Lecture 3

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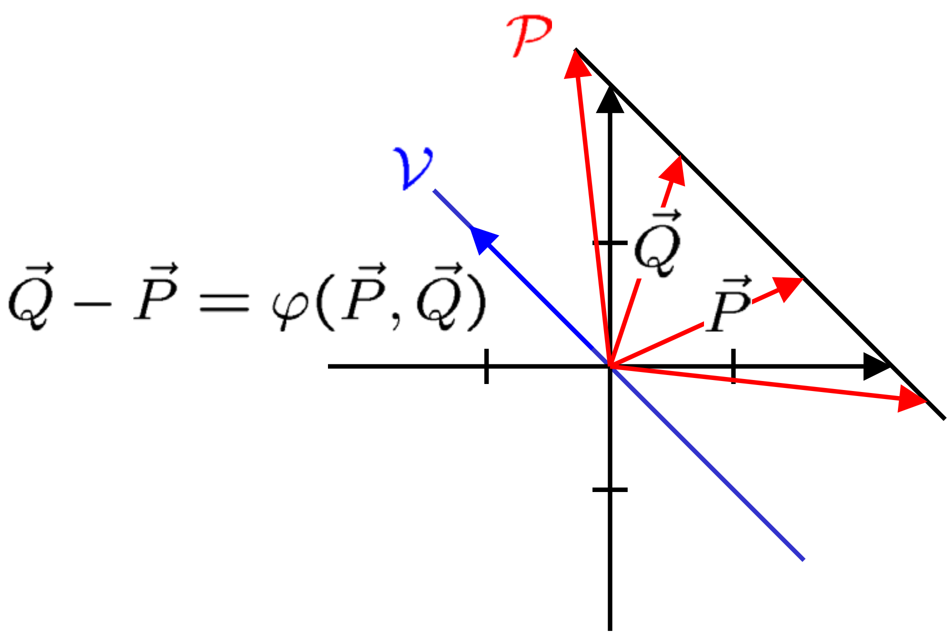
Affine space examples

1. Linear space over itself, i.e. $\mathcal{A} = \mathcal{V}$, $\mathcal{V} = \mathcal{V}$, $\varphi(\vec{x}, \vec{y}) = \vec{y} - \vec{x}$

2. If $\mathtt{A}$ is a matrix and $\vec{b}$ lies in its column space, the set of solutions of the equation $\mathtt{A}\,\vec{x} = \vec{b}$ is an affine space over the subspace of solutions of $\mathtt{A}\,\vec{x} = \vec{0}$.

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \vec{x} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \text{ has solution } \mathcal{P} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \tau \begin{bmatrix} 1 \\ -1 \end{bmatrix} \middle| \tau \in R \right\}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \vec{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ has solution } \mathcal{V} = \left\{ \tau \begin{bmatrix} 1 \\ -1 \end{bmatrix} \middle| \tau \in R \right\}$$



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Affine space examples

3. 4-point affine space, i.e. $\mathcal{P} = \mathcal{V} = \mathbb{Z}_2^2 = \{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \}$

$$\left\{ (P, Q) \,|\, \varphi(P, Q) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\} =$$

$$= \left\{ \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right), \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right), \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right), \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) \right\}$$

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Free Vectors

Let us find a "natural" description of the geometrical space around us and its geometrical objects (points, lines, . . .) with the formal mathematical object – the linear space.

To do so, we need to rely on our intuition and experience with real objects around us. We will take a few intuitive facts as granted and call them axioms. Then, using the formal rules of mathematical logic, we shall construct the connection to a suitable linear space.

Such a connection will allow us to use linear space and in particular coordinates to derive new facts about our geometrical space.

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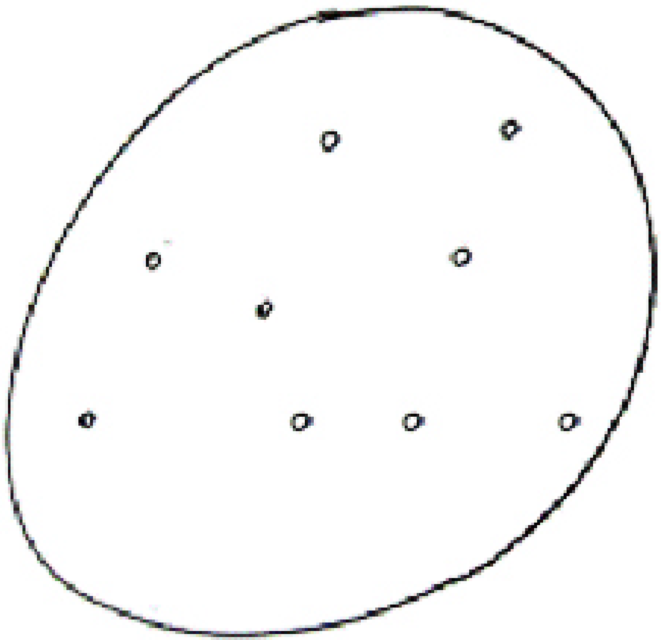
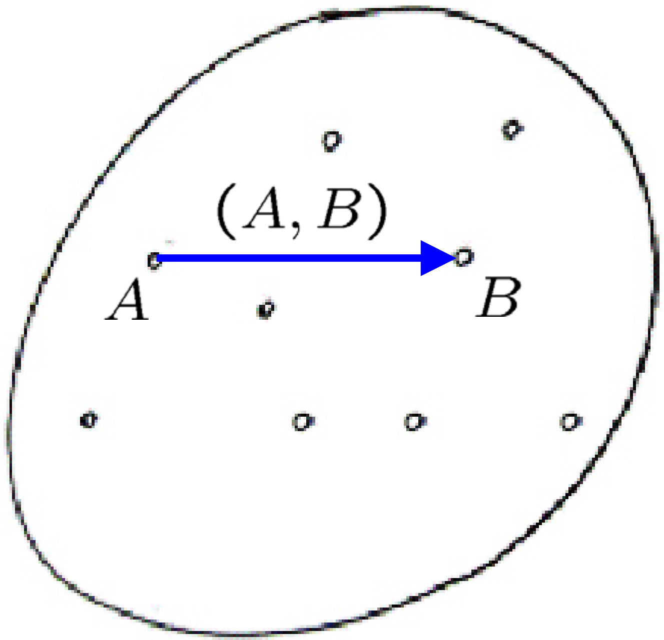
Points and oriented line segments

$\mathcal{P}$ . . . points of the 3D space



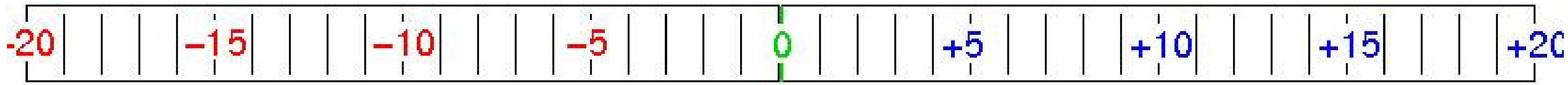
$\mathcal{P} \times \mathcal{P}$ . . . pairs of points are oriented line segments



Ruler

We know that:

1. There is a unique line passing through every two distinct points.
2. Rulers are lines with an origin and a scale. The scale is "linear", i.e. it "corresponds" to real numbers $\mathbb{R}$. Different rulers may have different units.



$\mathbb{R}$

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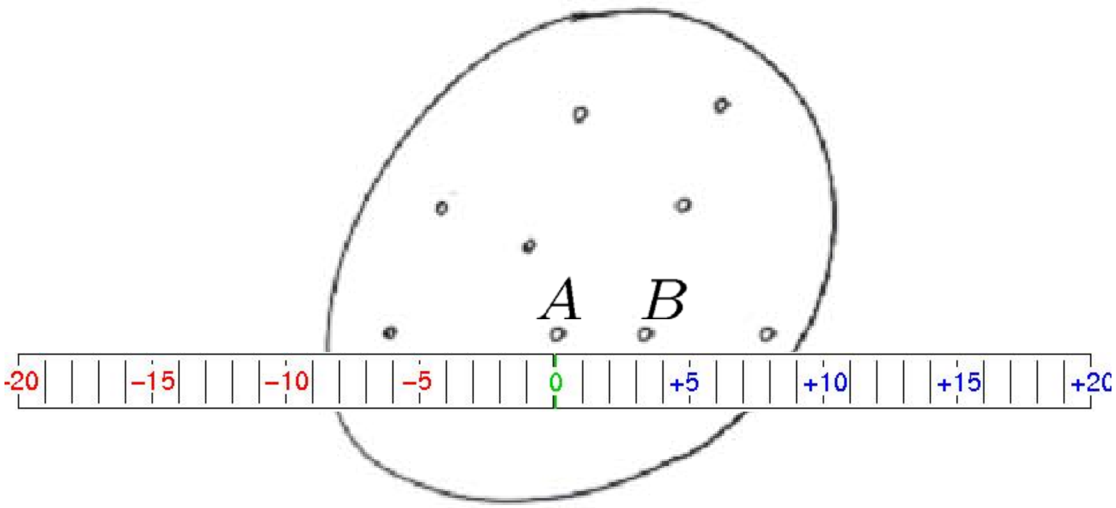
Length of line segment

Definition: length of line segment

$\delta : \mathcal{P} \times \mathcal{P} \rightarrow \mathbb{R}$

Property 1:

1. $\delta(A, B) = \delta(B, A)$



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Multiplication of a line segment by a real number

Definition :

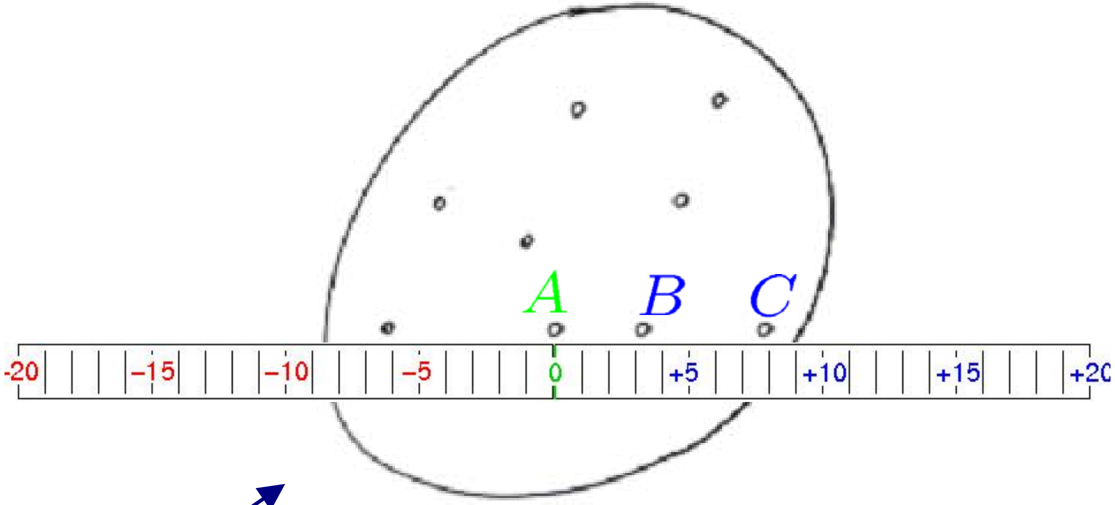
$\circ : \mathbb{R} \times (\mathcal{P} \times \mathcal{P}) \rightarrow (\mathcal{P} \times \mathcal{P})$

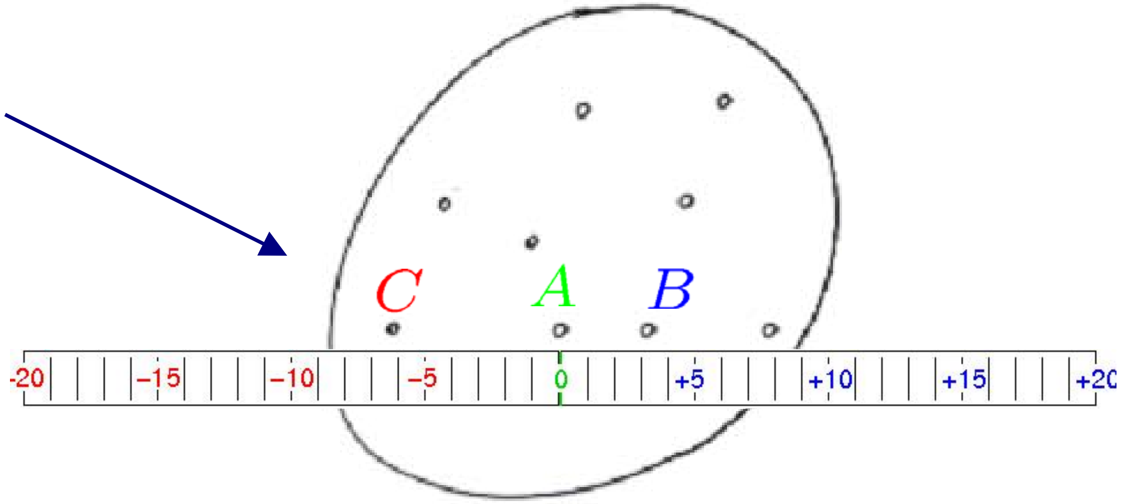
$\alpha \circ (A, B) \rightarrow (A, C)$ such that

$\alpha > 0$: $\quad \alpha\, \delta(A, B) = \delta(A, C)$

$\alpha = 0$: $\quad C = A$

$\alpha < 0$: $\quad -\alpha\, \delta(A, B) = \delta(A, C)$





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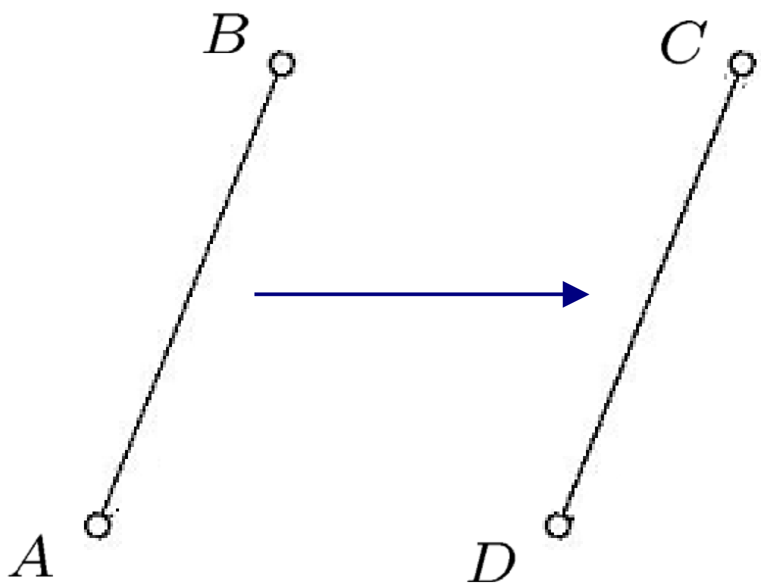
Translation relation

Definition: translation relation

$\leftrightarrow \subset \mathcal{P}^2 \times \mathcal{P}^2$

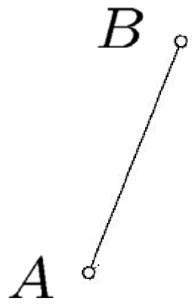
$(A, B) \leftrightarrow (C, D) \stackrel{def}{\equiv} (A, B)$ can be translated to (C, D)



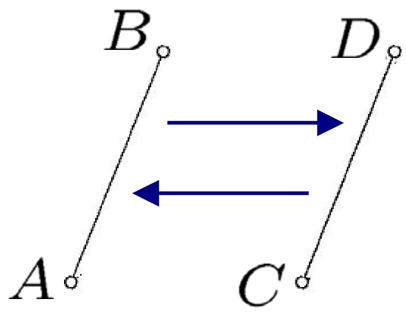
Translation relation is equivalence

Theorem: $\leftrightarrow$ is equivalence relation

1. $\leftrightarrow$ is reflexive: $(A, B) \leftrightarrow (A, B)$
   ... "zero" translation

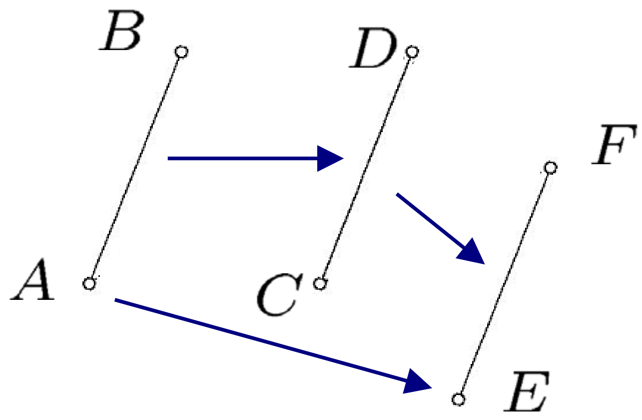


2. $\leftrightarrow$ is symmetric:
   $(A, B) \leftrightarrow (C, D) \Rightarrow (C, D) \leftrightarrow (A, B)$
   ... translations work both ways



3. $\leftrightarrow$ is transitive:

$$(A, B) \leftrightarrow (C, D) \quad \text{and} \quad (C, D) \leftrightarrow (E, F)$$
$$\Downarrow$$
$$(A, B) \quad \leftrightarrow \quad (E, F)$$



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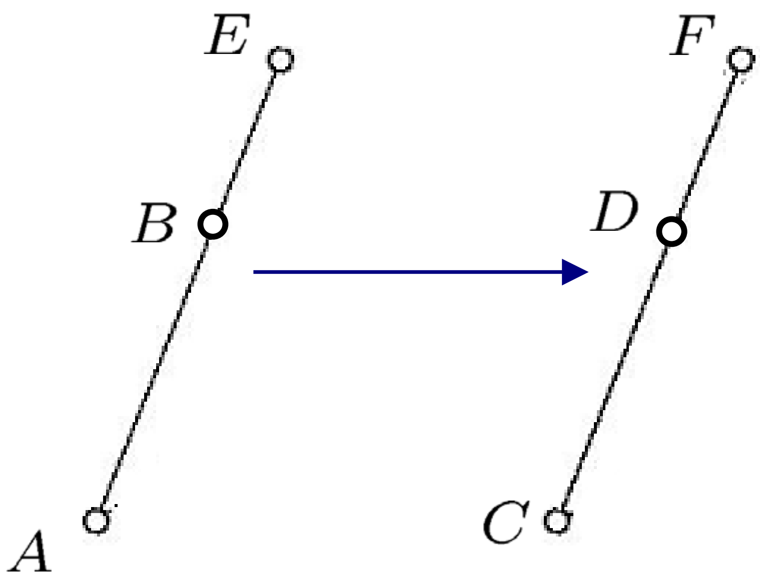
Translation commutes with scalar multiplication

$$(A,E) = a \circ (A,B) \text{ and } (C,F) \;=\; a \circ (C,D) \text{ and } (A,B) \leftrightarrow (C,D)$$

$$\Downarrow$$

$$a \circ (A,B) = (A,E) \;\leftrightarrow\; (C,F) = a \circ (C,D)$$



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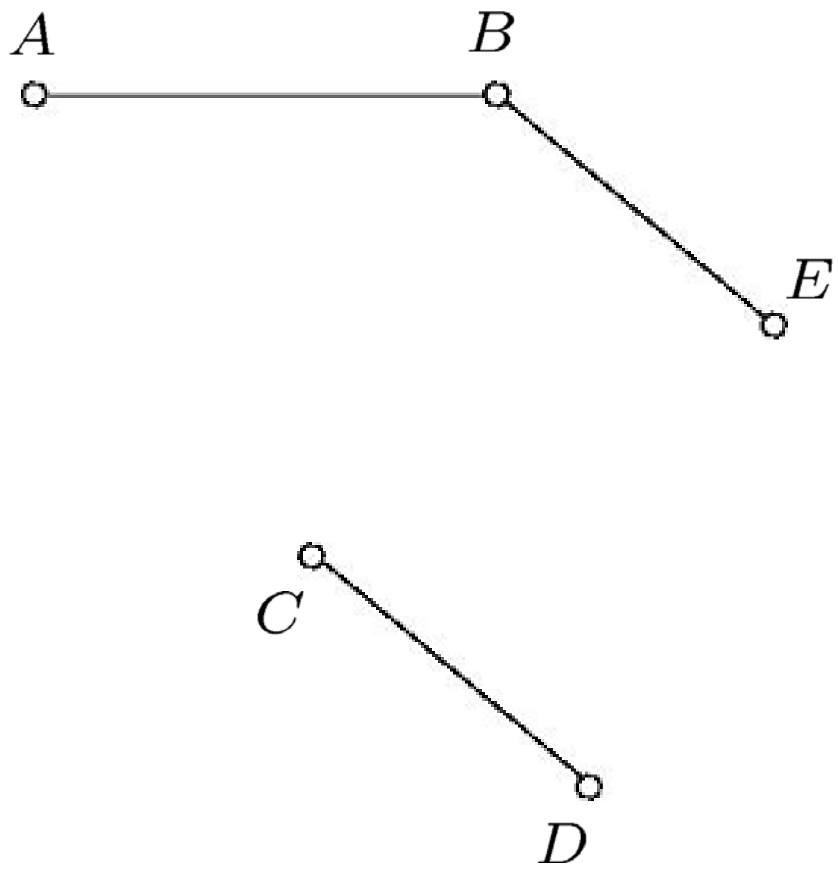
Adding line segments



Definition:

$\# : \mathcal{P}^2 \times \mathcal{P}^2 \rightarrow \mathcal{P}^2$

$(A, B)\#(C, D) \rightarrow (A, E)$

such that

$(B, E) \leftrightarrow (C, D)$

... is well defined since there is only one such E.

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Is $(\mathbb{R}, \mathcal{P}^2, \#, \circ)$ linear space?

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Axioms: of linear space $(\mathbb{R}, V, \#, \circ)$

1. $\forall u, v \in V$: $u \# v = v \# u$
2. $\forall u, v, w \in V$: $u \# (v \# w) = (u \# v) \# w$
3. $\forall u \in V \; \exists o \in V$: $u \# o = u$
4. $\forall u \in V \; \exists v \in V$: $u \# v = o$
5. $\forall u \in V$: and $1 \in \mathbb{R}$: $1 \circ u = u$
6. $\forall a, b \in \mathbb{R}$ and $\forall u \in V$: $(a\, b) \# u = a \circ (b \circ u)$
7. $\forall a, b \in \mathbb{R}$ and $\forall u \in V$: $(a + b) \circ u = (a \circ u) \# (b \circ u)$
8. $\forall a \in \mathbb{R}$ and $\forall u, v \in V$: $a \circ (u \# v) = (a \circ u) \# (a \circ v)$

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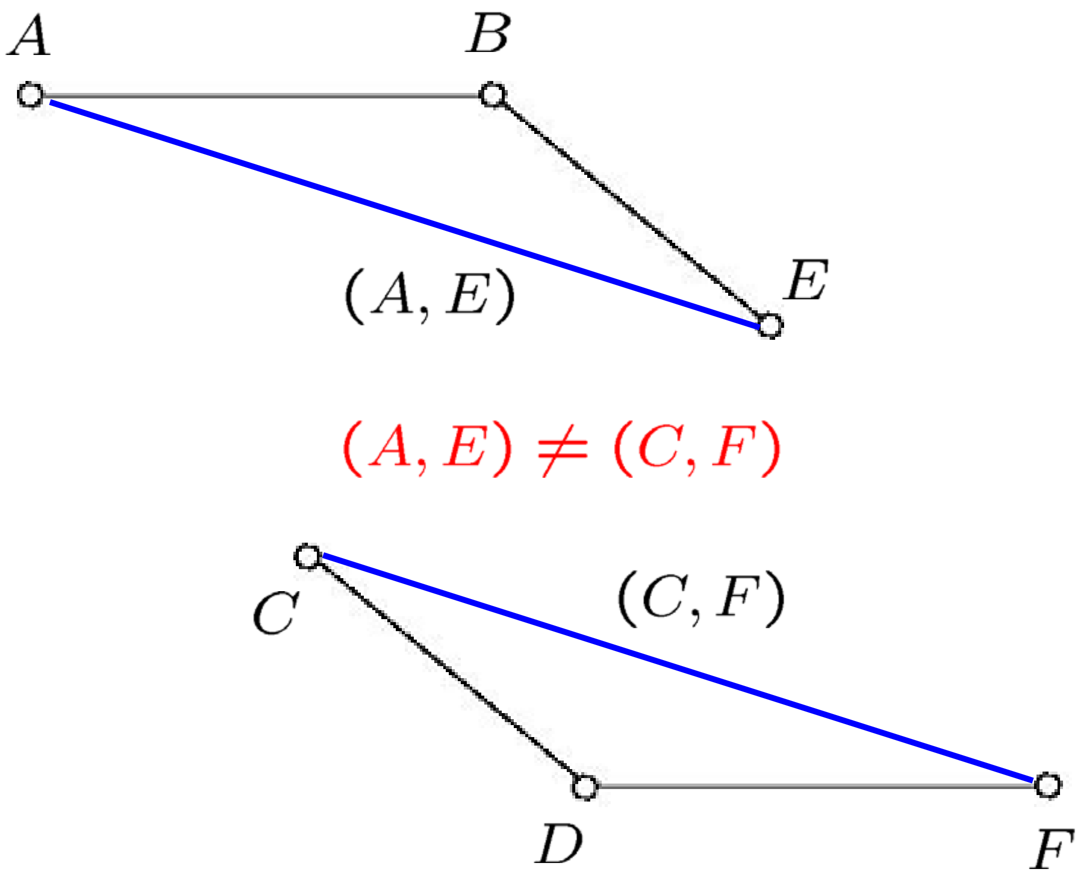
Axiom 1 does not hold

$\#: \mathcal{P}^2 \times \mathcal{P}^2 \rightarrow \mathcal{P}^2$

Axioms: of linear space $(\mathbb{R}, \mathcal{P}^2, \#, \circ)$

1. $\forall (A,B),(C,D) \in \mathcal{P}^2$: $(A,B)\#(C,D) = (C,D)\#(A,B)$

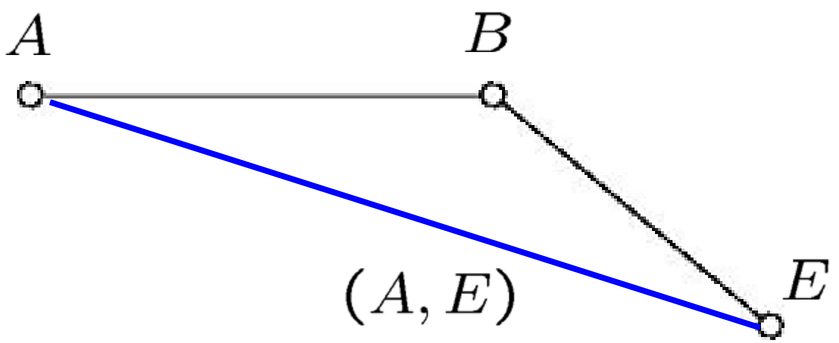


$(A,B)\#(C,D) \rightarrow (A,E)$
such that
$(B,E) \leftrightarrow (C,D)$

$(C,D)\#(A,B) \rightarrow (C,F)$
such that
$(D,F) \leftrightarrow (A,B)$

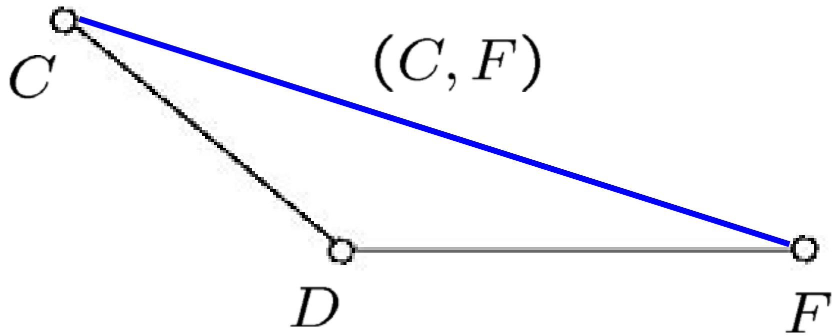
Interesting property of

$\# : \mathcal{P}^2 \times \mathcal{P}^2 \rightarrow \mathcal{P}^2$



$(A, B)\#(C, D) \rightarrow (A, E)$
such that
$(B, E) \leftrightarrow (C, D)$

$(A, E) \leftrightarrow (C, F)$



$(C, D)\#(A, B) \rightarrow (C, F)$
such that
$(D, F) \leftrightarrow (A, B)$

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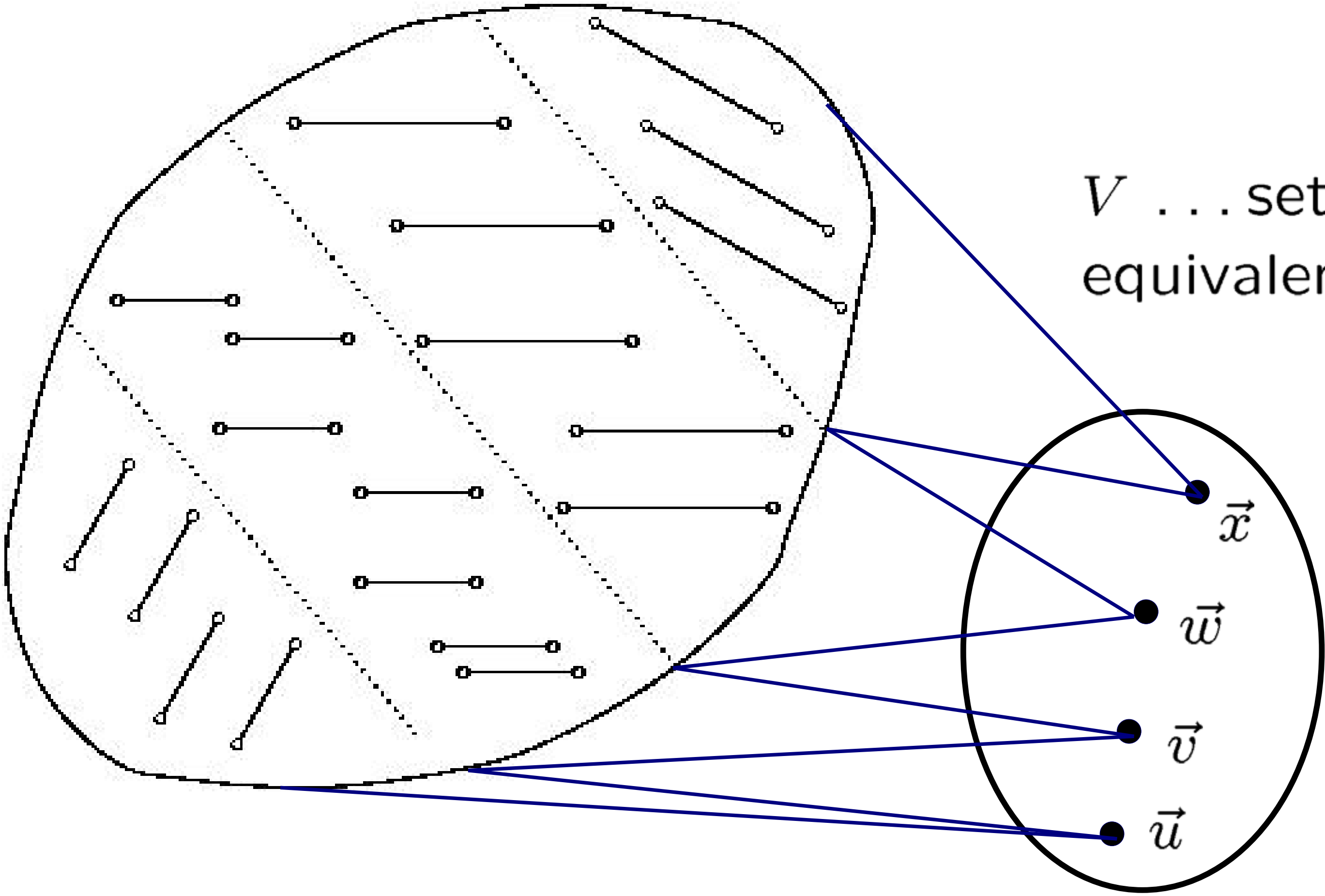
Equivalence classes of line segments

$\mathcal{P}^2$ ... oriented line segments



$$V = \left\{ \vec{v} \in \mathcal{P}^2 \,|\, \forall (A,B),(C,D) \in \vec{v}\colon\ (A,B) \leftrightarrow (C,D) \right\}$$

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Operations for V and $\mathbb{R}$

Addition in V: $\oplus : V \times V \to V$

Multiplication of V by $\mathbb{R}$: $\odot : \mathbb{R} \times V \to V$

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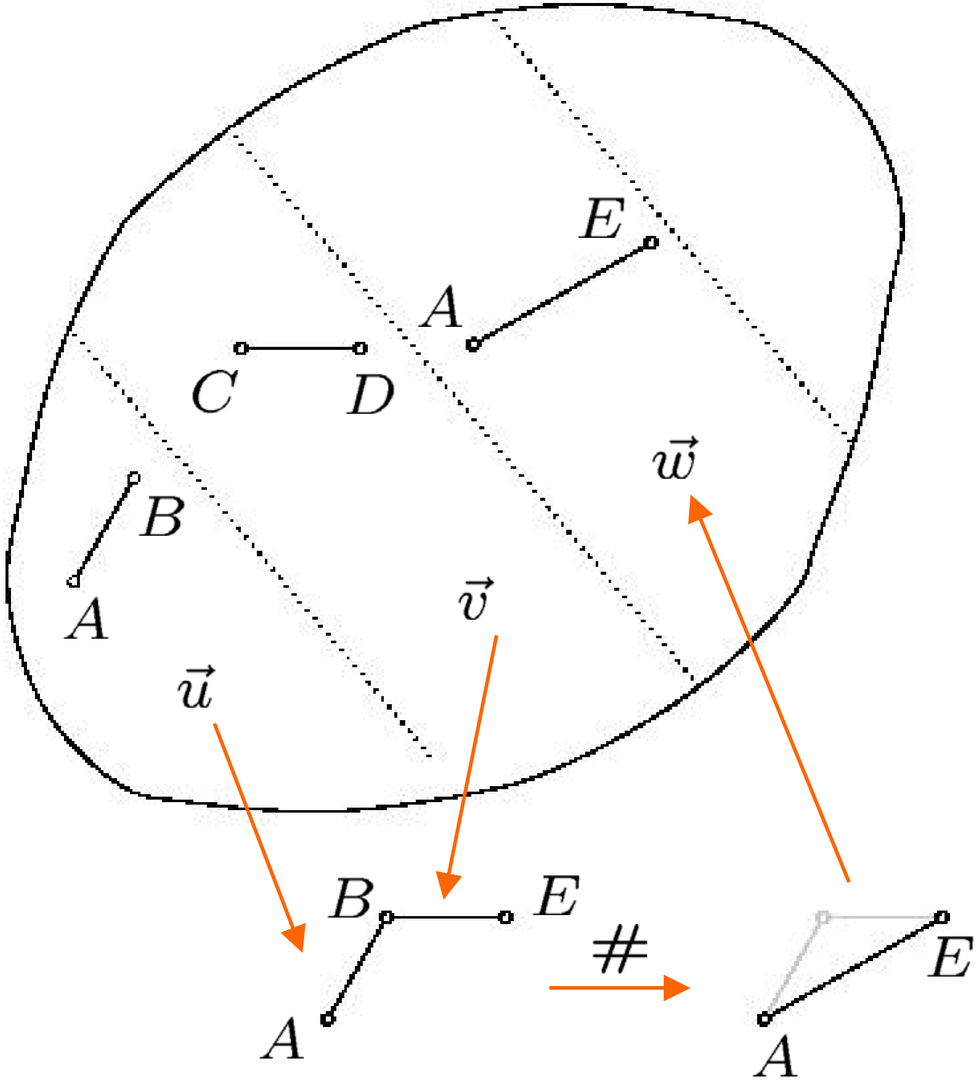
Addition in V

Addition: $\oplus : V \times V \to V$

$\vec{u} \oplus \vec{v} \to \vec{w}$



Take $(A, B) \in \vec{u}$ and $(C, D) \in \vec{v}$ and set $\vec{w}$ to the element of V such that $(A, B)\#(C, D) \to (A, E) \in \vec{w}$.

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Addition in V makes sense

$\vec{w} = \vec{u} \oplus \vec{v}$: Take $(A, B) \in \vec{u}$ and $(C, D) \in \vec{v}$ and set $\vec{w}$ to the element of V such that $(A, B)\#(C, D) \rightarrow (A, E) \in \vec{w}$.

We need to show that

the results obtained for different representatives in $\vec{u}$ and in $\vec{v}$ are in the same equivalence class $\vec{w}$.

Let

$(A, B), (A', B') \in \vec{u}$

$(C, D), (C', D') \in \vec{v}$

$(A, B)\#(C, D) \in \vec{w}$

$(A, B)\#(C, D) = (B, E)$

$(A', B')\#(C', D') = (B', E')$

Then

$(A, B) \leftrightarrow (A', B')$

$(C, D) \leftrightarrow (C', D')$

$(B, E) \leftrightarrow (C, D)$

$(B', E') \leftrightarrow (C', D')$

$(B, E) \leftrightarrow (B', E')$

$(A', B')\#(C', D') \in \vec{w}$

$\# : \mathcal{P}^2 \times \mathcal{P}^2 \rightarrow \mathcal{P}^2$ $(A, B)\#(C, D) \rightarrow (A, E)$ such that $(B, E) \leftrightarrow (C, D)$

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Operations for V and $\mathbb{R}$

Addition in V: $\oplus : V \times V \rightarrow V$

Multiplication of V by $\mathbb{R}$: $\odot : \mathbb{R} \times V \rightarrow V$

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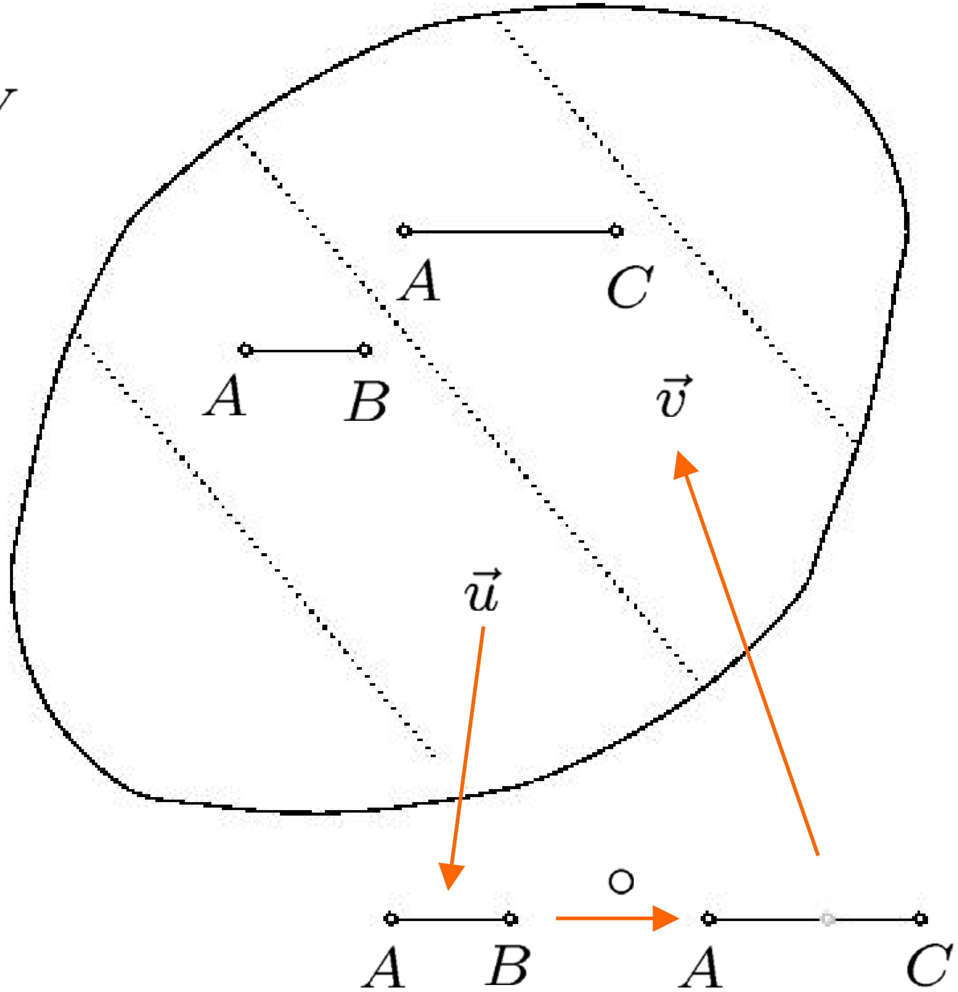
Scalar multiplication in V

Scalar multiplication: $\odot : \mathbb{R} \times V \rightarrow V$

$a \odot \vec{u} \rightarrow \vec{v}$



Take $(A, B) \in \vec{u}$ and set $\vec{v}$ to the element of V such that $a \circ (A, B) \in \vec{v}$.

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Affine space - scalar multiplication in $V = \mathcal{P}/\sim$

Take $(A, B) \in \vec{u}$ and set $\vec{v}$ to the element of V such that $a \circ (A, B) \in \vec{v}$.

We need to show that

the results obtained for different representatives of $\vec{u}$ are in the same equivalence class $\vec{v}$

Let

$a \in \mathbb{R}$

$(A, B), (A', B') \in \vec{u}$

$a \circ (A, B) \in \vec{v}$

Then

$(A, B) \leftrightarrow (A', B')$

$a \circ (A, B) \leftrightarrow a \circ (A', B')$

$a \circ (A', B') \in \vec{v}$

$$
\begin{array}{ccc}
(A, E) = a \circ (A, B) \text{ and } (C, F) & = & a \circ (C, D) \text{ and } (A, B) \leftrightarrow (C, D) \\
 & \Downarrow & \\
a \circ (A, B) = (A, E) & \leftrightarrow & (C, F) = a \circ (C, D)
\end{array}
$$

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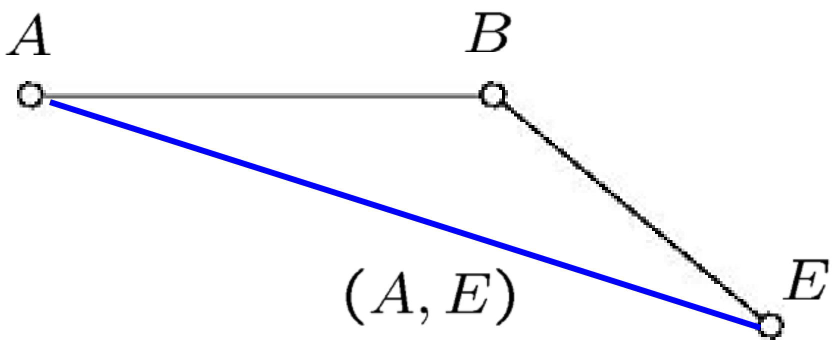
Axiom 1 holds

$\oplus : V \times V \rightarrow V$

Axioms: of linear space $(\mathbb{R}, V, \oplus, \odot)$

1. $\forall \vec{u}, \vec{v} \in V\colon\ \vec{u} \oplus \vec{v} = \vec{v} \oplus \vec{u}$

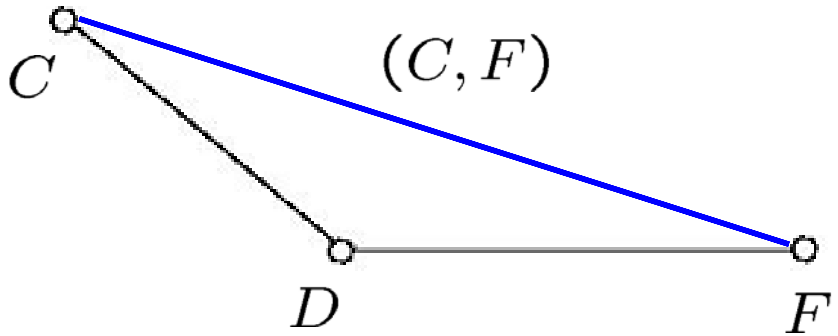


$$\underbrace{(A,B)}_{\in \vec{u}} \# \underbrace{(C,D)}_{\in \vec{v}} \rightarrow \underbrace{(A,E)}_{\in \vec{w}}$$

such that

$$(B,E) \leftrightarrow (C,D)$$

$$(A,E) \leftrightarrow (C,F)$$



$$\underbrace{(C,D)}_{\in \vec{v}} \# \underbrace{(A,B)}_{\in \vec{u}} \rightarrow \underbrace{(C,F)}_{\in \vec{w}}$$

such that

$$(D,F) \leftrightarrow (A,B)$$

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Space of free vectors V

Similarly, one can verify that axioms 2–8 hold, and thus

$$(\mathbb{R}, V, \oplus, \odot)$$

is a linear space

often called "the space of free vectors".

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The affine space with over the correponding space of free vectors V

Affine space $\mathcal{A} = (\mathcal{P}, V, \varphi)$

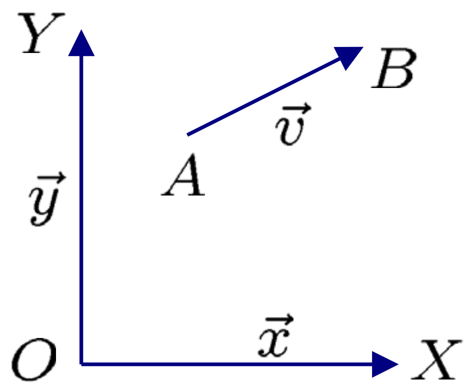
$\mathcal{P}$ ... set of points

V ... linear space of equivalence classes in $\mathcal{P}^2$, i.e. $\mathcal{P}^2/\leftrightarrow$

φ ... function $\varphi(A,B) \to \vec{u} \in V$ such that $(A,B) \in \vec{u}$

1. $\forall P, Q \in \mathcal{P}\ \exists! \vec{v} \in V : \varphi(P,Q) = \vec{v}$
2. $\forall P \in \mathcal{P}\ \forall \vec{v} \in V\ \exists! Q \in \mathcal{P} : \varphi(P,Q) = \vec{v}$ $\quad (\psi(P,\vec{v}) \to Q)$
3. $\forall P, Q, R \in \mathcal{P} : \varphi(P,Q) + \varphi(Q,R) - \varphi(P,R) = \vec{0}$

$\psi(P,\vec{v}) \to Q$ such that $(P,Q) \in \vec{v}$



Coordinates of (A,B) in $(O,X),(O,Y)$ $\equiv$ coordinates of $\vec{v}$ in the basis $\vec{x}, \vec{y}$, where $(A,B) \in \vec{v}$, $(O,X) \in \vec{x}$ and $(O,Y) \in \vec{y}$.

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Camera center

$\exists \mathtt{P} \in \mathbb{R}^{3\times 4}$, $rank\ \mathtt{P} = 3$, so that $\forall (u,v) \overset{corr}{\leftrightarrow} (x,y,z)\ \exists \alpha \in \mathbb{R}$:

$$\alpha \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \mathtt{P} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \quad \ldots \quad \alpha\, \mathbf{x} = \mathtt{P} \begin{bmatrix} \mathbf{X} \\ 1 \end{bmatrix} = \mathtt{A} \left[\mathtt{I} \,|\, -\mathbf{C} \right] \begin{bmatrix} \mathbf{X} \\ 1 \end{bmatrix}$$

$$0 = \mathtt{P} \begin{bmatrix} \mathbf{X} \\ 1 \end{bmatrix} = \mathtt{A} \left[\mathtt{I} \,|\, -\mathbf{C} \right] \begin{bmatrix} \mathbf{X} \\ 1 \end{bmatrix}$$

$\mathrm{rank}\, \mathtt{A} = 3 \quad \Updownarrow$

$$0 = \left[\mathtt{I} \,|\, -\mathbf{C} \right] \begin{bmatrix} \mathbf{X} \\ 1 \end{bmatrix} \quad \Rightarrow \mathbf{X} = \mathbf{C}$$

$$\left(0 = \left[\mathtt{I} \,|\, -\mathbf{C} \right] \begin{bmatrix} \mathbf{C} \\ 1 \end{bmatrix} \right)$$

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Camera Internal Calibration

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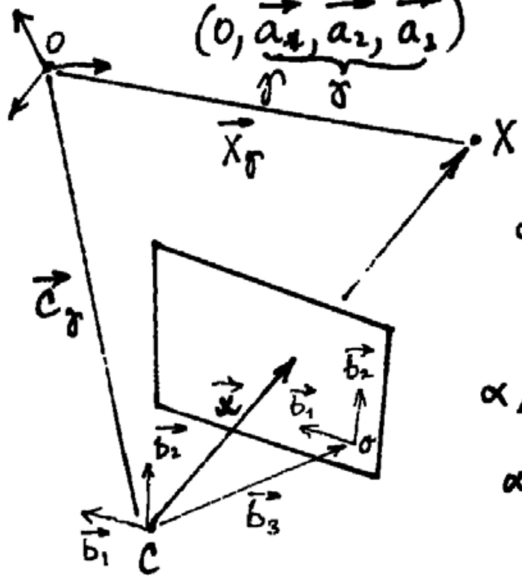
Camera rotation

Model promítání

$X \in A^3 \to x \in A^2$

↑ promítání

BÁZE $\underbrace{(0, \vec{a}_1, \vec{a}_2, \vec{a}_3)}_{\gamma}$



$\alpha x_\gamma = X_\gamma - c_\gamma$

$x_\gamma = A^{-1} x_\beta$

$\alpha A^{-1} x_\beta = X_\gamma - c_\gamma$

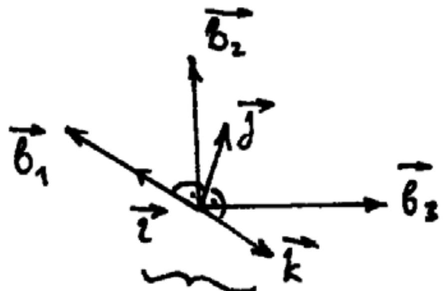
$\alpha x_\beta = A\,(X_\gamma - c_\gamma)$

$$\alpha x_\beta = \underbrace{(A \mid -A C_\gamma)}_{P \in \mathbb{R}^{3\times 4}} \begin{pmatrix} X_\gamma \\ 1 \end{pmatrix} \quad \leftarrow \mathbb{R}^{4\times 1}$$

↖ matice kamery

$\underbrace{(c, \vec{b}_1, \vec{b}_2, \vec{b}_3)}_{\beta}$ $\quad \begin{pmatrix} u \\ v \\ 1 \end{pmatrix} = x_\beta$

γ - ortonormální báze



$\delta = (\vec{i}, \vec{j}, \vec{k})$

$\delta \to \beta$

$\vec{i}_\beta \quad \vec{j}_\beta \quad$ ← obecný vektor

$$x_\beta = K x_\delta = \begin{pmatrix} k_{11} & k_{12} & k_{13} \\ 0 & k_{22} & k_{23} \\ 0 & 0 & k_{33} \end{pmatrix} x_\delta$$

$$\alpha x_\beta = (KR \mid -KRC) \begin{pmatrix} X \\ 1 \end{pmatrix}$$

$\gamma \to \delta \to \beta$

$x_\beta = K x_\gamma$

$x_\delta = R x_\gamma$

↑ ROTACE

$R^T \cdot R = E$

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Camera rotation

$\mathtt{A} = \mathtt{K}\,\mathtt{R}$ such that $\mathtt{R}^\top \mathtt{R} = \mathtt{I}$

$$\begin{aligned} \mathtt{A} &= \mathtt{K}\,\mathtt{R} \\ \begin{bmatrix} \mathbf{a}_1^\top \\ \mathbf{a}_2^\top \\ \mathbf{a}_3^\top \end{bmatrix} &= \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ 0 & k_{22} & k_{23} \\ 0 & 0 & k_{33} \end{bmatrix} \begin{bmatrix} \mathbf{r}_1^\top \\ \mathbf{r}_2^\top \\ \mathbf{r}_3^\top \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \mathbf{r}_1^\top \mathbf{r}_1 &= 1 \\ \mathbf{r}_1^\top \mathbf{r}_2 &= 0 \\ \mathbf{r}_1^\top \mathbf{r}_3 &= 0 \\ \mathbf{r}_2^\top \mathbf{r}_2 &= 1 \\ \mathbf{r}_2^\top \mathbf{r}_3 &= 0 \end{aligned}$$

$$\begin{aligned} \mathbf{a}_1^\top &= k_{11}\,\mathbf{r}_1^\top + k_{12}\,\mathbf{r}_2^\top + k_{13}\,\mathbf{r}_3^\top \\ \mathbf{a}_2^\top &= k_{22}\,\mathbf{r}_2^\top + k_{23}\,\mathbf{r}_3^\top \\ \mathbf{a}_3^\top &= k_{33}\,\mathbf{r}_3^\top \end{aligned}$$

$$\begin{aligned} \mathbf{a}_3^\top \mathbf{a}_3 &= k_{33}^2\,\mathbf{r}_3^\top \mathbf{r}_3 = k_{33}^2 & \rightarrow \quad & k_{33} \\ \mathbf{a}_2^\top \mathbf{a}_3 &= k_{22}\,k_{33}\,\mathbf{r}_2^\top \mathbf{r}_3 + k_{23}\,k_{33}\,\mathbf{r}_3^\top \mathbf{r}_3 = k_{23}\,k_{33} & \rightarrow \quad & k_{23} \\ \mathbf{a}_1^\top \mathbf{a}_3 &= k_{13}\,k_{33} & \rightarrow \quad & k_{13} \\ \mathbf{a}_2^\top \mathbf{a}_2 &= k_{22}^2 + k_{23}^2 & \rightarrow \quad & k_{22} \\ \mathbf{a}_1^\top \mathbf{a}_2 &= k_{12}\,k_{22} + k_{13}\,k_{23} & \rightarrow \quad & k_{12} \\ \mathbf{a}_1^\top \mathbf{a}_1 &= k_{11}^2 + k_{12}^2 + k_{13}^2 & \rightarrow \quad & k_{11} \end{aligned}$$

$$\rightarrow \mathtt{K} \rightarrow \mathtt{R} = \mathtt{K}^{-1}\mathtt{A}$$

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