

► Three-Point Exterior Orientation Problem (P3P)

Calibrated camera rotation and translation from Perspective images of 3 reference Points.

Problem: Given \mathbf{K} and three corresponding pairs $\{(m_i, X_i)\}_{i=1}^3$, find \mathbf{R} , \mathbf{C} by solving

$$\lambda_i \underline{\mathbf{m}}_i = \mathbf{K}\mathbf{R}(\mathbf{X}_i - \mathbf{C}), \quad i = 1, 2, 3$$

1. Transform $\underline{\mathbf{v}}_i \stackrel{\text{def}}{=} \mathbf{K}^{-1}\underline{\mathbf{m}}_i$. Then

$$\lambda_i \underline{\mathbf{v}}_i = \mathbf{R}(\mathbf{X}_i - \mathbf{C}). \quad (10)$$

2. Eliminate \mathbf{R} by taking rotation preserves length: $\|\mathbf{R}\mathbf{x}\| = \|\mathbf{x}\|$

$$|\lambda_i| \cdot \|\underline{\mathbf{v}}_i\| = \|\mathbf{X}_i - \mathbf{C}\| \stackrel{\text{def}}{=} z_i \quad (11)$$

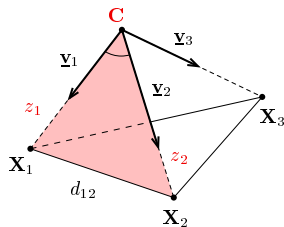
3. Consider only angles among $\underline{\mathbf{v}}_i$ and apply Cosine Law per triangle $(\mathbf{C}, \mathbf{X}_i, \mathbf{X}_j)$ $i, j = 1, 2, 3, i \neq j$

$$d_{ij}^2 = z_i^2 + z_j^2 - 2 z_i z_j c_{ij},$$

$$z_i = \|\mathbf{X}_i - \mathbf{C}\|, \quad d_{ij} = \|\mathbf{X}_j - \mathbf{X}_i\|, \quad c_{ij} = \cos(\angle \underline{\mathbf{v}}_i \underline{\mathbf{v}}_j)$$

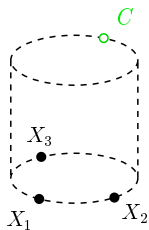
4. Solve system of 3 quadratic eqs in 3 unknowns z_i [Fischler & Bolles, 1981]
there may be no real root; there are up to 4 solutions that cannot be ignored (verify on additional points)
5. Compute \mathbf{C} by trilateration (3-sphere intersection) from \mathbf{X}_i and z_i ; then λ_i from (11) and \mathbf{R} from (10)

configuration w/o rotation in (11)



Similar problems (P4P with unknown f) at <http://cmp.felk.cvut.cz/minimal/> (with code)

Degenerate (Critical) Configurations for Exterior Orientation



unstable solution

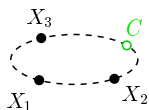
- center of projection C located on the orthogonal circular cylinder with base circumscribing the three points X_i

unstable: a small change of X_i results in a large change of C
can be detected by error propagation

degenerate

- camera C is coplanar with points (X_1, X_2, X_3) but is not on the circumscribed circle of (X_1, X_2, X_3)

camera sees point on a line



no solution

- C cocyclic with (X_1, X_2, X_3) camera sees points on a line

- additional critical configurations depend on the method to solve the quadratic equations

[Haralick et al. IJCV 1994]

► Populating A Little ZOO of Minimal Geometric Problems in CV

problem	given	unknown	slide
camera resection	6 world–img correspondences $\{(X_i, m_i)\}_{i=1}^6$	P	62
exterior orientation	K , 3 world–img correspondences $\{(X_i, m_i)\}_{i=1}^3$	R, C	66
relative orientation	3 world–world correspondences $\{(X_i, Y_i)\}_{i=1}^3$	R, t	69

- camera resection and exterior orientation are similar problems in a sense:
 - we do resectioning when our camera is uncalibrated
 - we do orientation when our camera is calibrated
- relative orientation involves no camera (see next)
- more problems to come

The Relative Orientation Problem

Problem: Given two point triples (X_1, X_2, X_3) and (Y_1, Y_2, Y_3) in a general position in \mathbf{R}^3 such that the correspondence $X_i \leftrightarrow Y_i$ is known, determine the relative orientation (\mathbf{R}, \mathbf{t}) that maps \mathbf{X}_i to \mathbf{Y}_i , i.e.

$$\mathbf{Y}_i = \mathbf{R}\mathbf{X}_i + \mathbf{t}, \quad i = 1, 2, 3.$$

Applies to:

- 3D scanners
- partial reconstructions from different viewpoints

Obs: Let the centroid be $\bar{\mathbf{X}} = \frac{1}{3} \sum_i \mathbf{X}_i$ and analogically for $\bar{\mathbf{Y}}$. Then

$$\bar{\mathbf{Y}} = \mathbf{R}\bar{\mathbf{X}} + \mathbf{t}.$$

Therefore

$$\mathbf{Z}_i \stackrel{\text{def}}{=} (\mathbf{Y}_i - \bar{\mathbf{Y}}) = \mathbf{R}(\mathbf{X}_i - \bar{\mathbf{X}}) \stackrel{\text{def}}{=} \mathbf{R}\mathbf{W}_i$$

If all dot products are equal, $\mathbf{Z}_i^\top \mathbf{Z}_j = \mathbf{W}_i^\top \mathbf{W}_j$ for $i, j = 1, 2, 3$, we have

$$\mathbf{R}^* = [\mathbf{W}_1 \quad \mathbf{W}_2 \quad \mathbf{W}_3]^{-1} [\mathbf{Z}_1 \quad \mathbf{Z}_2 \quad \mathbf{Z}_3]$$

Otherwise (in practice) we setup a minimization problem

$$\mathbf{R}^* = \arg \min_{\mathbf{R}} \sum_i \|\mathbf{Z}_i - \mathbf{R}\mathbf{W}_i\|^2 \quad \text{s.t.} \quad \mathbf{R}^\top \mathbf{R} = \mathbf{I}$$

$$\min_{\mathbf{R}} \sum_i \|\mathbf{Z}_i - \mathbf{R}\mathbf{W}_i\|^2 = \min_{\mathbf{R}} \sum_i \left(\|\mathbf{Z}_i\|^2 - 2\mathbf{Z}_i^\top \mathbf{R}\mathbf{W}_i + \|\mathbf{W}_i\|^2 \right) = \dots = \max_{\mathbf{R}} \sum_i \mathbf{Z}_i^\top \mathbf{R}\mathbf{W}_i$$

cont'd (What is Linear Algebra Telling Us?)

Obs 1: Let $\mathbf{A} : \mathbf{B} = \sum_{i,j} a_{ij}b_{ij}$ be the dot-product (Frobenius inner product) over real matrices. Then

$$\mathbf{A} : \mathbf{B} = \mathbf{B} : \mathbf{A} = \text{tr}(\mathbf{A}^\top \mathbf{B})$$

Obs 2: (cyclic property for matrix trace)

$$\text{tr}(\mathbf{ABC}) = \text{tr}(\mathbf{CAB})$$

Obs 3: ($\mathbf{Z}_i, \mathbf{W}_i$ are vectors)

$$\mathbf{Z}_i^\top \mathbf{R} \mathbf{W}_i = \text{tr}(\mathbf{Z}_i^\top \mathbf{R} \mathbf{W}_i) = \text{tr}(\mathbf{W}_i \mathbf{Z}_i^\top \mathbf{R}) = (\mathbf{Z}_i \mathbf{W}_i^\top) : \mathbf{R} = \mathbf{R} : (\mathbf{Z}_i \mathbf{W}_i^\top)$$

Let the SVD be

$$\sum_i \mathbf{Z}_i \mathbf{W}_i^\top \stackrel{\text{def}}{=} \mathbf{M} = \mathbf{U} \mathbf{D} \mathbf{V}^\top$$

Then

$$\mathbf{R} : \mathbf{M} = \mathbf{R} : (\mathbf{U} \mathbf{D} \mathbf{V}^\top) = \text{tr}(\mathbf{R}^\top \mathbf{U} \mathbf{D} \mathbf{V}^\top) = \text{tr}(\mathbf{V}^\top \mathbf{R}^\top \mathbf{U} \mathbf{D}) = (\mathbf{U}^\top \mathbf{R} \mathbf{V}) : \mathbf{D}$$

We are solving

$$\mathbf{R}^* = \arg \max_{\mathbf{R}} \sum_i \mathbf{z}_i^\top \mathbf{R} \mathbf{W}_i = \arg \max_{\mathbf{R}} \left(\mathbf{U}^\top \mathbf{R} \mathbf{V} \right) : \mathbf{D}$$

- It follows that $\mathbf{U}^\top \mathbf{R} \mathbf{V}$ must be (1) orthogonal, (2) diagonal, (3) positive definite
- Since \mathbf{U} , \mathbf{V} are orthogonal matrices then the solution to the problem among $\mathbf{R}^* = \mathbf{U} \mathbf{S} \mathbf{V}^\top$, where \mathbf{S} is diagonal and orthogonal, i.e. one of

$$\pm \text{diag}(1, 1, 1), \quad \pm \text{diag}(1, -1, -1), \quad \pm \text{diag}(-1, 1, -1), \quad \pm \text{diag}(-1, -1, 1)$$

- $\mathbf{U}^\top \mathbf{V}$ is not necessarily positive definite
- We choose \mathbf{S} so that $(\mathbf{R}^*)^\top \mathbf{R}^* = \mathbf{I}$

Alg:

1. Compute matrix $\mathbf{M} = \sum_i \mathbf{z}_i \mathbf{W}_i^\top$.
2. Compute SVD $\mathbf{M} = \mathbf{U} \mathbf{D} \mathbf{V}^\top$.
3. Compute all $\mathbf{R}_k = \mathbf{U} \mathbf{S}_k \mathbf{V}^\top$ that give $\mathbf{R}_k^\top \mathbf{R}_k = \mathbf{I}$.
4. Compute $\mathbf{t}_k = \bar{\mathbf{Y}} - \mathbf{R}_k \bar{\mathbf{X}}$.

- The algorithm can be used for more than 3 points
- Triple pairs can be pre-filtered based on motion invariants (lengths, angles)
- The P3P problem is very similar but not identical

Computing with a Camera Pair

- 4.1 Camera Motions Inducing Epipolar Geometry
- 4.2 Estimating Fundamental Matrix from 7 Correspondences
- 4.3 Estimating Essential Matrix from 5 Correspondences
- 4.4 Triangulation: 3D Point Position from a Pair of Corresponding Points

covered by

- [1] [H&Z] Secs: 9.1, 9.2, 9.6, 11.1, 11.2, 11.9, 12.2, 12.3, 12.5.1
- [2] H. Li and R. Hartley. Five-point motion estimation made easy. In *Proc ICPR 2006*, pp. 630–633

additional references

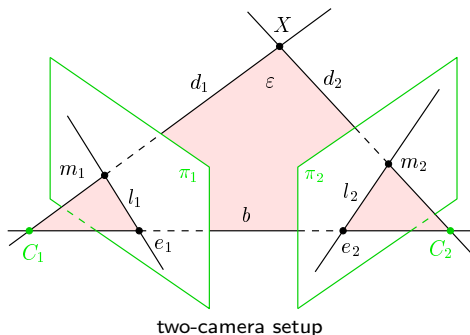


H. Longuet-Higgins. A computer algorithm for reconstructing a scene from two projections. *Nature*, 293 (5828):133–135, 1981.

► Geometric Model of a Camera Pair

Epipolar geometry:

- brings constraints necessary for inter-image matching
- its parametric form encapsulates information about the relative pose of two cameras



Description

- baseline b joins projection centers C_1, C_2
 $\mathbf{b} = \mathbf{C}_2 - \mathbf{C}_1$
- epipole $e_i \in \pi_i$ is the image of C_j :
 $\mathbf{e}_1 \simeq \mathbf{P}_1 \mathbf{C}_2, \quad \mathbf{e}_2 \simeq \mathbf{P}_2 \mathbf{C}_1$
- $l_i \in \pi_i$ is the image of epipolar plane
 $\varepsilon = (C_2, X, C_1)$
- l_j is the epipolar line in image π_j induced by m_i in image π_i

Epipolar constraint: corresponding d_2, b, d_1 are coplanar

a necessary condition $\rightarrow 86$

$$\mathbf{P}_i = [\mathbf{Q}_i \quad \mathbf{q}_i] = \mathbf{K}_i [\mathbf{R}_i \quad \mathbf{t}_i] = \mathbf{K}_i \mathbf{R}_i [\mathbf{I} \quad -\mathbf{C}_i] \quad i = 1, 2 \quad \rightarrow 31$$

Epipolar Geometry Example: Forward Motion

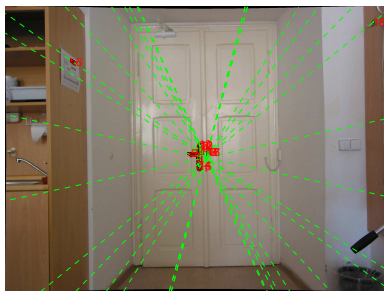


image 1

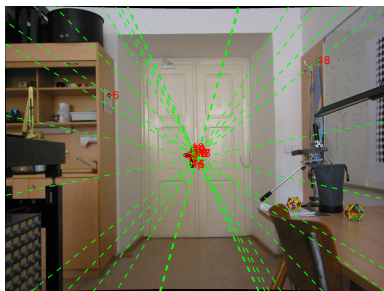
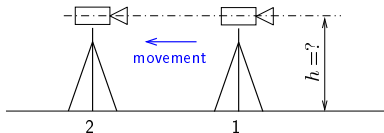


image 2

- red: correspondences
- green: epipolar line pairs per correspondence

click on the image to see their IDs
same ID in both images

How high was the camera above the floor?



► Cross Products and Maps by Skew-Symmetric 3×3 Matrices

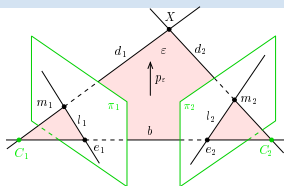
- There is an equivalence $\mathbf{b} \times \mathbf{m} = [\mathbf{b}]_{\times} \mathbf{m}$, where $[\mathbf{b}]_{\times}$ is a 3×3 skew-symmetric matrix

$$[\mathbf{b}]_{\times} = \begin{bmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{bmatrix}, \quad \text{assuming } \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Some properties

- $[\mathbf{b}]_{\times}^{\top} = -[\mathbf{b}]_{\times}$ the general antisymmetry property
- \mathbf{A} is skew-symmetric iff $\mathbf{x}^{\top} \mathbf{A} \mathbf{x} = 0$ for all \mathbf{x} skew-sym mtx generalizes cross products
- $[\mathbf{b}]_{\times}^3 = -\|\mathbf{b}\|^2 \cdot [\mathbf{b}]_{\times}$
- $\|[\mathbf{b}]_{\times}\|_F = \sqrt{2} \|\mathbf{b}\|$ Frobenius norm ($\|\mathbf{A}\|_F = \sqrt{\text{tr}(\mathbf{A}^{\top} \mathbf{A})} = \sqrt{\sum_{i,j} |a_{ij}|^2}$)
- $[\mathbf{b}]_{\times} \mathbf{b} = \mathbf{0}$
- $\text{rank} [\mathbf{b}]_{\times} = 2$ iff $\|\mathbf{b}\| > 0$ check minors of $[\mathbf{b}]_{\times}$
- eigenvalues of $[\mathbf{b}]_{\times}$ are $(0, \lambda, -\lambda)$
- for any regular \mathbf{B} : $\mathbf{B}^{\top} [\mathbf{B}\mathbf{z}]_{\times} \mathbf{B} = \det \mathbf{B} [\mathbf{z}]_{\times}$ follows from the factoring on $\rightarrow 38$
- in particular: if $\mathbf{R}\mathbf{R}^{\top} = \mathbf{I}$ then $[\mathbf{R}\mathbf{b}]_{\times} = \mathbf{R}[\mathbf{b}]_{\times} \mathbf{R}^{\top}$
 - note that if \mathbf{R}_b is rotation about \mathbf{b} then $\mathbf{R}_b \mathbf{b} = \mathbf{b}$
 - note $[\mathbf{b}]_{\times}$ is not a homography; it is not a rotation matrix it is a logarithm of a rotation mtx

► Expressing Epipolar Constraint Algebraically



$$\mathbf{P}_i = [\mathbf{Q}_i \quad \mathbf{q}_i] = \mathbf{K}_i [\mathbf{R}_i \quad \mathbf{t}_i], \quad i = 1, 2$$

\mathbf{R}_{21} – relative camera rotation, $\mathbf{R}_{21} = \mathbf{R}_2 \mathbf{R}_1^\top$

\mathbf{t}_{21} – relative camera translation, $\mathbf{t}_{21} = \mathbf{t}_2 - \mathbf{R}_{21} \mathbf{t}_1 = -\mathbf{R}_2 \mathbf{b} \rightarrow 73$

\mathbf{b} – baseline vector (world coordinate system)

remember: $\mathbf{C} = -\mathbf{Q}^{-1} \mathbf{q} = -\mathbf{R}^\top \mathbf{t}$

$\rightarrow 32$ and 34

$$0 = \mathbf{d}_2^\top \underbrace{\mathbf{p}_\varepsilon}_{\text{normal of } \varepsilon} \simeq \underbrace{(\mathbf{Q}_2^{-1} \mathbf{m}_2)^\top}_{\text{optical ray}} \underbrace{\mathbf{Q}_1^\top \mathbf{l}_1}_{\text{optical plane}} = \mathbf{m}_2^\top \underbrace{\mathbf{Q}_2^{-\top} \mathbf{Q}_1^\top (\mathbf{e}_1 \times \mathbf{m}_1)}_{\text{image of } \varepsilon \text{ in } \pi_2} = \mathbf{m}_2^\top \underbrace{(\mathbf{Q}_2^{-\top} \mathbf{Q}_1^\top [\mathbf{e}_1]_\times)}_{\text{fundamental matrix } \mathbf{F}} \mathbf{m}_1$$

Epipolar constraint $\mathbf{m}_2^\top \mathbf{F} \mathbf{m}_1 = 0$ is a point-line incidence constraint

- point \mathbf{m}_2 is incident on epipolar line $\mathbf{l}_2 \simeq \mathbf{F} \mathbf{m}_1$
- point \mathbf{m}_1 is incident on epipolar line $\mathbf{l}_1 \simeq \mathbf{F}^\top \mathbf{m}_2$
- $\mathbf{F} \mathbf{e}_1 = \mathbf{F}^\top \mathbf{e}_2 = \mathbf{0}$ (non-trivially)
- all epipolars meet at the epipole

$$\mathbf{e}_1 \simeq \mathbf{Q}_1 \mathbf{C}_2 + \mathbf{q}_1 = \mathbf{Q}_1 \mathbf{C}_2 - \mathbf{Q}_1 \mathbf{C}_1 = \mathbf{K}_1 \mathbf{R}_1 \mathbf{b} = -\mathbf{K}_1 \mathbf{R}_1 \mathbf{R}_2^\top \mathbf{t}_{21} = -\mathbf{K}_1 \mathbf{R}_{21}^\top \mathbf{t}_{21}$$

$$\mathbf{F} = \mathbf{Q}_2^{-\top} \mathbf{Q}_1^\top [\mathbf{e}_1]_\times = \mathbf{Q}_2^{-\top} \mathbf{Q}_1^\top [\mathbf{K}_1 \mathbf{R}_1 \mathbf{b}]_\times = \dots \simeq \mathbf{K}_2^{-\top} [-\mathbf{t}_{21}]_\times \mathbf{R}_{21} \mathbf{K}_1^{-1} \quad \text{fundamental}$$

$$\mathbf{E} = [-\mathbf{t}_{21}]_\times \mathbf{R}_{21} = \underbrace{[\mathbf{R}_2 \mathbf{b}]_\times}_{\text{baseline in Cam 2}} \mathbf{R}_{21} = \mathbf{R}_{21} \underbrace{[\mathbf{R}_1 \mathbf{b}]_\times}_{\text{baseline in Cam 1}} = \mathbf{R}_{21} [-\mathbf{R}_{21} \mathbf{t}_{21}]_\times \quad \text{essential}$$

► The Structure and the Key Properties of the Fundamental Matrix

$$\mathbf{F} = \underbrace{(\mathbf{Q}_2 \mathbf{Q}_1^{-1})^{-\top}}_{\text{epipolar homography } \mathbf{H}_e} [\mathbf{e}_1]_{\times} = \underbrace{\mathbf{K}_2^{-\top} \mathbf{R}_{21} \mathbf{K}_1^{\top}}_{\mathbf{H}_e^{-\top}} \overbrace{[\mathbf{e}_1]_{\times}}^{\text{left epipole}} \xrightarrow{75} \overbrace{[\mathbf{H}_e \mathbf{e}_1]_{\times}}^{\text{right epipole}} \mathbf{H}_e = \mathbf{K}_2^{-\top} \underbrace{[-\mathbf{t}_{21}]_{\times} \mathbf{R}_{21}}_{\text{essential matrix } \mathbf{E}} \mathbf{K}_1^{-1}$$

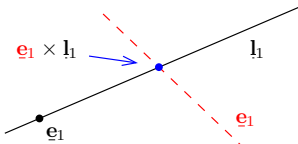
- \mathbf{E} captures relative camera pose only (the change of the world coordinate system does not change \mathbf{E}) [Longuet-Higgins 1981]

$$[\mathbf{R}'_i \quad \mathbf{t}'_i] = [\mathbf{R}_i \quad \mathbf{t}_i] \cdot \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0}^{\top} & 1 \end{bmatrix} = [\mathbf{R}_i \mathbf{R} \quad \mathbf{R}_i \mathbf{t} + \mathbf{t}_i],$$

then

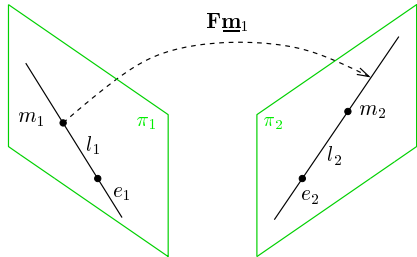
$$\mathbf{R}'_{21} = \mathbf{R}'_2 \mathbf{R}'_1{}^{\top} = \dots = \mathbf{R}_{21} \quad \mathbf{t}'_{21} = \mathbf{t}'_2 - \mathbf{R}'_{21} \mathbf{t}'_1 = \dots = \mathbf{t}_{21}$$

- the translation length \mathbf{t}_{21} is lost since \mathbf{E} is homogeneous
- \mathbf{F} maps points to lines and it is not a homography
- \mathbf{H}_e maps epipoles to epipoles, $\mathbf{H}_e^{-\top}$ epipolar lines to epipolar lines: $\mathbf{l}_2 \simeq \mathbf{H}_e^{-\top} \mathbf{l}_1$



- replacement for $\mathbf{H}_e^{-\top}$ for epipolar line map: $\mathbf{l}_2 \simeq \mathbf{F}[\mathbf{e}_1]_{\times} \mathbf{l}_1$
- proof by point/line 'transmutation' (left)
- point \mathbf{e}_1 does not lie on line \mathbf{e}_1 (dashed): $\mathbf{e}_1^{\top} \mathbf{e}_1 \neq 0$
- $\mathbf{F}[\mathbf{e}_1]_{\times}$ is not a homography, unlike $\mathbf{H}_e^{-\top}$ but it does the same job for epipolar line mapping

► Some Mappings by the Fundamental Matrix



$$0 = \underline{\mathbf{m}}_2^\top \mathbf{F} \underline{\mathbf{m}}_1$$

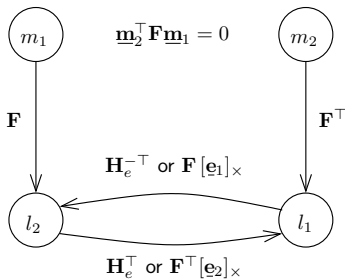
$$\underline{\mathbf{e}}_1 \simeq \text{null}(\mathbf{F}), \quad \underline{\mathbf{e}}_2 \simeq \text{null}(\mathbf{F}^\top)$$

$$\underline{\mathbf{e}}_1 \simeq \mathbf{H}_e^{-1} \underline{\mathbf{e}}_2 \quad \underline{\mathbf{e}}_2 \simeq \mathbf{H}_e \underline{\mathbf{e}}_1$$

$$\underline{\mathbf{l}}_1 \simeq \mathbf{F}^\top \underline{\mathbf{m}}_2 \quad \underline{\mathbf{l}}_2 \simeq \mathbf{F} \underline{\mathbf{m}}_1$$

$$\underline{\mathbf{l}}_1 \simeq \mathbf{H}_e^\top \underline{\mathbf{l}}_2 \quad \underline{\mathbf{l}}_2 \simeq \mathbf{H}_e^{-\top} \underline{\mathbf{l}}_1$$

$$\underline{\mathbf{l}}_1 \simeq \mathbf{F}^\top [\underline{\mathbf{e}}_2]_\times \underline{\mathbf{l}}_2 \quad \underline{\mathbf{l}}_2 \simeq \mathbf{F} [\underline{\mathbf{e}}_1]_\times \underline{\mathbf{l}}_1$$



- $\mathbf{F}[\underline{\mathbf{e}}_1]_\times$ maps lines to lines but it is not a homography
- $\mathbf{H}_e = \mathbf{Q}_2 \mathbf{Q}_1^{-1}$ is the epipolar homography → 77
 $\mathbf{H}_e^{-\top}$ maps epipolar lines to epipolar lines, where

$$\mathbf{H}_e = \mathbf{Q}_2 \mathbf{Q}_1^{-1} = \mathbf{K}_2 \mathbf{R}_{21} \mathbf{K}_1^{-1}$$

you have seen this → 59

Thank You

