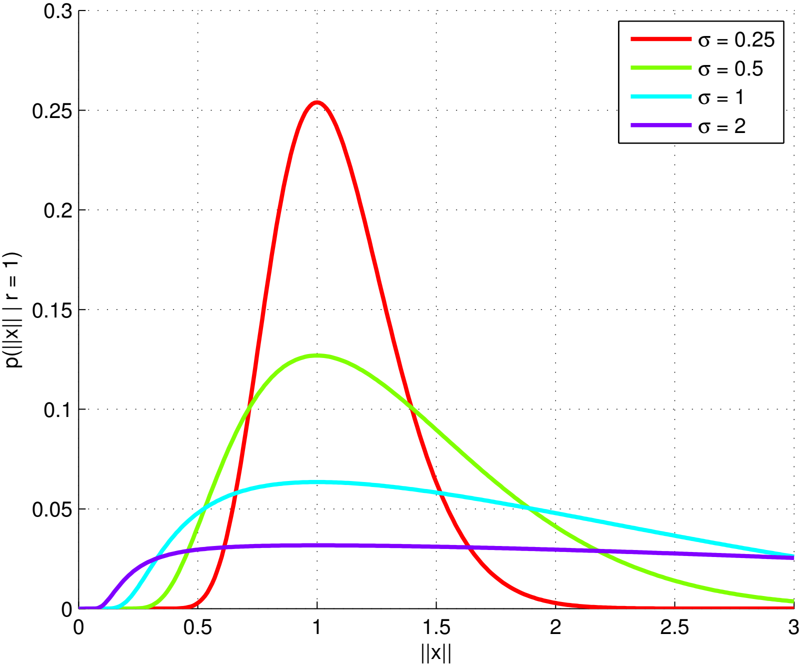


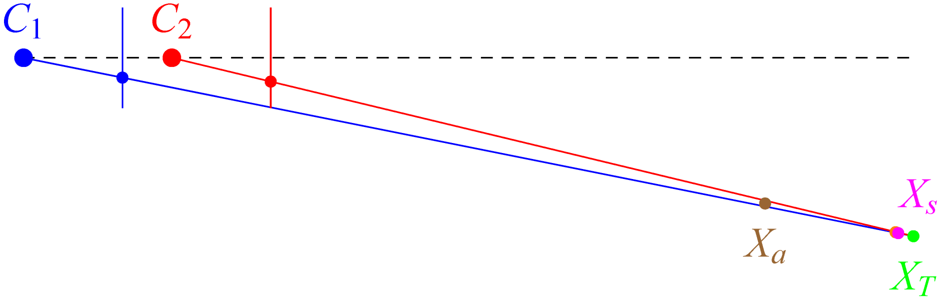


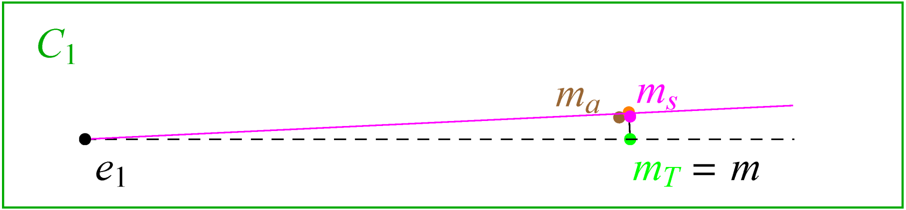

σ = 0.25
σ = 0.5
σ = 1
σ = 2
p(||x|| | r = 1)
||x||
0
0.05
0.1
0.15
0.2
0.25
0.3
0.5
1
1.5
2
2.5
3

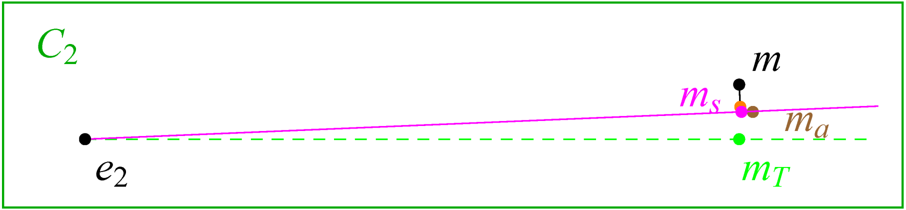


σ = 0.25
σ = 0.5
σ = 1
σ = 2
p(||x|| | r = 1)
||x||
0
0.05
0.1
0.15
0.2
0.25
0.3
0.5
1
1.5
2
2.5
3

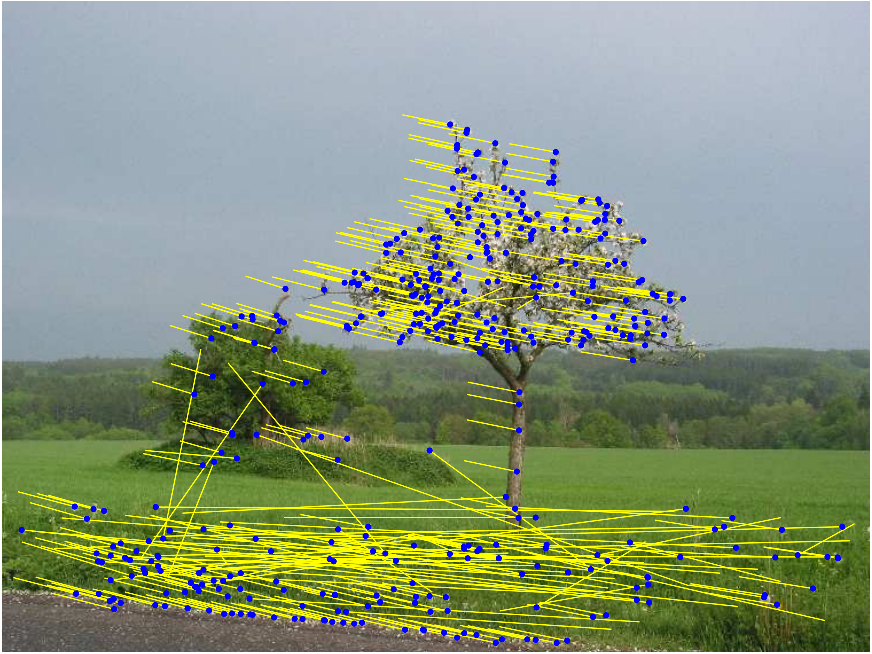

σ = 0.25
σ = 0.5
σ = 1
σ = 2
p(||x|| | r = 1)
||x||
0.3
0.25
0.2
0.15
0.1
0.05
0
0.5
1
1.5
2
2.5
3

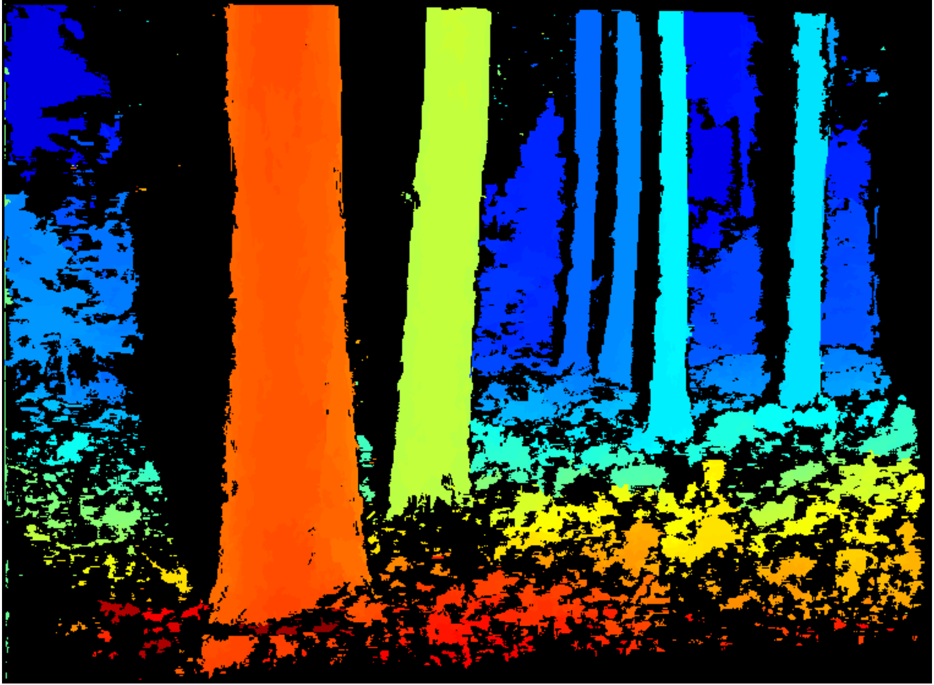

C_1
C_2
X_s
X_a
X_T

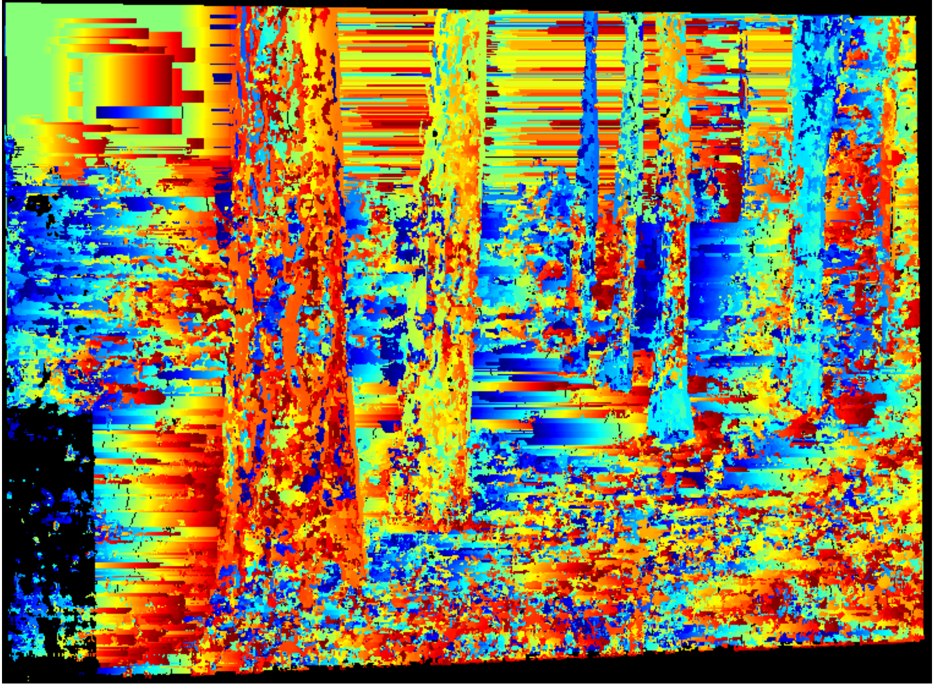

C_1
m_a
m_s
e_1
$m_T = m$

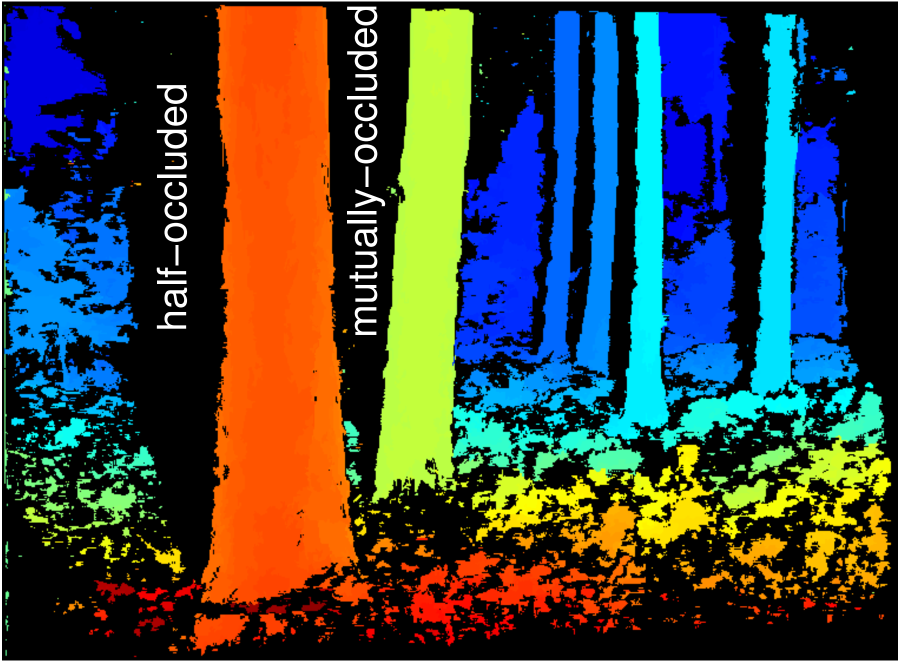


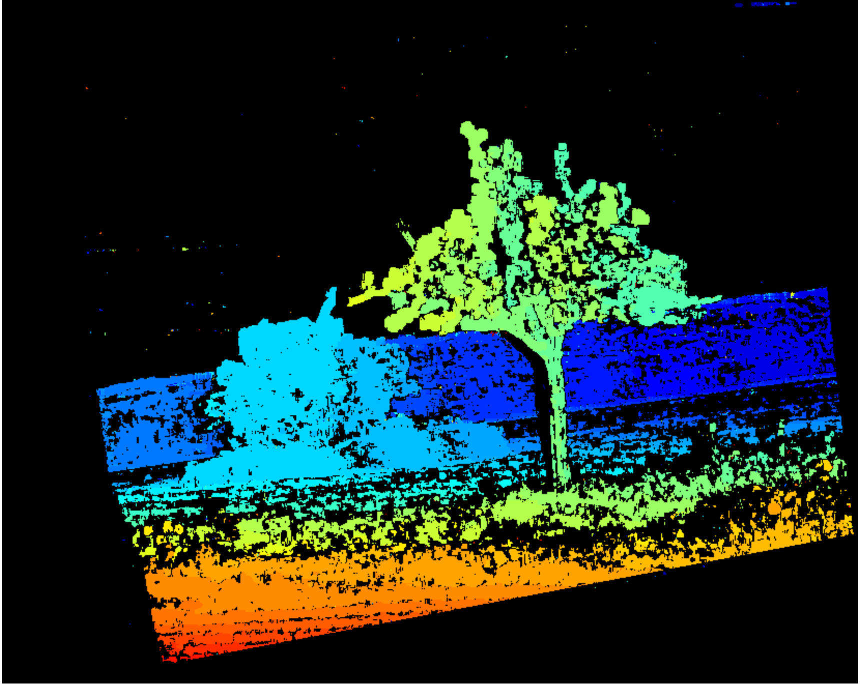
C_2
m
m_s
m_a
e_2
m_T

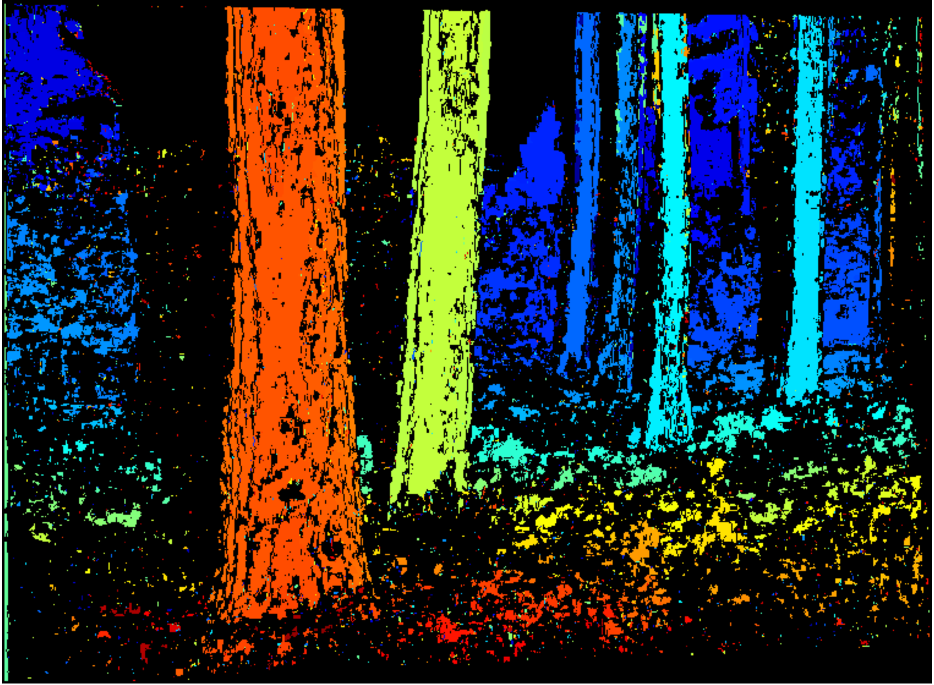


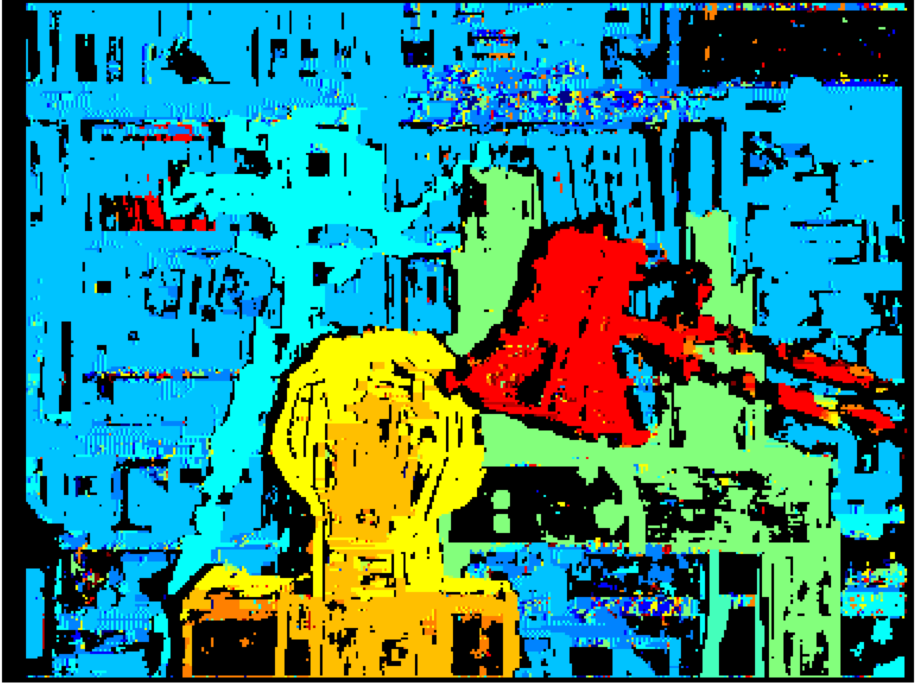


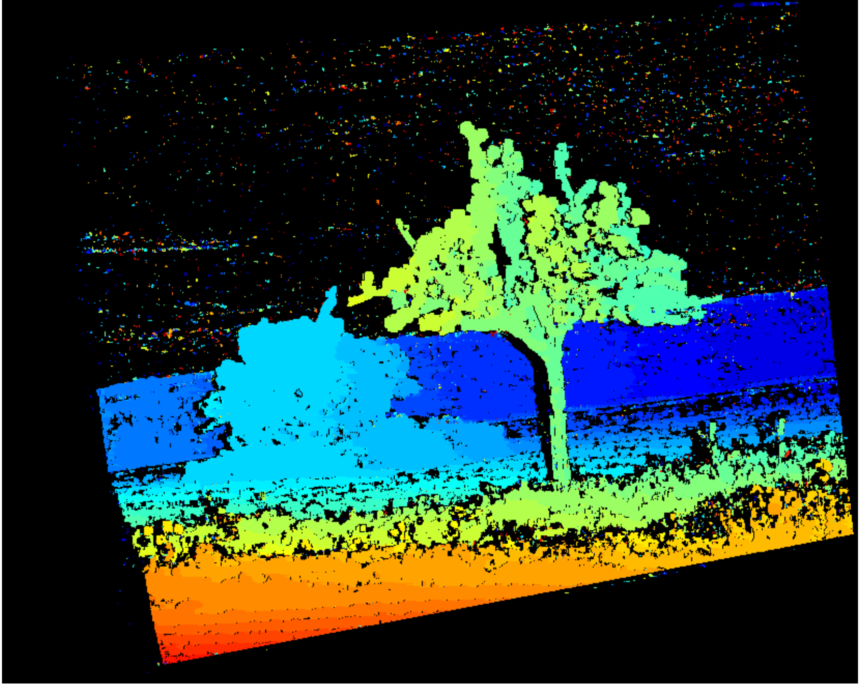


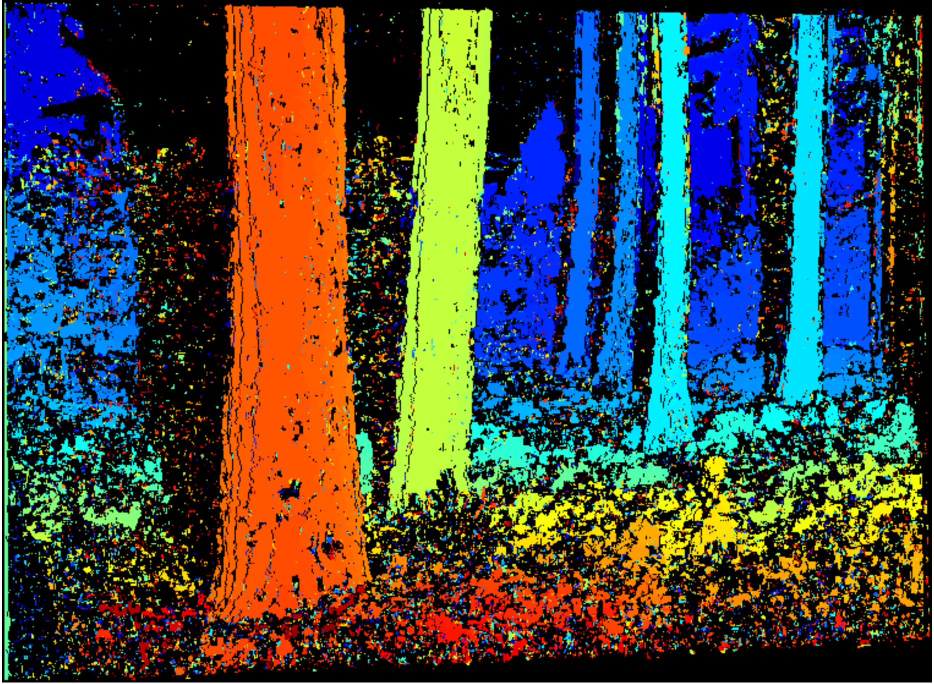

half–occluded
mutually–occluded

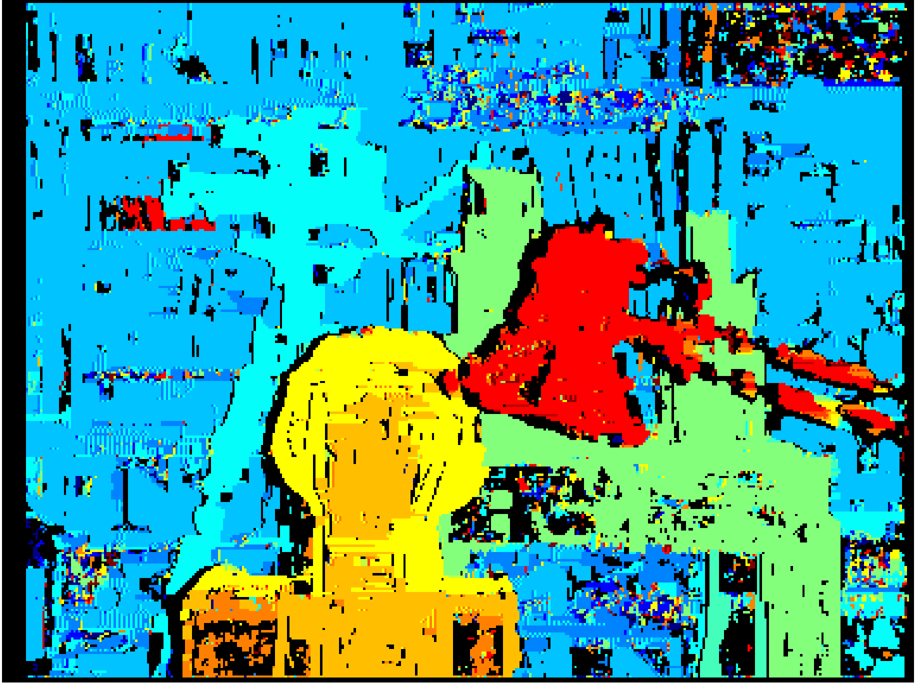


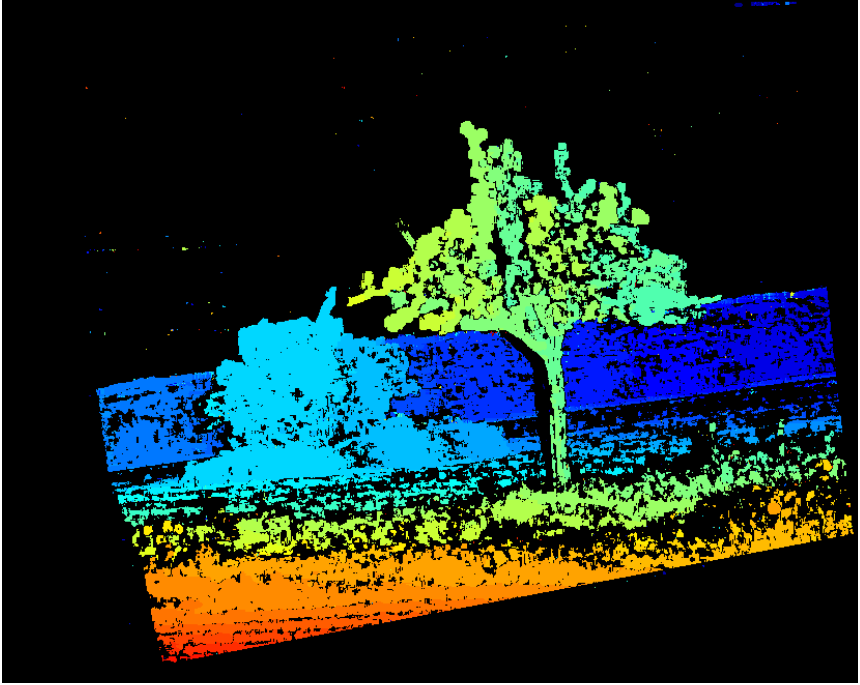


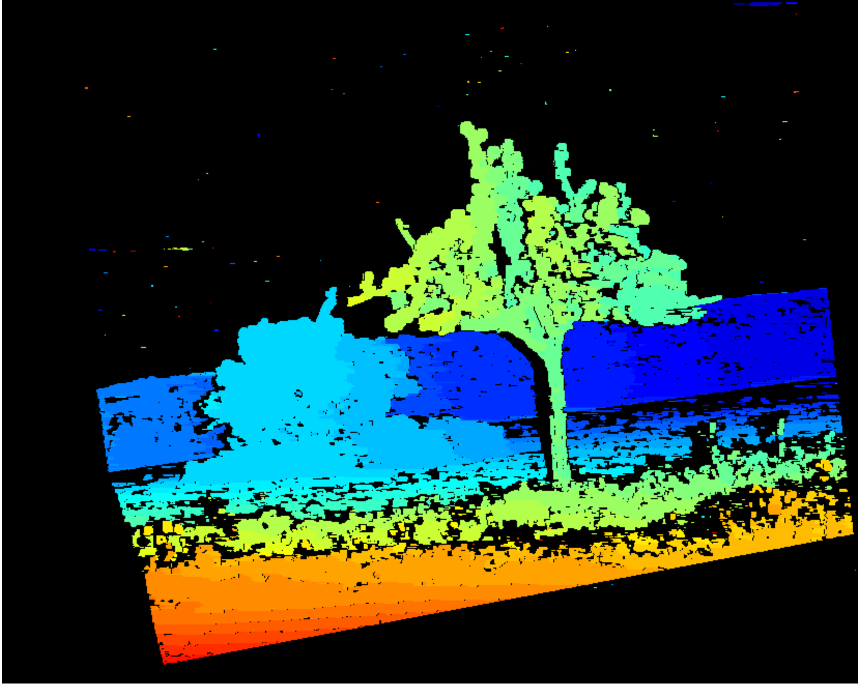


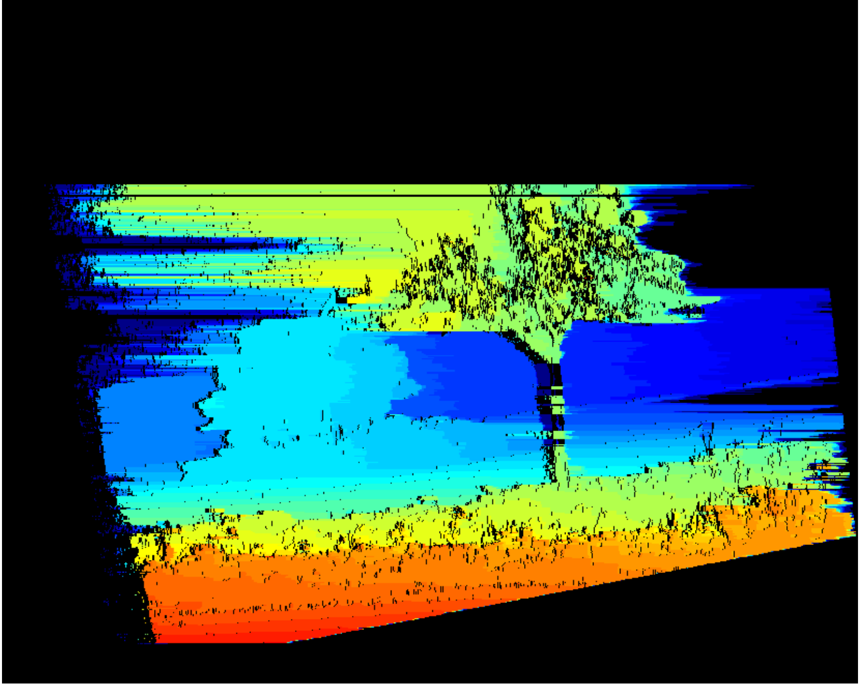


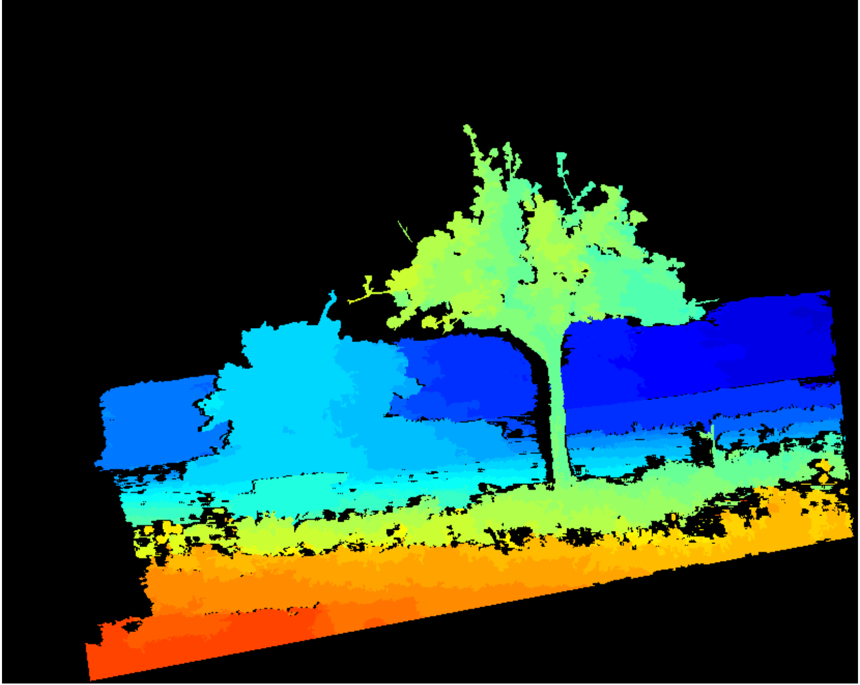


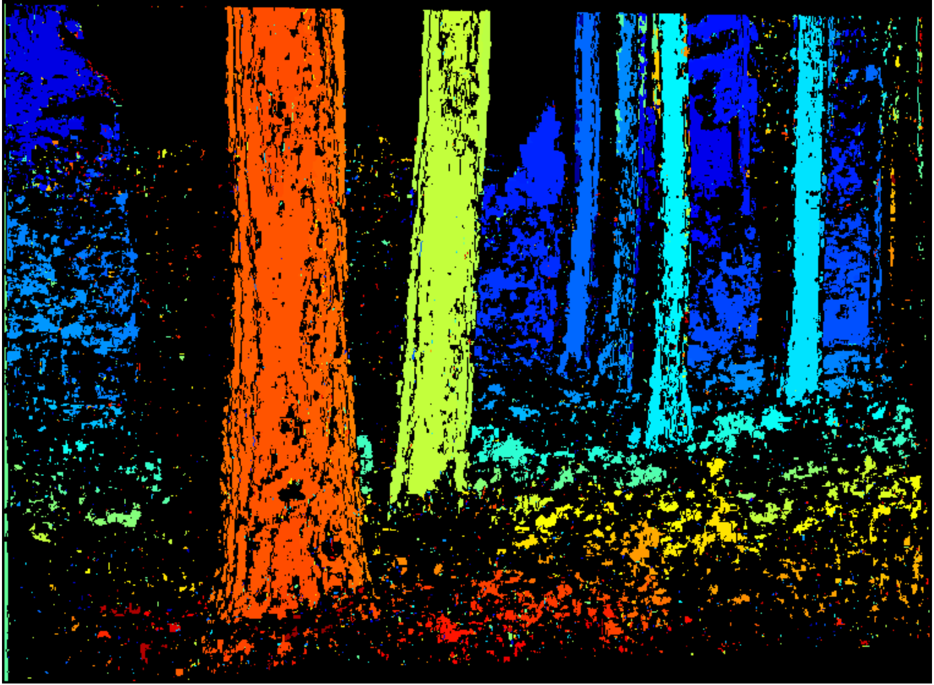


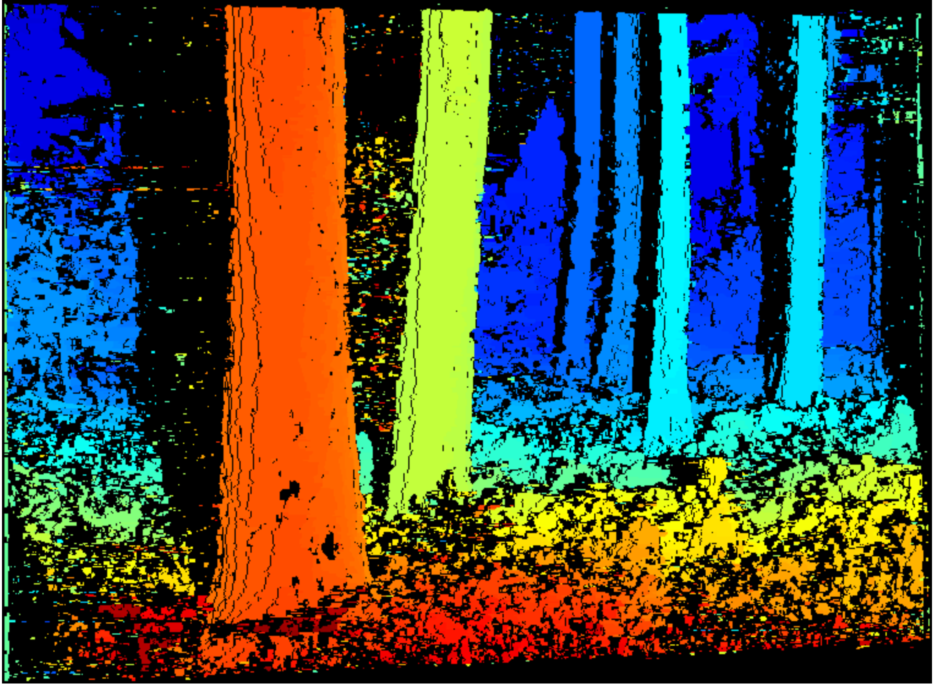


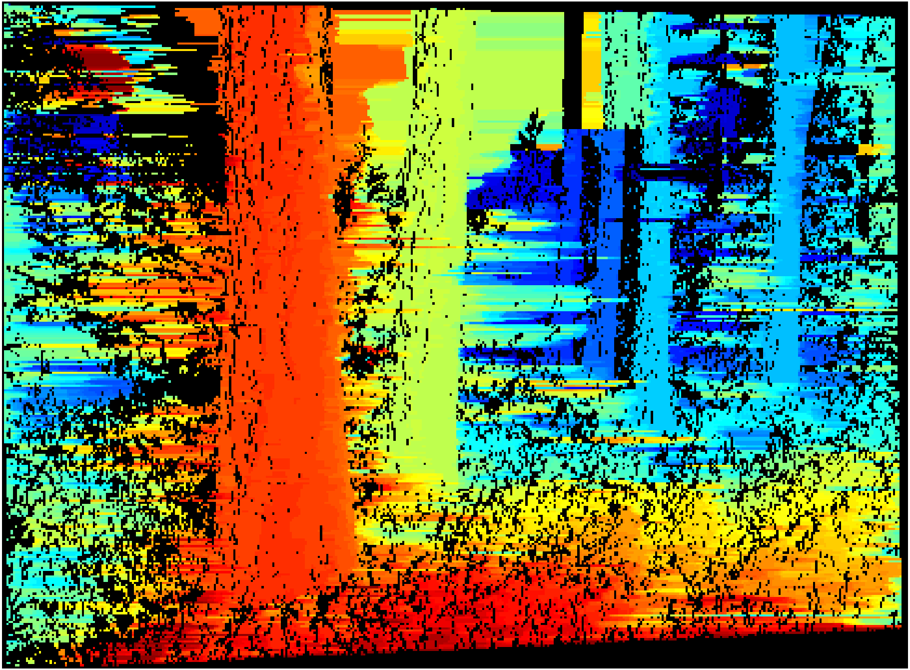


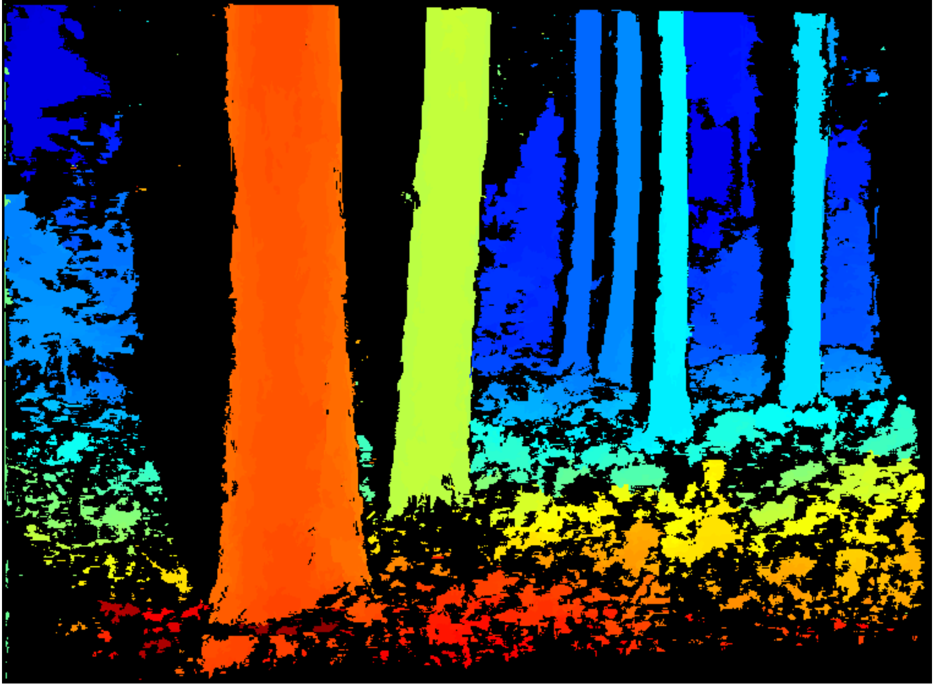


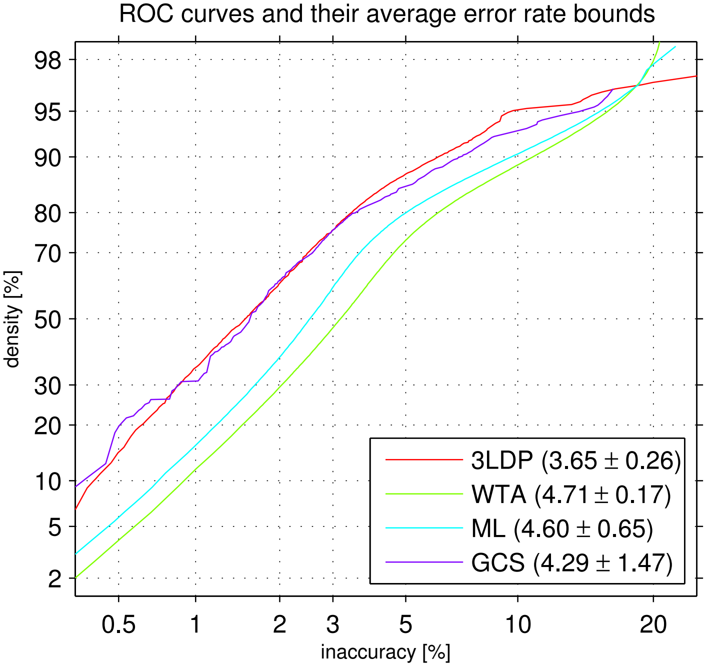











ROC curves and their average error rate bounds
density [%]
inaccuracy [%]
98
95
90
80
70
50
30
20
10
5
2
0.5
1
2
3
5
10
20
3LDP (3.65 ± 0.26)
WTA (4.71 ± 0.17)
ML (4.60 ± 0.65)
GCS (4.29 ± 1.47)