8 Projective space

8.1 Motivation – union of ideal points of all affine planes

Figure 8.1(a) shows a perspective image of three sets of parallel lines generated by sides of a cube in the three-dimensional real affine space. The images of the three sets of parallel lines converge to vanishing points $V_1$, $V_2$ and $V_3$. The cube has six faces. Each face generates two pairs of parallel lines and hence two vanishing points. Each face generates an affine plane which can be extended into a projective plane by adding a line of ideal points of that plane. The projection of the three ideal lines are vanishing lines $l_{12}, l_{23}, l_{31}$.

Imagine now all possible affine planes of the three-dimensional affine space and their corresponding ideal points. Let us take the union $V$ of the sets of ideal points of all planes. There is exactly one ideal point for every set of parallel lines in $V$, i.e. there is in one-to-one correspondence between elements of $V$ (ideal points) and directions in the three-dimensional affine space. Notice also that every plane $\pi$ generates one ideal line $l_{\pi}$ of its ideal points and that all other planes parallel with $\pi$ generate the same $l_{\pi}$, Figure 8.1.

It suggests itself to extend the three-dimensional affine space by adding the set $V$ to it, analogically to how we have extended the affine plane. In this new space, all parallel lines will intersect. We will call this space the three-dimensional real projective space and denote it $\mathbb{P}^3$. Let us develop an algebraic model of $\mathbb{P}^3$.

We start with the three-dimensional real affine space $\mathbb{A}^3$ and fix a coordinate system $(O, \delta)$ with $\delta = (\vec{d}_1, \vec{d}_2, \vec{d}_3)$. An affine plane $\pi$ is a set of points of $\mathbb{A}^3$ represented in $(O, \delta)$ by the set of vectors

$$\pi = \{[x, y, z] \mid a x + b y + c z + d = 0, a, b, c, d \in \mathbb{R}, a^2 + b^2 + c^2 \neq 0\} \quad (8.1)$$

We see that the point of $\pi$ represented by vector $[x, y, z] \mathbb{T}$ can also be represented by one-dimensional subspace $\{\lambda [x, y, z, 1] \mid \lambda \in \mathbb{R}\}$ of $\mathbb{R}^4$ and hence $\pi$ can be seen as the set

$$\pi = \{\lambda [x, y, z, 1] \mid \lambda \in \mathbb{R}\} \mid \{a, b, c, d\} [x, y, z, 1] \mathbb{T} = 0, a, b, c, d \in \mathbb{R}, a^2 + b^2 + c^2 \neq 0\} \quad (8.2)$$

of one-dimensional subspaces of $\mathbb{R}^4$.

Notice that we did not require $\lambda \neq 0$ in the above definition. This is because we establish the correspondence between a vector $[x, y, z]$ and the corresponding complete one-dimensional subspace $\{\lambda [x, y, z, 1] \mid \lambda \in \mathbb{R}\}$ of $\mathbb{R}^4$ and since every linear space contains zero vector, we admit zero $\lambda$.

Every $[x, y, z] \mathbb{T} \in \mathbb{R}^3$ represents in $(O, \delta)$ a point of $\mathbb{A}^3$ and hence the subset

$$\mathbb{A}^3 = \{\lambda [x, y, z, 1] \mid \lambda \in \mathbb{R}\} | x, y, z \in \mathbb{R}\} \quad (8.3)$$
Figure 8.1: (a) A perspective image of a cube generates three vanishing points $V_1$, $V_2$ and $V_3$ and hence also three vanishing lines $l_{12}$, $l_{23}$ and $l_{31}$. (b) Every plane adds one line of ideal points to the three-dimensional affine space. Every ideal point corresponds to one direction, i.e. to a set of parallel lines. Each ideal line corresponds to a set of parallel planes.

of one-dimensional subspaces of $\mathbb{R}^4$ represents $\mathbb{A}^3$.

We observe that we have not used all one-dimensional subspaces of $\mathbb{R}^4$ to represent $\mathbb{A}^3$. The subset

$$\pi_\infty = \{\{\lambda [x, y, z, 0]^T \mid \lambda \in \mathbb{R}\} \mid x, y, z \in \mathbb{R}, x^2 + y^2 + z^2 \neq 0\} \quad (8.4)$$

of one-dimensional subspaces of $\mathbb{R}^4$ is in one-to-one correspondence with all non-zero vectors of $\mathbb{A}^3$, i.e. in one-to-one correspondence with the set of directions in $\mathbb{A}^3$. This is the set of ideal points which we add to $\mathbb{A}^3$ to get the three-dimensional real projective space

$$\mathbb{P}^3 = \{\{\lambda [x, y, z, w]^T \mid \lambda \in \mathbb{R}\} \mid x, y, z, w \in \mathbb{R}, x^2 + y^2 + z^2 + w^2 \neq 0\} \quad (8.5)$$

which is the set of all one-dimensional subspaces of $\mathbb{R}^4$. Notice that $\mathbb{P}^3 = \mathbb{A}^3 \cup \pi_\infty$.

§ 41 Points Every non-zero vector of $\mathbb{R}^4$ generates a one-dimensional subspace and thus represents a point of $\mathbb{P}^3$. The zero vector $[0, 0, 0, 0]^T$ does not represent any point.

§ 42 Planes Affine planes $\pi_{\mathbb{A}^3}$, Equation 8.2, are in one-to-one correspondence to the subset

$$\pi_{\mathbb{A}^3} = \{\{\lambda [a, b, c, d]^T \mid \lambda \in \mathbb{R}\} \mid a, b, c, d \in \mathbb{R}, a^2 + b^2 + c^2 \neq 0\} \quad (8.6)$$

of the set of one-dimensional subspaces of $\mathbb{R}^4$. There is only one one-dimensional subspace of $\mathbb{R}^4$, $\{\lambda [0, 0, 0, 1]^T \mid \lambda \in \mathbb{R}\}$ missing in $\pi_{\mathbb{A}^3}$. It is exactly the one-dimensional subspace corresponding to the set $\pi_\infty$ of ideal points of $\mathbb{P}^3$

$$\pi_\infty = \{\{\lambda [x, y, z, w]^T \mid \lambda \in \mathbb{R}\} \mid x, y, z, w \in \mathbb{R}, x^2 + y^2 + z^2 \neq 0, [0, 0, 0, 1] [x, y, z, w]^T = 0\} \quad (8.7)$$
We can take another view upon planes and observe that affine planes are in one-to-one correspondence to the three-dimensional subspaces of $\mathbb{R}^4$. The set $\pi_\infty$ also corresponds to a three-dimensional subspace of $\mathbb{R}^4$. Hence $\pi_\infty$ can be considered another plane, the ideal plane of $\mathbb{P}^3$.

The set of planes of $\mathbb{P}^3$ can be hence represented by the set of one-dimensional subspaces of $\mathbb{R}^4$

$$\pi_{\mathbb{P}^3} = \{ \{ \lambda [a, b, c, d]^T | \lambda \in \mathbb{R} \} | a, b, c, d \in \mathbb{R}, a^2 + b^2 + c^2 + d^2 \neq 0 \} \quad (8.8)$$

but can also be viewed as the set of three-dimensional subspaces of $\mathbb{R}^4$.

We see that there is a duality between points and planes of $\mathbb{P}^3$. They both are represented by one-dimensional subspaces of $\mathbb{R}^4$ and we see that point $X$ represented by vector $\vec{X} = [x, y, x, w]^T$ is incident to plane $\pi$ represented by vector $\vec{\pi} = [a, b, c, d]^T$, i.e. $X \circ \pi$, when

$$\vec{\pi}^T \vec{X} = \begin{bmatrix} a & b & c & d \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = ax + by + cz + dw = 0 \quad (8.9)$$

§ 43 Lines Lines in $\mathbb{P}^3$ are represented by two-dimensional subspaces of $\mathbb{R}^4$. Unlike in $\mathbb{P}^2$, lines are not dual to points.

### 8.2 Perspective camera in projective space

Perspective camera model, Equation 5.11, is actually capable of working in $\mathbb{P}^3$. First, we used Equation 5.11 with $\vec{x}_\beta(3) = 1$ to describe the perspective projection from $\mathbb{A}^3$ to $\mathbb{A}^2$. Then we have extended the real affine plane into the real projective plane and allowed $\vec{x}_\beta(3) = 0$ to project points from the camera principal plane into ideal points in the image. Finally, now, we will use the same model to project all points, represented by $\vec{X}_\Delta$, from $\mathbb{P}^3$ into points, represented by $\vec{x}_\beta$, in $\mathbb{P}^2$

$$\eta \vec{x}_\beta = P \vec{X}_\Delta \quad (8.10)$$

with $\eta \in \mathbb{R}$, $\eta \neq 0$, $\vec{x}_\beta \in \mathbb{R}^3$, $P \in \mathbb{R}^{3 \times 4}$, rank $P = 3$ and $\vec{X}_\Delta \in \mathbb{R}^4$, $\vec{X}_\Delta \neq \vec{0}$. The left side of Equation 8.10 is identical with Equation 5.11. This is because we represented points in images right from the beginning by three-dimensional vectors in the camera coordinate system. The difference is on the right.

Vector $\vec{X}_\Delta$ can be understood as

$$\vec{X}_\Delta = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} \vec{X}_\delta \\ 1 \end{bmatrix}, \quad w \neq 0 \quad (8.11)$$

where $\vec{X}_\delta$ represents a point of $\mathbb{A}^3$ in basis $\delta$ and $\vec{Y}_\delta$ represents a direction in $\mathbb{A}^3$ in basis $\delta$. 

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Figure 8.2: Point \( X \) is projected to \( v_0 \) from the camera center \( C_0 \), which is \( f_0 \) far from the projection plane \( \pi \). When moving camera center \( C(f) \) away from \( \pi \) along the line perpendicular to \( \pi \) and passing through \( C_0 \), the projection of \( X \) tends to (but never reaches) point \( v \). Point \( v \) is obtained by projecting \( X \) along the ray \( p \), which is parallel to the direction of \( \vec{e}_3 \). This parallel projection is equal to the perspective projection model when considering the ideal projection center \( C(\infty) \).

This construction partially determines basis \( \Delta \), which can be then written as

\[
\Delta = \left( \begin{array}{c} u \vec{d}_1, u \vec{d}_2, u \vec{d}_3, \vec{d}_4 \end{array} \right) \quad \text{with} \quad \delta = \left( \vec{d}_1, \vec{d}_2, \vec{d}_3 \right) \quad \text{and} \quad u \neq 0
\]

Scalar \( u \) allows for a global change of scale on affine points and directions. We can write vectors of \( \delta \) in \( \Delta \) as

\[
\vec{d}_1\Delta = \begin{bmatrix} u \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{d}_2\Delta = \begin{bmatrix} 0 \\ u \\ 0 \\ 0 \end{bmatrix}, \quad \vec{d}_3\Delta = \begin{bmatrix} 0 \\ 0 \\ u \\ 0 \end{bmatrix}
\]

### 8.2.1 Parallel projection

In \( \mathbb{P}^3 \) we can generalize the interpretation of the camera projection matrix to model parallel projection. Consider a camera projection matrix, Equation 5.52

\[
P = \begin{bmatrix} \frac{1}{f} K R & -\frac{1}{f} K R \vec{C}_b \end{bmatrix}
\]

with elements of \( K \) given by Equations 5.44, 5.43, 5.41 and Equation 5.45

\[
K = \begin{bmatrix} \frac{f}{|b_1|} & \frac{f \cos \angle(b_1, b_2)}{|b_1|} & k_{13} \\ \frac{f \sin \angle(b_1, b_2)}{|b_1|} & k_{23} \\ 0 & 0 & 1 \end{bmatrix}
\]
To simplify the formulas, we introduce

\[
a = \frac{1}{|b_1|}, \quad b = -\frac{\cos \angle(b_1, b_2)}{|b_1| \sin \angle(b_1, b_2)}, \quad c = \frac{1}{|b_2| \sin \angle(b_1, b_2)} \tag{8.16}
\]

and write the projection equation 5.46 a follows

\[
\eta \vec{x}_\beta = \frac{1}{f} \begin{bmatrix} f & a & b & k_{13} \\ 0 & f & c & k_{23} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \begin{bmatrix} I \end{bmatrix} - \vec{C}_\delta \end{bmatrix} \begin{bmatrix} X_\delta \\ 1 \end{bmatrix}
\tag{8.17}
\]

Figure 8.2 shows that the parallel projection of point \(X\) along the direction \(\vec{e}_3\), which is perpendicular to the projection plane \(\pi\), corresponds to the perspective projection of \(X\) with the camera center \(C(\infty)\) at the ideal point corresponding to the direction \(\vec{e}_3\). Let us see an algebraic justification for this claim.

We use the camera orthonormal coordinate system with basis \(\epsilon = [\vec{e}_1, \vec{e}_2, \vec{e}_3]\), Figure 5.4, and parameterize the projection center \(C_\delta\), expressed in the world basis \(\delta\), by the focal length \(f\).

\[
\vec{C}_\delta(f) = \vec{C}_\delta(f_0) - R^{-1} \begin{bmatrix} 0 \\ 0 \\ f - f_0 \end{bmatrix} \tag{8.18}
\]

Rotation matrix \(R^{-1}\) transforms coordinates from \(\epsilon\) to \(\delta\), Equation 5.56, and hence its third column equals \(\vec{e}_3\). Parameter \(\eta\) in Equation 8.17 is the distance of \(X\) from \(C(f)\) along \(\vec{e}_3\) in multiplies of \(f\). Therefore, we can write

\[
\eta = \frac{f + g}{f} \tag{8.19}
\]

where \(g\) is the distance of \(X\) from \(\pi\). Coming back to Equation 8.17, we are getting

\[
\frac{f + g}{f} \vec{x}_\beta = \frac{1}{f} \begin{bmatrix} f & a & b & k_{13} \\ 0 & f & c & k_{23} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \begin{bmatrix} I \end{bmatrix} - \vec{C}_\delta(f_0) + R^{-1} \begin{bmatrix} 0 \\ 0 \\ f - f_0 \end{bmatrix} \end{bmatrix} \begin{bmatrix} X_\delta \\ 1 \end{bmatrix} \tag{8.20}
\]

We can now compute the limit of the equation for \(f\) going to \(\infty\)

\[
\vec{x}_\beta = \begin{bmatrix} a & b & 0 \\ 0 & c & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} R \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} R \vec{C}_\delta(f_0) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} X_\delta \\ 1 \end{bmatrix} \tag{8.21}
\]
Figure 8.3: The image plane of the parallel projection, which is perpendicular to $\vec{e}_3$, can be placed anywhere, e.g. to pass through the origin $O$ of the world coordinate system. The image coordinate system is placed to the perpendicular projection of $C$ into $\pi$.

where $r_1^T, r_2^T$ are the first two rows of $R$.

Let us now investigate the meaning of Equation 8.21. First of all observe that

$$
\begin{bmatrix}
0 \\
0
\end{bmatrix}
= 
\begin{bmatrix}
\begin{bmatrix} a & b & -r_1^T C_3(f_0) \\ 0 & c & -r_2^T C_3(f_0) \end{bmatrix} & \begin{bmatrix} r_1^T \\ r_2^T \\ 1 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
\end{bmatrix}
$$

(8.22)

where $r_3^T$ is the third row of $R$. We see that the camera center is an ideal point. To see which point it is, we realize that $R$ is a rotation matrix and therefore $R^{-1} = R^T$. Hence, $r_1 = \vec{e}_1$, $r_2 = \vec{e}_2$ and $r_3 = \vec{e}_3$. That leads to

$$
\vec{C}(x) = \begin{bmatrix} \vec{e}_3 \\ 0 \end{bmatrix}
$$

(8.23)

which is an ideal point representing the direction $\vec{r}_3$ in the world coordinate system.

Secondly, notice that $f$, $k_{13}$ and $k_{23}$ disappeared from the projection Equation 8.21. They do not have any geometrical meaning when considering parallel projection and hence can’t be meaningfully defined in that case.

Finally, observe that quantities $r_1^T \vec{C}_3(f_0)$ and $r_2^T \vec{C}_3(f_0)$ are the first two coordinates of $\vec{C}(f_0)$ in basis $\epsilon$. We can thus define vector

$$
\vec{x}_0 = \begin{bmatrix} r_1^T \vec{C}_3(f_0) \\ r_2^T \vec{C}_3(f_0) \end{bmatrix}
$$

(8.24)

which represents the origin of the image coordinate system, which is placed into the perpendicular projection of $C$ into $\pi$, Figure 8.3.

Parallel projection can be represented by the same formula as perspective projection by allowing the projection center to become an ideal point. We can therefore associate parallel projections of points represented in an orthonormal world coordinate system of $\mathbb{R}^3$ with the of $3 \times 4$ matrices $P$ with rank $P = 3$, which have the third row equal to $[0, 0, 0, 1]$. 

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