Lecture 9

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Let us have two vectors $\vec{x}$, $\vec{y}$ with coordinates w.r.t. basis β

$$\begin{aligned}\vec{x}_\beta &= \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^\top \\ \vec{y}_\beta &= \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix}^\top\end{aligned}$$

Then the "matrix multiplication" of $\vec{x}_\beta$, $\vec{y}_\beta$ yields the following formula

$$\vec{x}_\beta^\top \vec{y}_\beta = x_1 y_1 + x_2 y_2 + x_3 y_3$$

Let us investigate the behavior of this formula when changing the basis w.r.t. which we write coordinates of vectors $\vec{x}$, $\vec{y}$.

Let matrix $\mathtt{A} \in \mathbb{R}^{3\times 3}$ transforms the coordinates when passing from the basis β to basis β'

$$\begin{aligned}\vec{x}_{\beta'} &= \mathtt{A}\,\vec{x}_\beta \\ \vec{y}_{\beta'} &= \mathtt{A}\,\vec{y}_\beta\end{aligned}$$

then

$$\vec{x}_{\beta'}^\top \vec{y}_{\beta'} = \vec{x}_\beta^\top \mathtt{A}^\top \mathtt{A}\,\vec{y}_\beta$$

We see that in general $\vec{x}_\beta^\top \mathtt{A}^\top \mathtt{A}\,\vec{y}_\beta \neq \vec{x}_\beta^\top \vec{y}_\beta$ since, for instance, when

$$\mathtt{A} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad \vec{x}_\beta = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \vec{y}_\beta = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

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then

$$
\begin{aligned}
\vec{x}_{\beta'}^\top \vec{y}_{\beta'} &= \vec{x}_\beta^\top \mathtt{A}^\top \mathtt{A}\, \vec{y}_\beta = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 4 \\
&\neq 1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \vec{x}_\beta^\top \vec{y}_\beta
\end{aligned}
$$

However, we also see that for all matrices $\mathtt{A}$ such that $\mathtt{A}^\top \mathtt{A} = \mathtt{I}$ we get

$$
\vec{x}_{\beta'}^\top \vec{y}_{\beta'} = \vec{x}_\beta^\top \mathtt{A}^\top \mathtt{A}\, \vec{y}_\beta = \vec{x}_\beta^\top \vec{y}_\beta
$$

Interestingly, when we assume that

$$
\vec{x}_\beta^\top \mathtt{A}^\top \mathtt{A}\, \vec{y}_\beta = \vec{x}_\beta^\top \vec{y}_\beta \qquad \text{for all vectors } \vec{x}_\beta,\ \vec{y}_\beta \in \mathbb{R}^3
$$

then $\mathtt{A}^\top \mathtt{A} = \mathtt{I}$ must be true! See the next argument.

Since we assume that the above equation holds true for all vectors in $\mathbb{R}^3$, we can choose some particular vectors for which it holds true. Let us choose all vectors with exactly one non-zero coordinate equal 1 and construct all equations with them. For instance, when taking

$$
\vec{x}_\beta = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \vec{y}_\beta = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}
$$

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we then get

$$\begin{bmatrix}1 & 0 & 0\end{bmatrix} \mathtt{A}^\top \mathtt{A} \begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\end{bmatrix} \begin{bmatrix}0\\1\\0\end{bmatrix} = 0$$

which means that $(\mathtt{A}^\top\mathtt{A})_{12}$ element of matrix $\mathtt{A}^\top\mathtt{A}$ is equal to 0. We see that by choosing all possible combinations of vectors with only one nonzero element, which is equal to 1, we can "pick" all elements of $\mathtt{A}^\top\mathtt{A}$ and make them equal to elements of identity matrix $\mathtt{I}$. It is also clear that the above argument works for any dimension, not only for three-dimensional vectors.

The above observation shows the importance of the role which play orthonormal matrices $\mathtt{A}^\top\mathtt{A} = \mathtt{I}$. We see that

$$\vec{x}_\beta^\top \mathtt{A}^\top \mathtt{A}\, \vec{y}_\beta = \vec{x}_\beta^\top \vec{y}_\beta \text{ for all vectors } \vec{x}_\beta, \vec{y}_\beta \in \mathbb{R}^n \Leftrightarrow \mathtt{A}^\top\mathtt{A} = \mathtt{I}$$

Now, let us derive the geometrical meaning of the formula

$$\vec{x}_\beta^\top \vec{y}_\beta \;=\; x_1\, y_1 + x_2\, y_2 + \cdots + x_n\, y_n$$

in the three-dimensional space.

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Let us start with two vectors with coordinates

$$\vec{x}_\beta = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^\top$$
$$\vec{y}_\beta = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix}^\top$$

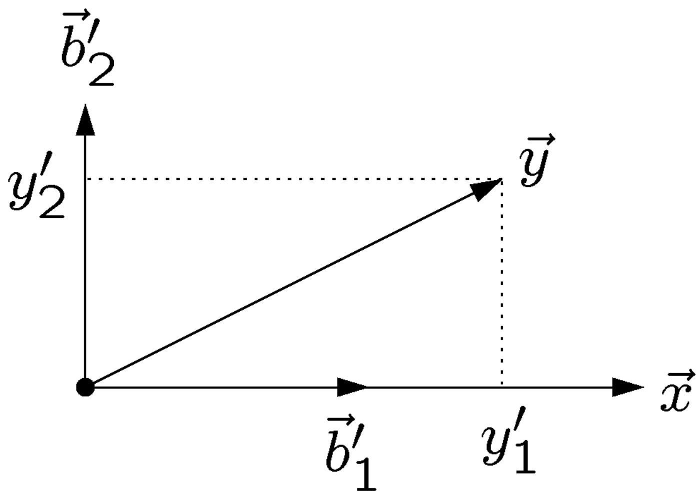
w.r.t. an orthonormal basis. We can choose a new orhonormal basis β' such that

$$\vec{x}_{\beta'} = \begin{bmatrix} x'_1 & 0 & 0 \end{bmatrix}^\top$$
$$\vec{y}_{\beta'} = \begin{bmatrix} y'_1 & y'_2 & 0 \end{bmatrix}^\top$$

and evaluate the formula

$$\frac{\vec{x}_{\beta'}^\top \vec{y}_{\beta'}}{(\vec{x}_{\beta'}^\top \vec{x}_{\beta'})^{\frac{1}{2}} (\vec{y}_{\beta'}^\top \vec{y}_{\beta'})^{\frac{1}{2}}} = \frac{x'_1 y'_1}{x'_1 (y'^2_1 + y'^2_2)^{\frac{1}{2}}} = \frac{y'_1}{(y'^2_1 + y'^2_2)^{\frac{1}{2}}} = \cos \measuredangle(\vec{x}, \vec{y})$$

thanks to the Pythagoras theorem:



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Since bases β, β' are orthonormal, we conclude that

$$\frac{\vec{x}_\beta^\top \vec{y}_\beta}{(\vec{x}_\beta^\top \vec{x}_\beta)^{\frac{1}{2}} (\vec{y}_\beta^\top \vec{y}_\beta)^{\frac{1}{2}}} = \frac{\vec{x}_{\beta'}^\top \vec{y}_{\beta'}}{(\vec{x}_{\beta'}^\top \vec{x}_{\beta'})^{\frac{1}{2}} (\vec{y}_{\beta'}^\top \vec{y}_{\beta'})^{\frac{1}{2}}} = \cos \measuredangle(\vec{x}, \vec{y})$$

We see that in an orthonormal basis β the formula

$$\vec{x}_\beta^\top \vec{y}_\beta = (\vec{x}_\beta^\top \vec{x}_\beta)^{\frac{1}{2}} (\vec{y}_\beta^\top \vec{y}_\beta)^{\frac{1}{2}} \cos \measuredangle(\vec{x}, \vec{y})$$

allows to measure the angle between vectors $\vec{x}$ and $\vec{y}$.

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Let us next look at the behaviour of the formula for the vector product under a change of the basis.

Let us have two vectors $\vec{x}$, $\vec{y}$ with coordinates w.r.t. basis β

$$\vec{x}_\beta = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^\top$$
$$\vec{y}_\beta = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix}^\top$$

and vector $\vec{z}_\beta$ constructed by using the vector product formula

$$\vec{z}_\beta \equiv \left| \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ i & j & k \end{bmatrix} \right| = \begin{bmatrix} i & j & k \end{bmatrix} \begin{bmatrix} x_2\, y_3 - x_3\, y_2 \\ -x_1\, y_3 + x_3\, y_1 \\ x_1\, y_2 - x_2\, y_1 \end{bmatrix} \equiv \begin{bmatrix} x_2\, y_3 - x_3\, y_2 \\ -x_1\, y_3 + x_3\, y_1 \\ x_1\, y_2 - x_2\, y_1 \end{bmatrix}$$

Let matrix $\mathtt{A} \in \mathbb{R}^{3\times3}$ transforms the coordinates when passing from the basis β to basis β'

$$\vec{x}_{\beta'} = \mathtt{A}\,\vec{x}_\beta$$
$$\vec{y}_{\beta'} = \mathtt{A}\,\vec{y}_\beta$$

we shall now construct vector $\vec{w}_{\beta'}$ in using the vector product formula on coordinates in β' and investigate its relationship to

$$\vec{z}_{\beta'} = \mathtt{A}\,\vec{z}_\beta$$

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Let us introduce

$$\vec{s} = \begin{bmatrix} i & j & k \end{bmatrix}^\top$$

and construct

$$\begin{aligned}
\vec{w}_{\beta'} &\equiv \left| \begin{bmatrix} \vec{x}_{\beta'}^\top \\ \vec{y}_{\beta'}^\top \\ \vec{s}^\top \end{bmatrix} \right| = \left| \begin{bmatrix} \vec{x}_{\beta}^\top \mathtt{A}^\top \\ \vec{y}_{\beta}^\top \mathtt{A}^\top \\ \vec{s}^\top \end{bmatrix} \right| = \left| \begin{bmatrix} \vec{x}_{\beta}^\top \\ \vec{y}_{\beta}^\top \\ \vec{s}^\top \mathtt{A}^{-\top} \end{bmatrix} \mathtt{A}^\top \right| = \left| \begin{bmatrix} \vec{x}_{\beta}^\top \\ \vec{y}_{\beta}^\top \\ \vec{s}^\top \mathtt{A}^{-\top} \end{bmatrix} \right| \left| \mathtt{A}^\top \right| \\
&= \vec{s}^\top \mathtt{A}^{-\top} \begin{bmatrix} x_2\, y_3 - x_3\, y_2 \\ -x_1\, y_3 + x_3\, y_1 \\ x_1\, y_2 - x_2\, y_1 \end{bmatrix} \left| \mathtt{A}^\top \right| \\
&\equiv \frac{\mathtt{A}^{-\top}}{|\mathtt{A}^{-\top}|} \begin{bmatrix} x_2\, y_3 - x_3\, y_2 \\ -x_1\, y_3 + x_3\, y_1 \\ x_1\, y_2 - x_2\, y_1 \end{bmatrix} \\
&= \frac{\mathtt{A}^{-\top}}{|\mathtt{A}^{-\top}|} \vec{z}_\beta \\
&= \frac{\mathtt{A}^{-\top}\mathtt{A}^{-1}}{|\mathtt{A}^{-\top}|} \mathtt{A}\, \vec{z}_\beta \\
&= \frac{\mathtt{A}^{-\top}\mathtt{A}^{-1}}{|\mathtt{A}^{-\top}|} \vec{z}_{\beta'}
\end{aligned}$$

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Hence we conclude

$$\vec{w}_{\beta'} \;=\; \frac{\mathtt{A}^{-\top}\mathtt{A}^{-1}}{|\mathtt{A}^{-\top}|}\,\vec{z}_{\beta'}$$

This again shows the importance of orthonormal matrices since when

$$\mathtt{A}^{\top}\mathtt{A} = \mathtt{I}\ \&\ |\mathtt{A}| = 1 \quad\Rightarrow\quad \vec{w}_{\beta'} \;=\; \frac{\mathtt{A}^{-\top}\mathtt{A}^{-1}}{|\mathtt{A}^{-\top}|}\,\vec{z}_{\beta'} = (\mathtt{A}\,\mathtt{A}^{-1})\ |\mathtt{A}|\ \vec{z}_{\beta'} = \vec{z}_{\beta'}$$

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Internally Calibrated Camera

We say that a camera is internally calibrated if the camera coordinate system is constructed by using an orhogonal basis δ. In such a case, we can measure the angle betwen projection rays generated by vectors $\vec{x}$, $\vec{y}$ by the formula

$$\cos \measuredangle(\vec{x}, \vec{y}) = \frac{\vec{x}_\delta^\top \vec{y}_\delta}{(\vec{x}_\delta^\top \vec{x}_\delta)^{\frac{1}{2}} (\vec{y}_\delta^\top \vec{y}_\delta)^{\frac{1}{2}}}$$

In general, the camera basis β, derived from the image basis and the projection center, is often not orthogonal. Then, we need to use

$$\vec{x}_\beta = \mathtt{K}\,\vec{x}_\delta$$

and evaluate

$$\cos \measuredangle(\vec{x}, \vec{y}) = \frac{\vec{x}_\beta^\top \mathtt{K}^{-\top}\mathtt{K}^{-1} \vec{y}_\beta}{(\vec{x}_\beta^\top \mathtt{K}^{-\top}\mathtt{K}^{-1} \vec{x}_\beta)^{\frac{1}{2}} (\vec{y}_\beta^\top \mathtt{K}^{-\top}\mathtt{K}^{-1} \vec{y}_\beta)^{\frac{1}{2}}} = \frac{\vec{x}_\beta^\top \omega\, \vec{y}_\beta}{(\vec{x}_\beta^\top \omega\, \vec{x}_\beta)^{\frac{1}{2}} (\vec{y}_\beta^\top \omega\, \vec{y}_\beta)^{\frac{1}{2}}}$$

with

$$\omega = \mathtt{K}^{-\top}\mathtt{K}^{-1}$$

Once we have matrix ω, we can recover matrix $\mathtt{K}$ from it.

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Assuming

$$\mathtt{K} = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ 0 & k_{22} & k_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

we get

$$\mathtt{K}^{-1} = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ 0 & k_{22} & k_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{k_{11}} & \frac{-k_{12}}{k_{11}k_{22}} & \frac{k_{12}\,k_{23}-k_{13}\,k_{22}}{k_{11}\,k_{22}\,k_{23}} \\ 0 & \frac{1}{k_{22}} & \frac{-k_{23}}{k_{22}} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ 0 & m_{22} & m_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathtt{K} = \begin{bmatrix} \frac{1}{m_{11}} & \frac{-m_{12}}{m_{11}m_{22}} & \frac{m_{12}\,m_{23}-m_{13}\,m_{22}}{m_{11}\,m_{22}\,m_{23}} \\ 0 & \frac{1}{m_{22}} & \frac{-m_{23}}{m_{22}} \\ 0 & 0 & 1 \end{bmatrix}$$

$$\omega = \mathtt{K}^{-\top}\mathtt{K}^{-1}$$

$$\begin{bmatrix} o_{11} & o_{12} & o_{13} \\ o_{12} & o_{22} & o_{23} \\ o_{13} & o_{23} & o_{33} \end{bmatrix} = \begin{bmatrix} m_{11}^2 & m_{11}\,m_{12} & m_{11}\,m_{13} \\ m_{11}\,m_{12} & m_{12}^2 + m_{22}^2 & m_{12}\,m_{13} + m_{22}\,m_{23} \\ m_{11}\,m_{13} & m_{12}\,m_{13} + m_{22}\,m_{23} & m_{13}^2 + m_{23}^2 + 1 \end{bmatrix}$$

which can be solved for $\mathtt{K}^{-1}$ up to the sign of the rows of $\mathtt{K}^{-1}$ as follows.

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$$\begin{bmatrix} o_{11} & o_{12} & o_{13} \\ o_{12} & o_{22} & o_{23} \\ o_{13} & o_{23} & o_{33} \end{bmatrix} = \begin{bmatrix} m_{11}^2 & m_{11}\, m_{12} & m_{11}\, m_{13} \\ m_{11}\, m_{12} & m_{12}^2 + m_{22}^2 & m_{12}\, m_{13} + m_{22}\, m_{23} \\ m_{11}\, m_{13} & m_{12}\, m_{13} + m_{22}\, m_{23} & m_{13}^2 + m_{23}^2 + 1 \end{bmatrix}$$

provides equations

$$\begin{aligned}
o_{11} = m_{11}^2 \quad &\Rightarrow \quad m_{11} = s_1 \sqrt{o_{11}} \\
o_{12} = m_{11}\, m_{12} \quad &\Rightarrow \quad m_{12} = o_{12}/(s_1 \sqrt{o_{11}}) = s_1\, o_{12}/\sqrt{o_{11}} \\
o_{13} = m_{11}\, m_{13} \quad &\Rightarrow \quad m_{13} = o_{13}/(s_1 \sqrt{o_{11}}) = s_1\, o_{13}/\sqrt{o_{11}} \\
o_{22} = m_{12}^2 + m_{22}^2 \quad &\Rightarrow \quad m_{22} = s_2 \sqrt{o_{22} - m_{12}^2} = s_2 \sqrt{o_{22} - o_{12}^2/o_{11}} \\
o_{23} = m_{12}\, m_{13} + m_{22}\, m_{23} \quad &\Rightarrow \quad m_{23} = s_2\, (o_{23} - o_{12}\, o_{13}/o_{11})/\sqrt{o_{22} - o_{12}^2/o_{11}} \\
&\qquad\qquad = s_2\, (o_{11}\, o_{23} - o_{12}\, o_{13})/\sqrt{o_{11}^2 o_{22} - o_{11}\, o_{12}^2}
\end{aligned}$$

for $s_1 = \pm 1$ and $s_2 = \pm 1$.

Hence

$$\mathtt{K} = \begin{bmatrix} s_1 \sqrt{o_{11}} & s_1\, o_{12}/\sqrt{o_{11}} & s_1\, o_{13}/\sqrt{o_{11}} \\ 0 & s_2 \sqrt{o_{22} - o_{12}^2/o_{11}} & s_2\, (o_{23} - o_{12}\, o_{13}/o_{11})/\sqrt{o_{22} - o_{12}^2/o_{11}} \\ 0 & 0 & 1 \end{bmatrix}^{-1}$$

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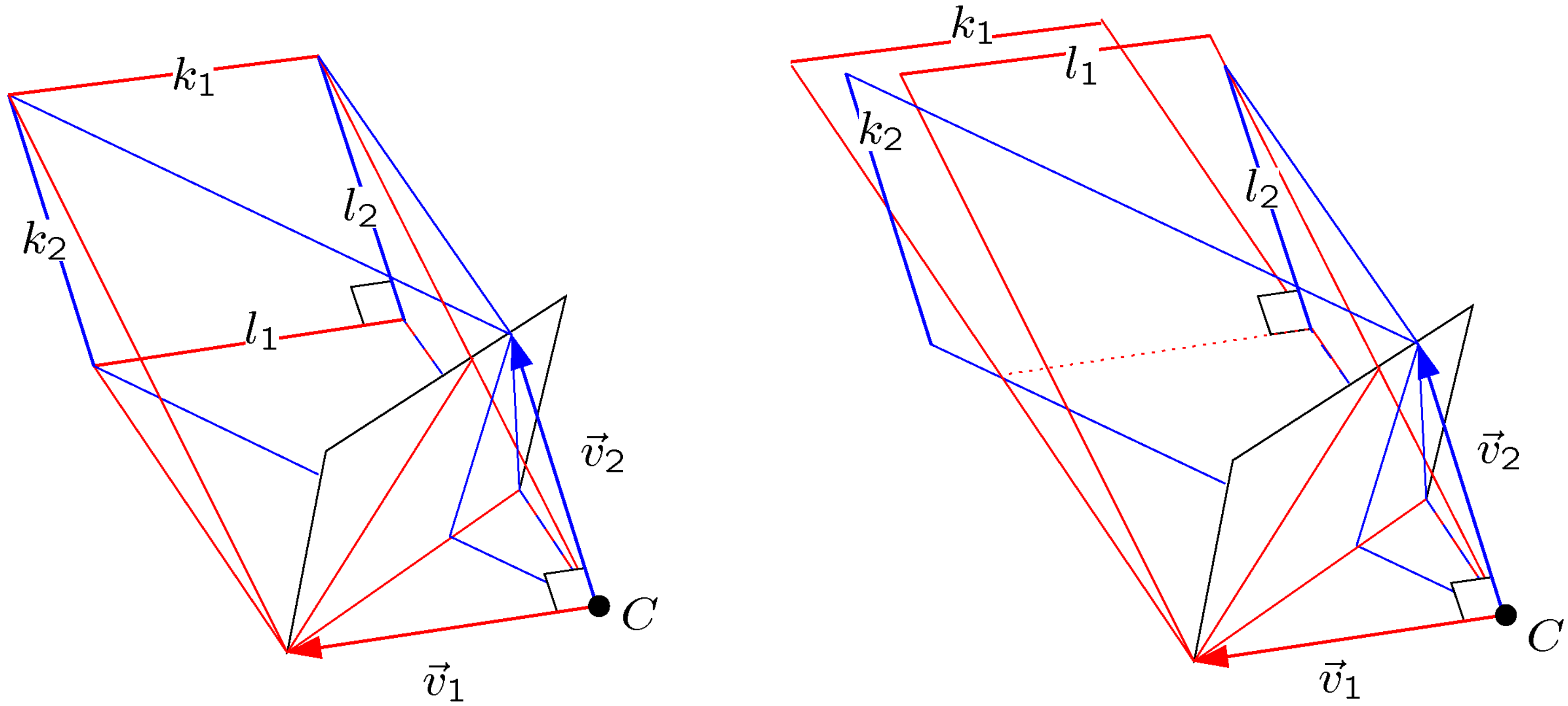
Camera calibration from vanishing points



We will now show how to calibrate the camera by fining the matrix $\omega = \mathtt{K}^{-\top}\mathtt{K}^{-1}$ from at least three vanishing points in general position.

Let us have two pairs of parallel lines in space, such that they are also orthogonal, i.e. let k_1 be parallel with l_1 and k_2 be parallel with l_2 and at the same time let k_1 be orthogonal to k_2 and l_1 be otrthogonal to l_2.

This, for instance, happens when lines k_1, l_2, k_2, l_2 form a rectangle but they also may be arranged in the three-dimensional space as non-intersecting.

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Let us have two pairs of parallel lines in space, such that they are also orthogonal, i.e. let k_1 be parallel with l_1 and k_2 be parallel with l_2 and at the same time let k_1 be orthogonal to k_2 and l_1 be otrthogonal to l_2.

This, for instance, happens when lines k_1, l_2, k_2, l_2 form a rectangle but they also may be arranged in the three-dimensional space as non-intersecting.

Let lines k_1, l_1, k_2, l_2 be represented by the corresponding vetors $\vec{k_1}_\beta, \vec{l_1}_\beta, \vec{k_2}_\beta, \vec{l_2}_\beta$ in the camera coordinates system with (in general non-orthogonal) basis β.

Parallel lines k_1 and l_1, resp. k_2 and l_2, generate vanishing points

$$\begin{aligned}\vec{v}_{1\beta} &= \vec{k_1}_\beta \times \vec{l_1}_\beta \\ \vec{v}_{2\beta} &= \vec{k_2}_\beta \times \vec{l_2}_\beta\end{aligned}$$

Vector $\vec{v}_1$ is a direction vector of the line through C, which is parallel with line l_1. Vector $\vec{v}_2$ is a direction vector of the line through C, which is parallel with line l_2. Lines l_1 and l_2 are perpendicular. Threfore, vector $\vec{v}_1$ is perpendicular to vector $\vec{v}_2$.

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The perpendicularity of $\vec{v}_1$ to $\vec{v}_2$ is, in the camera orthogonal basis δ, modeled by

$$\vec{v}_{1\delta}^\top \vec{v}_{2\delta} = 0$$

We therefore get

$$\begin{aligned} \vec{v}_{1\beta}^\top \mathtt{K}^{-1} \mathtt{K}^{-1} \vec{v}_{2\beta} &= 0 \\ \vec{v}_{1\beta}^\top \omega \, \vec{v}_{2\beta} &= 0 \end{aligned}$$

which is a linear homogeneous equation on ω.

There are 6 unknowns in ω and hence we need 5 pairs of perpendicular vanishing points spaning $\mathbb{R}^3$ to recover the one-dimensional space of matrices ω.

Often, when working with digital cameras, we can assume that pixels are square and hence the canonical choice of coordinates in the image plane, i.e. setting the corners of a pixel to $(0,0)^\top$, $(1,0)^\top$, $(0,1)^\top$, $(1,1)^\top$, gives an orthogonal basis in the image coordinate system and consequently, the camera coordinate system basis β has the first two vectors orthogonal. This leads to a more special $\mathtt{K}$ matrix, which is then as

$$\mathtt{K} = \begin{bmatrix} k_{11} & 0 & k_{13} \\ 0 & k_{11} & k_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

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The corresponding

$$\omega = \frac{1}{k_{11}^2}\begin{bmatrix} 1 & 0 & -k_{13} \\ 0 & 1 & -k_{23} \\ -k_{13} & -k_{23} & k_{11}^2 + k_{13}^2 + k_{23}^2 \end{bmatrix}$$

then provides equation

$$\begin{aligned}
\vec{v}_{1\beta}^\top \omega \, \vec{v}_{2\beta} &= 0 \\
\begin{bmatrix} v_{11} & v_{12} & v_{13} \end{bmatrix} \begin{bmatrix} 1 & 0 & o_1 \\ 0 & 1 & o_2 \\ o_1 & o_2 & o_3 \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \\ v_{23} \end{bmatrix} &= 0 \\
\begin{bmatrix} v_{23}\, v_{11} + v_{21}\, v_{13} & v_{23}\, v_{12} + v_{22}\, v_{13} & v_{23}\, v_{13} \end{bmatrix} \begin{bmatrix} o_1 \\ o_2 \\ o_3 \end{bmatrix} &= v_{21}\, v_{11} + v_{22}\, v_{12}
\end{aligned}$$

Now, we need only 3 pairs of perpendicular vanishing points, e.g. to observe 3 rectangles not all in one plane to compute o_1, o_2, o_3 and then

$$\begin{aligned}
k_{13} &= -o_1 \\
k_{23} &= -o_2 \\
k_{11} &= \sqrt{o_3 - k_{13}^2 - k_{23}^2}
\end{aligned}$$

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