3D Computer Vision

Radim Šára   Martin Matoušek

Center for Machine Perception
Department of Cybernetics
Faculty of Electrical Engineering
Czech Technical University in Prague

https://cw.fel.cvut.cz/wiki/courses/tdv/start

http://cmp.felk.cvut.cz
mailto:sara@cmp.felk.cvut.cz
phone ext. 7203

rev. October 26, 2021

Open Informatics Master's Course

▶Three-Point Exterior Orientation Problem (P3P)

Calibrated camera rotation and translation from Perspective images of 3 reference Points.

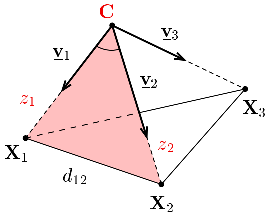
Problem: Given $\mathbf{K}$ and three corresponding pairs $\{(m_i,\, X_i)\}_{i=1}^{3}$, find $\mathbf{R}$, $\mathbf{C}$ by solving

$$\lambda_i \underline{\mathbf{m}}_i = \mathbf{K}\mathbf{R}\,(\mathbf{X}_i - \mathbf{C}), \qquad i = 1, 2, 3 \qquad \mathbf{X}_i \text{ Cartesian}$$

1. Transform $\underline{\mathbf{v}}_i \stackrel{\text{def}}{=} \mathbf{K}^{-1}\underline{\mathbf{m}}_i$. Then

$$\lambda_i \underline{\mathbf{v}}_i = \mathbf{R}\,(\mathbf{X}_i - \mathbf{C}). \tag{10}$$

2. If there was no rotation in (10), the situation would look like this



3. and we could shoot 3 lines from the given points $\mathbf{X}_i$ in given directions $\underline{\mathbf{v}}_i$ to get $\mathbf{C}$
4. given $\mathbf{C}$ we solve (10) for λ_i, $\mathbf{R}$

▶P3P cont'd

If there is rotation $\mathbf{R}$

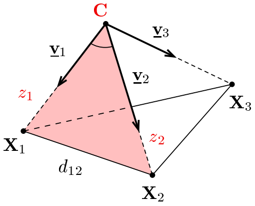
1. Eliminate $\mathbf{R}$ by taking   rotation preserves length: $\|\mathbf{Rx}\| = \|\mathbf{x}\|$

$$|\lambda_i| \cdot \|\underline{\mathbf{v}}_i\| = \|\mathbf{X}_i - \mathbf{C}\| \stackrel{\text{def}}{=} z_i \qquad (11)$$

2. Consider only angles among $\underline{\mathbf{v}}_i$ and apply Cosine Law per triangle $(\mathbf{C}, \mathbf{X}_i, \mathbf{X}_j)$ $i, j = 1, 2, 3,\ i \neq j$

$$d_{ij}^2 = z_i^2 + z_j^2 - 2\, z_i\, z_j\, c_{ij},$$

$$z_i = \|\mathbf{X}_i - \mathbf{C}\|,\ d_{ij} = \|\mathbf{X}_j - \mathbf{X}_i\|,\ c_{ij} = \cos(\angle \underline{\mathbf{v}}_i\, \underline{\mathbf{v}}_j)$$



4. Solve the system of 3 quadratic eqs in 3 unknowns z_i [Fischler & Bolles, 1981]

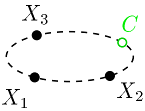
   there may be no real root; there are up to 4 solutions that cannot be ignored (verify on additional points)

5. Compute $\mathbf{C}$ by trilateration (3-sphere intersection) from $\mathbf{X}_i$ and z_i; then λ_i from (11)

6. Compute $\mathbf{R}$ from (10)   we will solve this problem next →70

Similar problems (P4P with unknown f) at http://aag.ciirc.cvut.cz/minimal/ (papers, code)

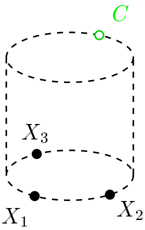
Degenerate (Critical) Configurations for Exterior Orientation



no solution

1. C cocyclic with (X_1, X_2, X_3) camera sees points on a line

---



unstable solution

- center of projection C located on the orthogonal circular cylinder with base circumscribing the three points X_i

unstable: a small change of X_i results in a large change of C
can be detected by error propagation

degenerate

- camera C is coplanar with points (X_1, X_2, X_3) but is not on the circumscribed circle of (X_1, X_2, X_3)

camera sees points on a line

- additional critical configurations depend on the quadratic equations solver

[Haralick et al. IJCV 1994]

►Populating A Little ZOO of Minimal Geometric Problems in CV

problem	given	unknown	slide
camera resection	6 world–img correspondences $\{(X_i, m_i)\}_{i=1}^{6}$	$\mathbf{P}$	→62
exterior orientation	$\mathbf{K}$, 3 world–img correspondences $\{(X_i, m_i)\}_{i=1}^{3}$	$\mathbf{R}$, $\mathbf{C}$	→66
relative orientation	3 world-world correspondences $\{(X_i, Y_i)\}_{i=1}^{3}$	$\mathbf{R}$, $\mathbf{t}$	→70

- camera resection and exterior orientation are similar problems in a sense:
  - we do resectioning when our camera is uncalibrated
  - we do orientation when our camera is calibrated
- relative orientation involves no camera (see next) it is a recurring problem in 3D vision
- more problems to come

▶The Relative Orientation Problem

Problem: Given point triples (X_1, X_2, X_3) and (Y_1, Y_2, Y_3) in a general position in $\mathbf{R}^3$ such that the correspondence $X_i \leftrightarrow Y_i$ is known, determine the relative orientation $(\mathbf{R}, \mathbf{t})$ that maps $\mathbf{X}_i$ to $\mathbf{Y}_i$, i.e.

$$\mathbf{Y}_i = \mathbf{R}\mathbf{X}_i + \mathbf{t}, \quad i = 1, 2, 3\,.$$

Applies to:

- 3D scanners
- merging partial reconstructions from different viewpoints
- generalization of the last step of P3P

Obs: Let the centroid be $\bar{\mathbf{X}} = \frac{1}{3}\sum_i \mathbf{X}_i$ and analogically for $\bar{\mathbf{Y}}$. Then

$$\bar{\mathbf{Y}} = \mathbf{R}\bar{\mathbf{X}} + \mathbf{t}.$$

Therefore

$$\mathbf{Z}_i \stackrel{\text{def}}{=} (\mathbf{Y}_i - \bar{\mathbf{Y}}) = \mathbf{R}(\mathbf{X}_i - \bar{\mathbf{X}}) \stackrel{\text{def}}{=} \mathbf{R}\mathbf{W}_i$$

If all dot products are equal, $\mathbf{Z}_i^\top \mathbf{Z}_j = \mathbf{W}_i^\top \mathbf{W}_j$ for $i, j = 1, 2, 3$, we have

$$\mathbf{R}^* = \begin{bmatrix}\mathbf{W}_1 & \mathbf{W}_2 & \mathbf{W}_3\end{bmatrix}^{-1} \begin{bmatrix}\mathbf{Z}_1 & \mathbf{Z}_2 & \mathbf{Z}_3\end{bmatrix}$$

Poor man's solver:

- normalize $\mathbf{W}_i$, $\mathbf{Z}_i$ to unit length and then use the above formula
- but this is equivalent to a non-optimal objective — it ignores errors in vector lengths

An Optimal Algorithm for Relative Orientation

We setup a minimization problem

$$\mathbf{R}^* = \arg\min_{\mathbf{R}} \sum_{i=1}^{3} \|\mathbf{Z}_i - \mathbf{R}\mathbf{W}_i\|^2 \quad \text{s.t.} \quad \mathbf{R}^\top\mathbf{R} = \mathbf{I}, \quad \det \mathbf{R} = 1$$

$$\arg\min_{\mathbf{R}} \sum_i \|\mathbf{Z}_i - \mathbf{R}\mathbf{W}_i\|^2 = \arg\min_{\mathbf{R}} \sum_i \left(\|\mathbf{Z}_i\|^2 - 2\mathbf{Z}_i^\top \mathbf{R}\mathbf{W}_i + \|\mathbf{W}_i\|^2\right) = \cdots$$

$$\cdots = \arg\max_{\mathbf{R}} \sum_i \mathbf{Z}_i^\top \mathbf{R}\mathbf{W}_i$$

Obs 1: Let $\mathbf{A} : \mathbf{B} = \sum_{i,j} a_{ij} b_{ij}$ be the dot-product (Frobenius inner product) over real matrices. Then

$$\mathbf{A} : \mathbf{B} = \mathbf{B} : \mathbf{A} = \operatorname{tr}(\mathbf{A}^\top \mathbf{B})$$

Obs 2: (cyclic property for matrix trace)

$$\operatorname{tr}(\mathbf{ABC}) = \operatorname{tr}(\mathbf{CAB})$$

Obs 3: ($\mathbf{Z}_i$, $\mathbf{W}_i$ are vectors)

$$\mathbf{Z}_i^\top \mathbf{R}\mathbf{W}_i = \operatorname{tr}(\mathbf{Z}_i^\top \mathbf{R}\mathbf{W}_i) \overset{\text{O2}}{=} \operatorname{tr}(\mathbf{W}_i \mathbf{Z}_i^\top \mathbf{R}) \overset{\text{O1}}{=} (\mathbf{Z}_i \mathbf{W}_i^\top) : \mathbf{R} = \mathbf{R} : (\mathbf{Z}_i \mathbf{W}_i^\top)$$

Let there be SVD of

$$\sum_i \mathbf{Z}_i \mathbf{W}_i^\top \overset{\text{def}}{=} \mathbf{M} = \mathbf{U}\mathbf{D}\mathbf{V}^\top$$

Then

$$\mathbf{R} : \mathbf{M} = \mathbf{R} : (\mathbf{U}\mathbf{D}\mathbf{V}^\top) \overset{\text{O1}}{=} \operatorname{tr}(\mathbf{R}^\top \mathbf{U}\mathbf{D}\mathbf{V}^\top) \overset{\text{O2}}{=} \operatorname{tr}(\mathbf{V}^\top \mathbf{R}^\top \mathbf{U}\mathbf{D}) \overset{\text{O1}}{=} (\mathbf{U}^\top \mathbf{R}\mathbf{V}) : \mathbf{D}$$

cont'd: The Algorithm

We are solving

$$\mathbf{R}^* = \arg\max_{\mathbf{R}} \sum_i \mathbf{Z}_i^\top \mathbf{R} \mathbf{W}_i = \arg\max_{\mathbf{R}} \left(\mathbf{U}^\top \mathbf{R} \mathbf{V} \right) : \mathbf{D}$$

A particular solution is found as follows:

- $\mathbf{U}^\top \mathbf{R} \mathbf{V}$ must be (1) orthogonal, and most similar to (2) diagonal, (3) positive definite
- Since $\mathbf{U}$, $\mathbf{V}$ are orthogonal matrices then the solution to the problem is among $\mathbf{R}^* = \mathbf{U}\mathbf{S}\mathbf{V}^\top$, where $\mathbf{S}$ is diagonal and orthogonal, i.e. one of

$$\pm\,\mathrm{diag}(1,1,1), \quad \pm\,\mathrm{diag}(1,-1,-1), \quad \pm\,\mathrm{diag}(-1,1,-1), \quad \pm\,\mathrm{diag}(-1,-1,1)$$

- $\mathbf{U}^\top \mathbf{V}$ is not necessarily positive definite
- We choose $\mathbf{S}$ so that $(\mathbf{R}^*)^\top \mathbf{R}^* = \mathbf{I}$

Alg:

1. Compute matrix $\mathbf{M} = \sum_i \mathbf{Z}_i \mathbf{W}_i^\top$.
2. Compute SVD $\mathbf{M} = \mathbf{U}\mathbf{D}\mathbf{V}^\top$.
3. Compute all $\mathbf{R}_k = \mathbf{U}\mathbf{S}_k\mathbf{V}^\top$ that give $\mathbf{R}_k^\top \mathbf{R}_k = \mathbf{I}$.
4. Compute $\mathbf{t}_k = \bar{\mathbf{Y}} - \mathbf{R}_k \bar{\mathbf{X}}$.

- The algorithm can be used for more than 3 points
- Triple pairs can be pre-filtered based on motion invariants (lengths, angles)
- Can be used for the last step of the exterior orientation (P3P) problem →66

Module IV

Computing with a Camera Pair

4.1 Camera Motions Inducing Epipolar Geometry

4.2 Estimating Fundamental Matrix from 7 Correspondences

4.3 Estimating Essential Matrix from 5 Correspondences

4.4 Triangulation: 3D Point Position from a Pair of Corresponding Points

covered by

[1] [H&Z] Secs: 9.1, 9.2, 9.6, 11.1, 11.2, 11.9, 12.2, 12.3, 12.5.1

[2] H. Li and R. Hartley. Five-point motion estimation made easy. In *Proc ICPR* 2006, pp. 630–633

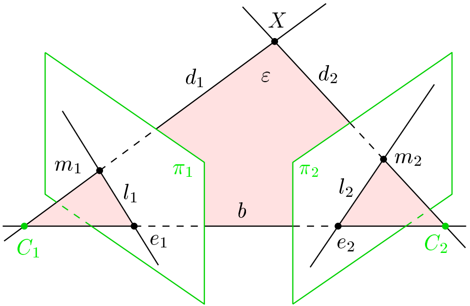
additional references

H. Longuet-Higgins. A computer algorithm for reconstructing a scene from two projections. *Nature*, 293 (5828):133–135, 1981.

▶Geometric Model of a Camera Stereo Pair

Epipolar geometry:

- brings constraints necessary for inter-image matching
- its parametric form encapsulates information about the relative pose of two cameras



two-camera setup

Description

- baseline b joins projection centers C_1, C_2

$$\mathbf{b} = \mathbf{C}_2 - \mathbf{C}_1$$

- epipole $e_i \in \pi_i$ is the image of C_j:

$$\underline{\mathbf{e}}_1 \simeq \mathbf{P}_1 \underline{\mathbf{C}}_2, \quad \underline{\mathbf{e}}_2 \simeq \mathbf{P}_2 \underline{\mathbf{C}}_1$$

- $l_i \in \pi_i$ is the image of epipolar plane

$$\varepsilon = (C_2, X, C_1)$$

- l_j is the epipolar line ('epipolar') in image π_j induced by m_i in image π_i

Epipolar constraint: corresponding d_2, b, d_1 are coplanar

a necessary condition →87

$$\mathbf{P}_i = \begin{bmatrix} \mathbf{Q}_i & \mathbf{q}_i \end{bmatrix} = \mathbf{K}_i \begin{bmatrix} \mathbf{R}_i & \mathbf{t}_i \end{bmatrix} = \mathbf{K}_i \mathbf{R}_i \begin{bmatrix} \mathbf{I} & -\mathbf{C}_i \end{bmatrix} \qquad i = 1, 2$$

→31

Epipolar Geometry Example: Forward Motion

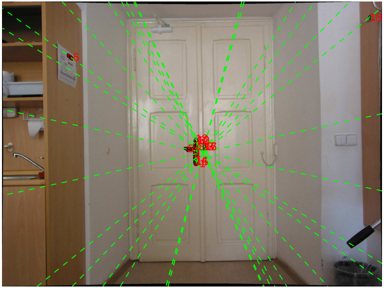


image 1

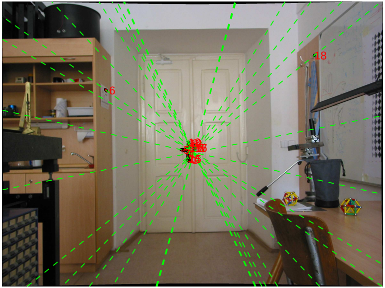
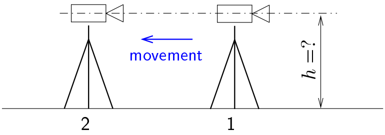


image 2

- red: correspondences
- green: epipolar line pairs per correspondence

click on the image to see their IDs
same ID in both images

How high was the camera above the floor?



▶Cross Products and Maps by Skew-Symmetric 3×3 Matrices

- There is an equivalence $\mathbf{b} \times \mathbf{m} = [\mathbf{b}]_\times \mathbf{m}$, where $[\mathbf{b}]_\times$ is a 3×3 skew-symmetric matrix

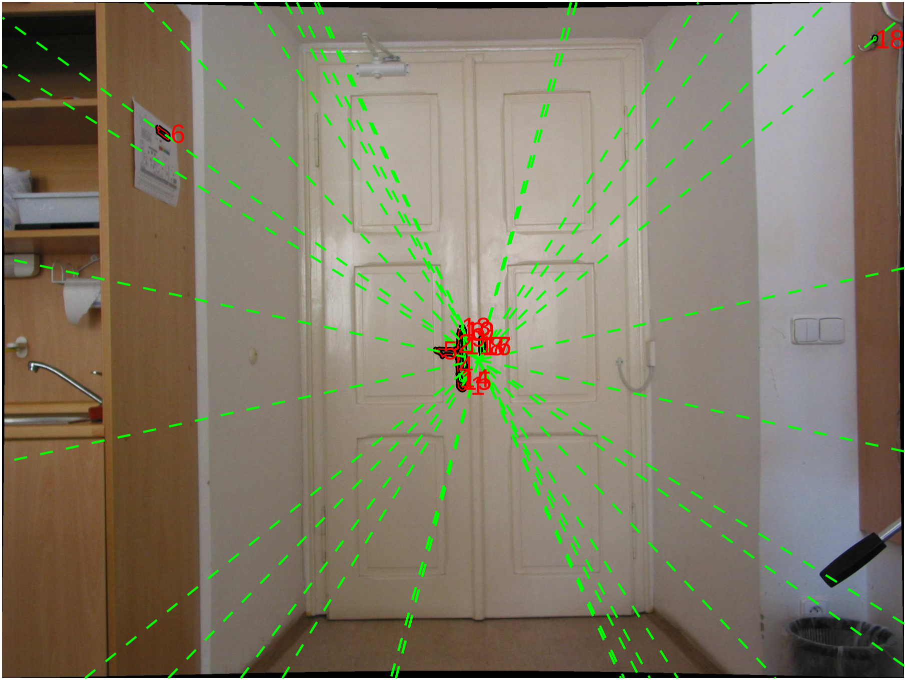
$$[\mathbf{b}]_\times = \begin{bmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{bmatrix}, \qquad \text{assuming} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Some properties

1. $[\mathbf{b}]_\times^\top = -[\mathbf{b}]_\times$ — the general antisymmetry property
2. $\mathbf{A}$ is skew-symmetric iff $\mathbf{x}^\top \mathbf{A} \mathbf{x} = 0$ for all $\mathbf{x}$ — skew-sym mtx generalizes cross products
3. $[\mathbf{b}]_\times^3 = -\|\mathbf{b}\|^2 \cdot [\mathbf{b}]_\times$
4. $\|[\mathbf{b}]_\times\|_F = \sqrt{2}\,\|\mathbf{b}\|$ — Frobenius norm ($\|\mathbf{A}\|_F = \sqrt{\mathrm{tr}(\mathbf{A}^\top \mathbf{A})} = \sqrt{\sum_{i,j} |a_{ij}|^2}$)
5. $\mathrm{rank}\,[\mathbf{b}]_\times = 2$ iff $\|\mathbf{b}\| > 0$ — check minors of $[\mathbf{b}]_\times$
6. $[\mathbf{b}]_\times \mathbf{b} = \mathbf{0}$
7. eigenvalues of $[\mathbf{b}]_\times$ are $(0, \lambda, -\lambda)$
8. for any 3×3 regular $\mathbf{B}$: $\mathbf{B}^\top [\mathbf{B}\mathbf{z}]_\times \mathbf{B} = \det \mathbf{B}\, [\mathbf{z}]_\times$ — follows from the factoring on →39
9. in particular: if $\mathbf{R}\mathbf{R}^\top = \mathbf{I}$ then $[\mathbf{R}\mathbf{b}]_\times = \mathbf{R}[\mathbf{b}]_\times \mathbf{R}^\top$

- note that if $\mathbf{R}_b$ is rotation about $\mathbf{b}$ then $\mathbf{R}_b \mathbf{b} = \mathbf{b}$
- note $[\mathbf{b}]_\times$ is not a homography; it is not a rotation matrix — it is the logarithm of a rotation mtx

Thank You


6
18