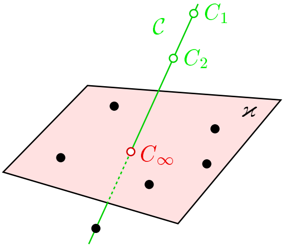
▶Degenerate (Critical) Configurations for Camera Resection

Let $\mathcal{X} = \{X_i;\, i = 1, \ldots\}$ be a set of points and $\mathbf{P}_1 \not\simeq \mathbf{P}_j$ be two regular (rank-3) cameras. Then two configurations $(\mathbf{P}_1, \mathcal{X})$ and $(\mathbf{P}_j, \mathcal{X})$ are image-equivalent if

$$\mathbf{P}_1 \underline{\mathbf{X}}_i \simeq \mathbf{P}_j \underline{\mathbf{X}}_i \quad \text{for all} \quad X_i \in \mathcal{X}$$

there is a non-trivial set of other cameras that see the same image



Case 4

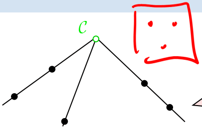
- **importantly**: If all calibration points $X_i \in \mathcal{X}$ lie on a plane $\varkappa$ then camera resection is non-unique and all image-equivalent camera centers lie on a spatial line $\mathcal{C}$ with the $C_\infty = \varkappa \cap \mathcal{C}$ excluded

  this also means we cannot resect if all X_i are infinite
- by adding points $X_i \in \mathcal{X}$ to $\mathcal{C}$ we gain nothing
- there are additional image-equivalent configurations, see next

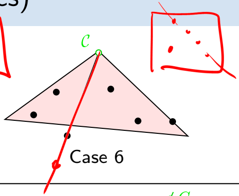
proof sketch in [H&Z, Sec. 22.1.2]

Note that if $\mathbf{Q}$, $\mathbf{T}$ are suitable homographies then $\mathbf{P}_1 \simeq \mathbf{Q}\mathbf{P}_0\mathbf{T}$, where $\mathbf{P}_0$ is canonical and the analysis can be made with $\hat{\mathbf{P}}_j \simeq \mathbf{Q}^{-1}\mathbf{P}_j$

$$\mathbf{P}_0 \underbrace{\mathbf{T}\underline{\mathbf{X}}_i}_{\underline{\mathbf{Y}}_i} \simeq \hat{\mathbf{P}}_j \underbrace{\mathbf{T}\underline{\mathbf{X}}_i}_{\underline{\mathbf{Y}}_i} \quad \text{for all} \quad Y_i \in \mathcal{Y}$$

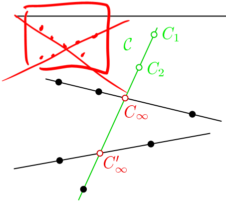


Case 5

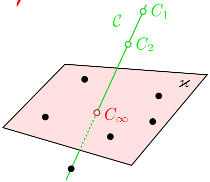


Case 6

- cameras C_1, C_2 co-located at point $\mathcal{C}$
- points on three optical rays or one optical ray and one optical plane
- Case 5: camera sees 3 isolated point images
- Case 6: cam. sees a line of points and an isolated point



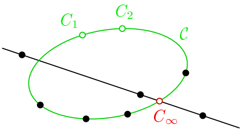
Case 3

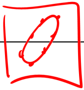


Case 4

- cameras lie on a line $\mathcal{C} \setminus \{C_\infty, C'_\infty\}$
- points lie on $\mathcal{C}$ and
  1. on two lines meeting $\mathcal{C}$ at C_∞, C'_∞
  2. or on a plane meeting $\mathcal{C}$ at C_∞
- Case 3: camera sees 2 lines of points

Case 2

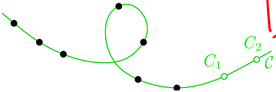




- cameras lie on a planar conic $\mathcal{C} \setminus \{C_\infty\}$

  not necessarily an ellipse
- points lie on $\mathcal{C}$ and an additional line meeting the conic at C_∞
- Case 2: camera sees 2 lines of points

Case 1



- cameras and points all lie on a twisted cubic $\mathcal{C}$
- Case 1: camera sees a conic

►Three-Point Exterior Orientation Problem (P3P)

Calibrated camera rotation and translation from Perspective images of 3 reference Points.

Problem: Given $\mathbf{K}$ and three corresponding pairs $\{(m_i,\, X_i)\}_{i=1}^{3}$, find $\mathbf{R}$, $\mathbf{C}$ by solving

$$\lambda_i \underline{\mathbf{m}}_i = \mathbf{K}\mathbf{R}\,(\mathbf{X}_i - \mathbf{C}), \qquad i = 1,2,3$$

$R^T R = I$

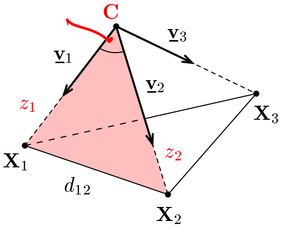
1. Transform $\underline{\mathbf{v}}_i \overset{\text{def}}{=} \mathbf{K}^{-1}\underline{\mathbf{m}}_i$. Then

$$\lambda_i \underline{\mathbf{v}}_i = \mathbf{R}\,(\mathbf{X}_i - \mathbf{C}). \tag{10}$$

2. Eliminate $\mathbf{R}$ by taking   rotation preserves length: $\|\mathbf{R}\mathbf{x}\| = \|\mathbf{x}\|$

$$|\lambda_i| \cdot \|\underline{\mathbf{v}}_i\| = \|\mathbf{X}_i - \mathbf{C}\| \overset{\text{def}}{=} z_i \tag{11}$$

configuration w/o rotation in (11)



3. Consider only angles among $\underline{\mathbf{v}}_i$ and apply Cosine Law per triangle $(\mathbf{C}, \mathbf{X}_i, \mathbf{X}_j)$ $i, j = 1, 2, 3,\ i \neq j$

$$d_{ij}^2 = z_i^2 + z_j^2 - 2\, z_i\, z_j\, c_{ij},$$

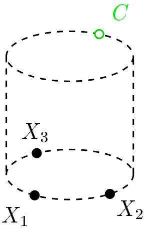
$$z_i = \|\mathbf{X}_i - \mathbf{C}\|,\ \ d_{ij} = \|\mathbf{X}_j - \mathbf{X}_i\|,\ \ c_{ij} = \cos(\angle \underline{\mathbf{v}}_i\, \underline{\mathbf{v}}_j)$$

4. Solve system of 3 quadratic eqs in 3 unknowns z_i [Fischler & Bolles, 1981]
   there may be no real root; there are up to 4 solutions that cannot be ignored   (verify on additional points)

5. Compute $\mathbf{C}$ by trilateration (3-sphere intersection) from $\mathbf{X}_i$ and z_i; then λ_i from (11) and $\mathbf{R}$ from (10)

Similar problems (P4P with unknown f) at `http://cmp.felk.cvut.cz/minimal/` (with code)

Degenerate (Critical) Configurations for Exterior Orientation



unstable solution

- center of projection C located on the orthogonal circular cylinder with base circumscribing the three points X_i

unstable: a small change of X_i results in a large change of C
can be detected by error propagation

degenerate

- camera C is coplanar with points (X_1, X_2, X_3) but is not on the circumscribed circle of (X_1, X_2, X_3) camera sees a line

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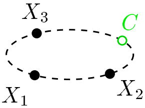


no solution

1. C cocyclic with (X_1, X_2, X_3) camera sees a line

- additional critical configurations depend on the method to solve the quadratic equations

[Haralick et al. IJCV 1994]

►Populating A Little ZOO of Minimal Geometric Problems in CV

problem	**given**	**unknown**	slide
camera resection	6 world–img correspondences $\{(X_i, m_i)\}_{i=1}^{6}$	$\mathbf{P}$	64
exterior orientation	$\mathbf{K}$, 3 world–img correspondences $\{(X_i, m_i)\}_{i=1}^{3}$	$\mathbf{R}$, $\mathbf{C}$	68

- camera resection and exterior orientation are similar problems in a sense:
  - we do resectioning when our camera is uncalibrated
  - we do orientation when our camera is calibrated
- more problems to come

Part IV

Computing with a Camera Pair

4.1 Camera Motions Inducing Epipolar Geometry

4.2 Estimating Fundamental Matrix from 7 Correspondences

4.3 Estimating Essential Matrix from 5 Correspondences

4.4 Triangulation: 3D Point Position from a Pair of Corresponding Points

covered by

[1] [H&Z] Secs: 9.1, 9.2, 9.6, 11.1, 11.2, 11.9, 12.2, 12.3, 12.5.1

[2] H. Li and R. Hartley. Five-point motion estimation made easy. In *Proc ICPR* 2006, pp. 630–633

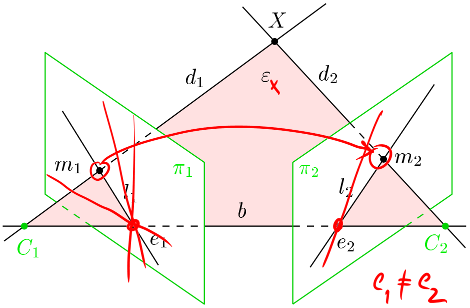
additional references

H. Longuet-Higgins. A computer algorithm for reconstructing a scene from two projections. *Nature*, 293 (5828):133–135, 1981.

▶Geometric Model of a Camera Pair

Epipolar geometry:

- brings constraints necessary for inter-image matching
- its parametric form encapsulates information about the relative pose of two cameras



two-camera setup

Description

- baseline b joins projection centers C_1, C_2

$$\mathbf{b} = \mathbf{C}_2 - \mathbf{C}_1$$

- epipole $e_i \in \pi_i$ is the image of C_j:

$$\underline{\mathbf{e}}_1 \simeq \mathbf{P}_1 \underline{\mathbf{C}}_2, \quad \underline{\mathbf{e}}_2 \simeq \mathbf{P}_2 \underline{\mathbf{C}}_1$$

- $l_i \in \pi_i$ is the image of epipolar plane

$$\varepsilon = (C_2, X, C_1)$$

- l_j is the epipolar line in image π_j induced by m_i in image π_i

Epipolar constraint: corresponding d_2, b, d_1 are coplanar a necessary condition, see →87

Epipolar Geometry Example: Forward Motion

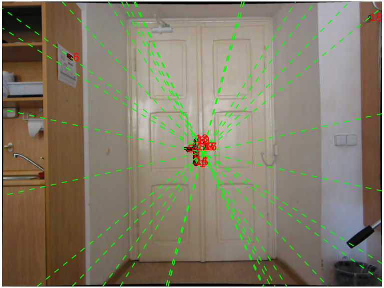


image 1

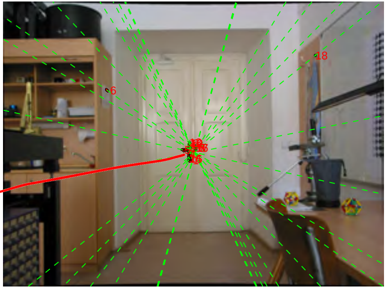
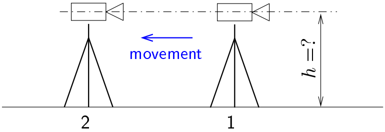


image 2

- red: correspondences
- green: epipolar line pairs per correspondence

click on the image to see their IDs
same ID in both images

How high was the camera above the floor?



▶Cross Products and Maps by Skew-Symmetric 3×3 Matrices

- There is an equivalence $\mathbf{b} \times \mathbf{m} = [\mathbf{b}]_\times \mathbf{m}$, where $[\mathbf{b}]_\times$ is a 3×3 skew-symmetric matrix

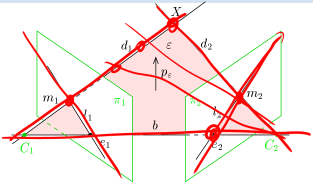
$$[\mathbf{b}]_\times = \begin{bmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{bmatrix}, \qquad \text{assuming} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Some properties

1. $[\mathbf{b}]_\times^\top = -[\mathbf{b}]_\times$ — the general antisymmetry property
2. $\mathbf{A}$ is skew-symmetric iff $\mathbf{x}^\top \mathbf{A} \mathbf{x} = 0$ for all $\mathbf{x}$ — skew-sym mtx generalizes cross products
3. $[\mathbf{b}]_\times^3 = -\|\mathbf{b}\|^2 \cdot [\mathbf{b}]_\times$
4. $\|[\mathbf{b}]_\times\|_F = \sqrt{2}\,\|\mathbf{b}\|$ — Frobenius norm ($\|\mathbf{A}\|_F = \sqrt{\mathrm{tr}(\mathbf{A}^\top \mathbf{A})} = \sqrt{\sum_{i,j} |a_{ij}|^2}$)
5. $[\mathbf{b}]_\times \mathbf{b} = \mathbf{0}$ $\quad b \times b = 0$
6. $\mathrm{rank}\,[\mathbf{b}]_\times = 2$ iff $\|\mathbf{b}\| > 0$ — check minors of $[\mathbf{b}]_\times$
7. eigenvalues of $[\mathbf{b}]_\times$ are $(0, \lambda, -\lambda)$
8. for any regular $\mathbf{B}$: $[\mathbf{B}\mathbf{z}]_\times \mathbf{B} = \det \mathbf{B} \cdot \mathbf{B}^{-\top} [\mathbf{z}]_\times$ — follows from the factoring on →40
9. special case: if $\mathbf{R}\mathbf{R}^\top = \mathbf{I}$ then $[\mathbf{R}\mathbf{b}]_\times = \mathbf{R}\,[\mathbf{b}]_\times\,\mathbf{R}^\top$

- note that if $\mathbf{R}_b$ is rotation about $\mathbf{b}$ then $\mathbf{R}_b \mathbf{b} = \mathbf{b}$
- note $[\mathbf{b}]_\times$ is not a homography; it is not a rotation matrix — it is a logarithm of a rotation mtx

▶Expressing Epipolar Constraint Algebraically



$$\mathbf{P}_i = \begin{bmatrix}\mathbf{Q}_i & \mathbf{q}_i\end{bmatrix} = \mathbf{K}_i \begin{bmatrix}\mathbf{R}_i & \mathbf{t}_i\end{bmatrix}, \; i = 1, 2$$

$\mathbf{R}_{21}$ – relative camera rotation, $\mathbf{R}_{21} = \mathbf{R}_2\mathbf{R}_1^\top$

$\mathbf{t}_{21}$ – relative camera translation, $\mathbf{t}_{21} = \mathbf{t}_2 - \mathbf{R}_{21}\mathbf{t}_1 = -\mathbf{R}_2\mathbf{b}$

$\mathbf{b}$ – baseline (world coordinate system)

remember: $\mathbf{C} = -\mathbf{Q}^{-1}\mathbf{q} = -\mathbf{R}^\top\mathbf{t}$ →34 and 36

$$0 = \mathbf{d}_2^\top \underbrace{\mathbf{p}_\varepsilon}_{\text{normal of } \varepsilon} \simeq \underbrace{(\mathbf{Q}_2^{-1}\underline{\mathbf{m}}_2)^\top}_{\text{optical ray}} \underbrace{\mathbf{Q}_1^\top \mathbf{l}_1}_{\text{optical plane}} = \underline{\mathbf{m}}_2^\top \underbrace{\mathbf{Q}_2^{-\top}\mathbf{Q}_1^\top(\underline{\mathbf{e}}_1 \times \underline{\mathbf{m}}_1)}_{\text{image of } \varepsilon \text{ in } \pi_2} = \underline{\mathbf{m}}_2^\top \underbrace{(\mathbf{Q}_2^{-\top}\mathbf{Q}_1^\top[\underline{\mathbf{e}}_1]_\times)}_{\text{fundamental matrix } \mathbf{F}} \underline{\mathbf{m}}_1$$

Epipolar constraint $\underline{\mathbf{m}}_2^\top \mathbf{F}\, \underline{\mathbf{m}}_1 = 0$ is a point-line incidence constraint

- point $\underline{\mathbf{m}}_2$ is incident on epipolar line $\mathbf{l}_2 \simeq \mathbf{F}\underline{\mathbf{m}}_1$
- point $\underline{\mathbf{m}}_1$ is incident on epipolar line $\mathbf{l}_1 \simeq \mathbf{F}^\top\underline{\mathbf{m}}_2$
- $\mathbf{F}\underline{\mathbf{e}}_1 = \mathbf{F}^\top\underline{\mathbf{e}}_2 = \mathbf{0}$ (non-trivially)
- all epipolars meet at the epipole

$$\underline{\mathbf{e}}_1 \simeq \mathbf{Q}_1\mathbf{C}_2 + \mathbf{q}_1 = \mathbf{Q}_1\mathbf{C}_2 - \mathbf{Q}_1\mathbf{C}_1 = \mathbf{K}_1\mathbf{R}_1\mathbf{b} = -\mathbf{K}_1\mathbf{R}_1\mathbf{R}_2^\top\mathbf{t}_{21} = -\mathbf{K}_1\mathbf{R}_{21}^\top\mathbf{t}_{21}$$

$$\mathbf{F} = \mathbf{Q}_2^{-\top}\mathbf{Q}_1^\top[\underline{\mathbf{e}}_1]_\times = \mathbf{Q}_2^{-\top}\mathbf{Q}_1^\top[\mathbf{K}_1\mathbf{R}_1\mathbf{b}]_\times \overset{\circledast 1}{=} \cdots \simeq \mathbf{K}_2^{-\top}[-\mathbf{t}_{21}]_\times\mathbf{R}_{21}\mathbf{K}_1^{-1}$$ fundamental

$$\mathbf{E} = [-\mathbf{t}_{21}]_\times\mathbf{R}_{21} = \underbrace{[\mathbf{R}_2\mathbf{b}]_\times}_{\text{baseline in Cam 2}}\mathbf{R}_{21} = \mathbf{R}_{21}\underbrace{[\mathbf{R}_1\mathbf{b}]_\times}_{\text{baseline in Cam 1}} = \mathbf{R}_{21}[-\mathbf{R}_{21}\mathbf{t}_{21}]_\times$$ essential

►The Structure and the Key Properties of the Fundamental Matrix

$l_L = H^{-\top} l_1 \qquad H^{-\top}(e_1 \times m_1)$

$$\mathbf{F} = \underbrace{(\mathbf{Q}_2\mathbf{Q}_1^{-1})}_{\text{epipolar homography } \mathbf{H}}{}^{-\top}[\mathbf{e}_1]_\times = \underbrace{\mathbf{K}_2^{-\top}\mathbf{R}_{21}\mathbf{K}_1^{\top}}_{\mathbf{H}^{-\top}}[\overbrace{\mathbf{e}_1}^{\text{left epipole}}]_\times \simeq [\overbrace{\mathbf{H}\mathbf{e}_1}^{\text{right epipole}}]_\times \mathbf{H} = \mathbf{K}_2^{-\top}\underbrace{[-\mathbf{t}_{21}]_\times \mathbf{R}_{21}}_{\text{essential matrix } \mathbf{E}}\mathbf{K}_1^{-1}$$

1. $\mathbf{E}$ captures relative camera pose only [Longuet-Higgins 1981]
   (the change of the world coordinate system does not change $\mathbf{E}$)

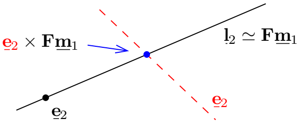
$$\begin{bmatrix}\mathbf{R}'_i & \mathbf{t}'_i\end{bmatrix} = \begin{bmatrix}\mathbf{R}_i & \mathbf{t}_i\end{bmatrix} \cdot \begin{bmatrix}\mathbf{R} & \mathbf{t} \\ \mathbf{0}^\top & 1\end{bmatrix} = \begin{bmatrix}\mathbf{R}_i\mathbf{R} & \mathbf{R}_i\mathbf{t} + \mathbf{t}_i\end{bmatrix},$$

   then

$$\mathbf{R}'_{21} = \mathbf{R}'_2\mathbf{R}'^{\top}_1 = \cdots = \mathbf{R}_{21}$$
$$\mathbf{t}'_{21} = \mathbf{t}'_2 - \mathbf{R}'_{21}\mathbf{t}'_1 = \cdots = \mathbf{t}_{21}$$

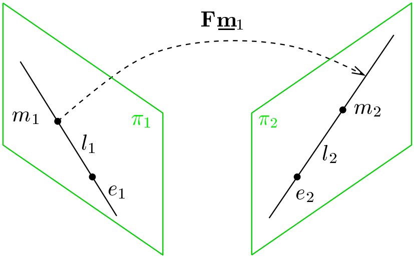
$l_1 \simeq (F^\top [e_2]_\times)\, l_2$

2. the translation length $\mathbf{t}_{21}$ is lost since $\mathbf{E}$ is homogeneous
3. $\mathbf{F}$ maps points to lines and it is not a homography
4. $\underline{\mathbf{e}}_2 \times (\underline{\mathbf{e}}_2 \times \mathbf{F}\underline{\mathbf{m}}_1) \simeq \mathbf{F}\underline{\mathbf{m}}_1$, in general $\mathbf{F} \simeq [\underline{\mathbf{e}}_2]_\times^{2a}\mathbf{F}\,[\underline{\mathbf{e}}_1]_\times^{2b}$ for any $a, b \in \mathbb{N}$



- by point/line 'transmutation' (left)
- point $\underline{\mathbf{e}}_2$ does not lie on line $\underline{\mathbf{e}}_2$ (dashed): $\underline{\mathbf{e}}_2^\top\underline{\mathbf{e}}_2 \neq 0$
- application: $\mathbf{F}^\top(\underline{\mathbf{e}}_2 \times \underline{\mathbf{l}}_2) \simeq \mathbf{F}^\top(\underline{\mathbf{e}}_2 \times \mathbf{F}\underline{\mathbf{m}}_1) \simeq \mathbf{F}^\top\underline{\mathbf{m}}_2 \simeq \underline{\mathbf{l}}_1$
- $\mathbf{F}^\top[\underline{\mathbf{e}}_2]_\times$ maps epipolar lines to epi. lines but it is not a homography

►Some Mappings by the Fundamental Matrix

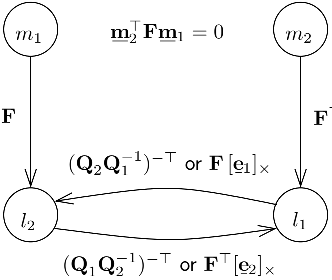


$$0 = \underline{\mathbf{m}}_2^\top \mathbf{F}\, \underline{\mathbf{m}}_1$$

$$\underline{\mathbf{e}}_1 \simeq \mathrm{null}(\mathbf{F}), \qquad \underline{\mathbf{e}}_2 \simeq \mathrm{null}(\mathbf{F}^\top)$$

$$\underline{\mathbf{l}}_2 = \mathbf{F}\underline{\mathbf{m}}_1 \qquad \underline{\mathbf{l}}_1 = \mathbf{F}^\top \underline{\mathbf{m}}_2$$

$$\underline{\mathbf{l}}_2 = \mathbf{F}[\underline{\mathbf{e}}_1]_\times \underline{\mathbf{l}}_1 \qquad \underline{\mathbf{l}}_1 = \mathbf{F}^\top [\underline{\mathbf{e}}_2]_\times \underline{\mathbf{l}}_2$$

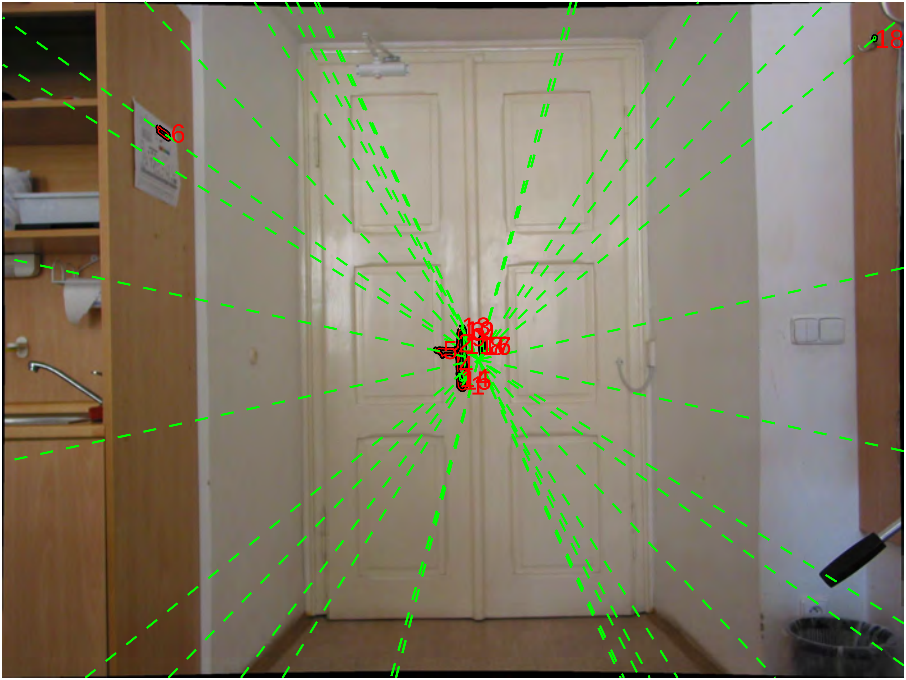


- $\underline{\mathbf{l}}_2 \simeq \mathbf{F}\,[\underline{\mathbf{e}}_1]_\times\, \underline{\mathbf{l}}_1$: by 'transmutation' →78
- $\mathbf{F}[\underline{\mathbf{e}}_1]_\times$ maps lines to lines but it is not a homography
- $\mathbf{H} = \mathbf{Q}_2\mathbf{Q}_1^{-1}$ is the epipolar homography→78 mapping epipolar lines to epipolar lines, hence

$$\mathbf{H} = \mathbf{Q}_2\mathbf{Q}_1^{-1} = \mathbf{K}_2\mathbf{R}_{21}\mathbf{K}_1^{-1}$$

you have seen this →61

Thank You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