Homography from point pairs

Projective transformation, Collineation

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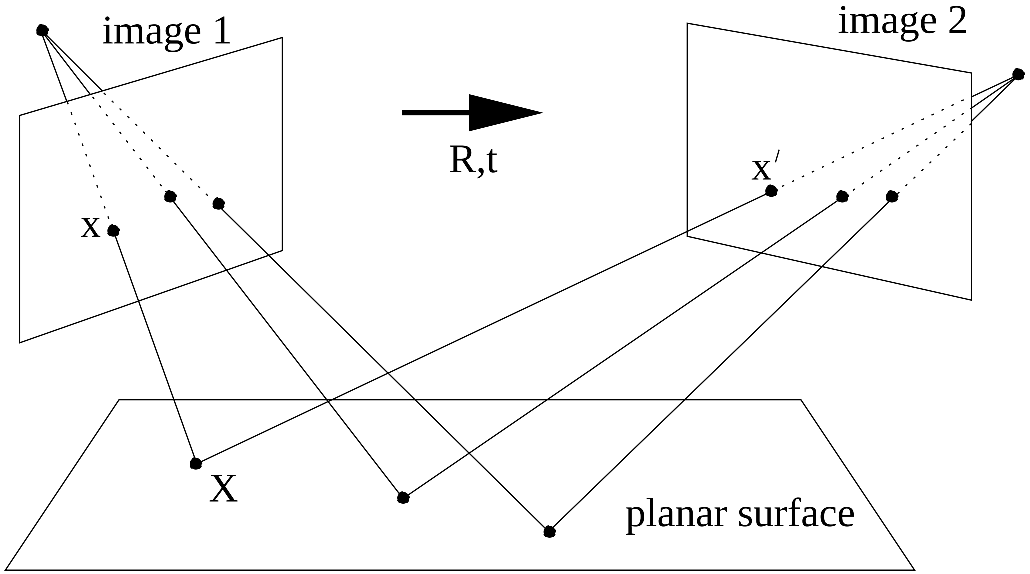
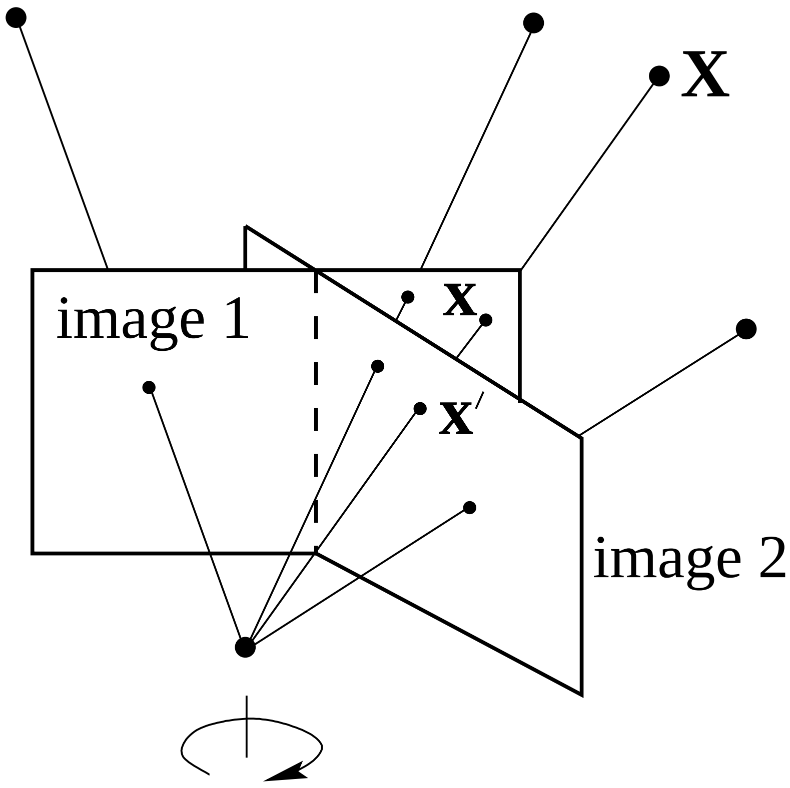
Where are the homographies?



induced by a world plane to cameras



camera rotating around its center

Sketches borrowed from [2]

Problem

How are the images related?

2D Projective transformation

Sometimes called homography or collineation.

A projective transformation is a special transformation which relates the coordinate system $\mathbf{x} = [x_1, x_2]^\top$ with $\mathbf{u} = [u_1, u_2]^\top$ by

$$x_1 = \frac{a_{11}u_1 + a_{12}u_2 + a_{13}}{a_{31}u_1 + a_{32}u_2 + a_{33}} \quad \text{and} \quad x_2 = \frac{a_{21}u_1 + a_{22}u_2 + a_{23}}{a_{31}u_1 + a_{32}u_2 + a_{33}} \,.$$

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Such expressions can be written in a compact way as

$$x_1 = \frac{\mathbf{A}_1[\mathbf{u}^\top, 1]^\top}{\mathbf{A}_3[\mathbf{u}^\top, 1]^\top}, \quad x_2 = \frac{\mathbf{A}_2[\mathbf{u}^\top, 1]^\top}{\mathbf{A}_3[\mathbf{u}^\top, 1]^\top},$$

where

$$\mathtt{A} = \begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \\ \mathbf{A}_3 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} .$$

Computation of $\mathtt{A}$ from point correspondences

Computation of $\mathtt{A}$ from point correspondences arranging equations

Let us have n corresponding point pairs $(\mathbf{x}^i, \mathbf{u}^i)$. We know that for each pair should hold

$$x_1^i = \frac{\mathbf{A}_1 \left[{\mathbf{u}^i}^\top, 1\right]^\top}{\mathbf{A}_3 \left[{\mathbf{u}^i}^\top, 1\right]^\top}, \quad x_2^i = \frac{\mathbf{A}_2 \left[{\mathbf{u}^i}^\top, 1\right]^\top}{\mathbf{A}_3 \left[{\mathbf{u}^i}^\top, 1\right]^\top}.$$

We may manipulate the equations to get

$$\begin{aligned}
\mathbf{A}_3 \left[{\mathbf{u}^i}^\top 1\right]^\top x_1^i - \mathbf{A}_1 \left[{\mathbf{u}^i}^\top 1\right]^\top &= 0\,, \\
\mathbf{A}_3 \left[{\mathbf{u}^i}^\top 1\right]^\top x_2^i - \mathbf{A}_2 \left[{\mathbf{u}^i}^\top 1\right]^\top &= 0\,.
\end{aligned}$$

Each pair $(\mathbf{x}^i, \mathbf{u}^i)$ creates thus a pair of homogenous equations. Manipulation continues . . .

Computation of A from point correspondences LSQ formulation

We form a vector $\mathbf{a}$ from elements of the matrix A:
$\mathbf{a} = [\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3] = [a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}, a_{31}, a_{32}, a_{33}]^\top$. Using all correspondences having at hand, the homogeneous equations are written as

$$\begin{pmatrix} u_1^1 & u_2^1 & 1 & 0 & 0 & 0 & -x_1^1 u_1^1 & -x_1^1 u_2^1 & -x_1^1 \\ 0 & 0 & 0 & u_1^1 & u_2^1 & 1 & -x_2^1 u_1^1 & -x_2^1 u_2^1 & -x_2^1 \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ u_1^n & u_2^n & 1 & 0 & 0 & 0 & -x_1^n u_1^n & -x_1^n u_2^n & -x_1^n \\ 0 & 0 & 0 & u_1^n & u_2^n & 1 & -x_2^n u_1^n & -x_2^n u_2^n & -x_2^n \end{pmatrix} \mathbf{a} = \mathtt{C}\mathbf{a} = \mathbf{0}\,.$$

LSQ formulation

$$\mathbf{a} = \operatorname*{argmin}_{\|\mathbf{a}^*\|=1} \|\mathtt{C}\mathbf{a}^*\|^2$$

Solution of the above problem is covered by the constrained_lsq[1] talk.

[1] http://cmp.felk.cvut.cz/cmp/courses/EZS/Lectures/constrained_lsq.pdf

Application—Image Panoramas!



If the images are taken by a camera which rotates around its center of projection than they are related by exactly the same transformation we have just described.

Application—Image Panoramas!

Further reading

- The textbook [3] provides readable explanation of the homography and its computation.
- Mathematical book [1] gives you more insight into the matrix manipulation and solution of the constrained LSQ problem.

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[1] Gene H. Golub and Charles F. Van Loan. Matrix Computation. Johns Hopkins Studies in the Mathematical Sciences. Johns Hopkins University Press, Baltimore, USA, 3rd edition, 1996.

[2] R. Hartley and A. Zisserman. Multiple View Geometry in Computer Vision. Cambridge University Press, Cambridge, UK, 2000. On-line resources at: http://www.robots.ox.ac.uk/~vgg/hzbook/hzbook1.html.

[3] Richard Hartley and Andrew Zisserman. Multiple view geometry in computer vision. Cambridge University, Cambridge, 2nd edition, 2003.

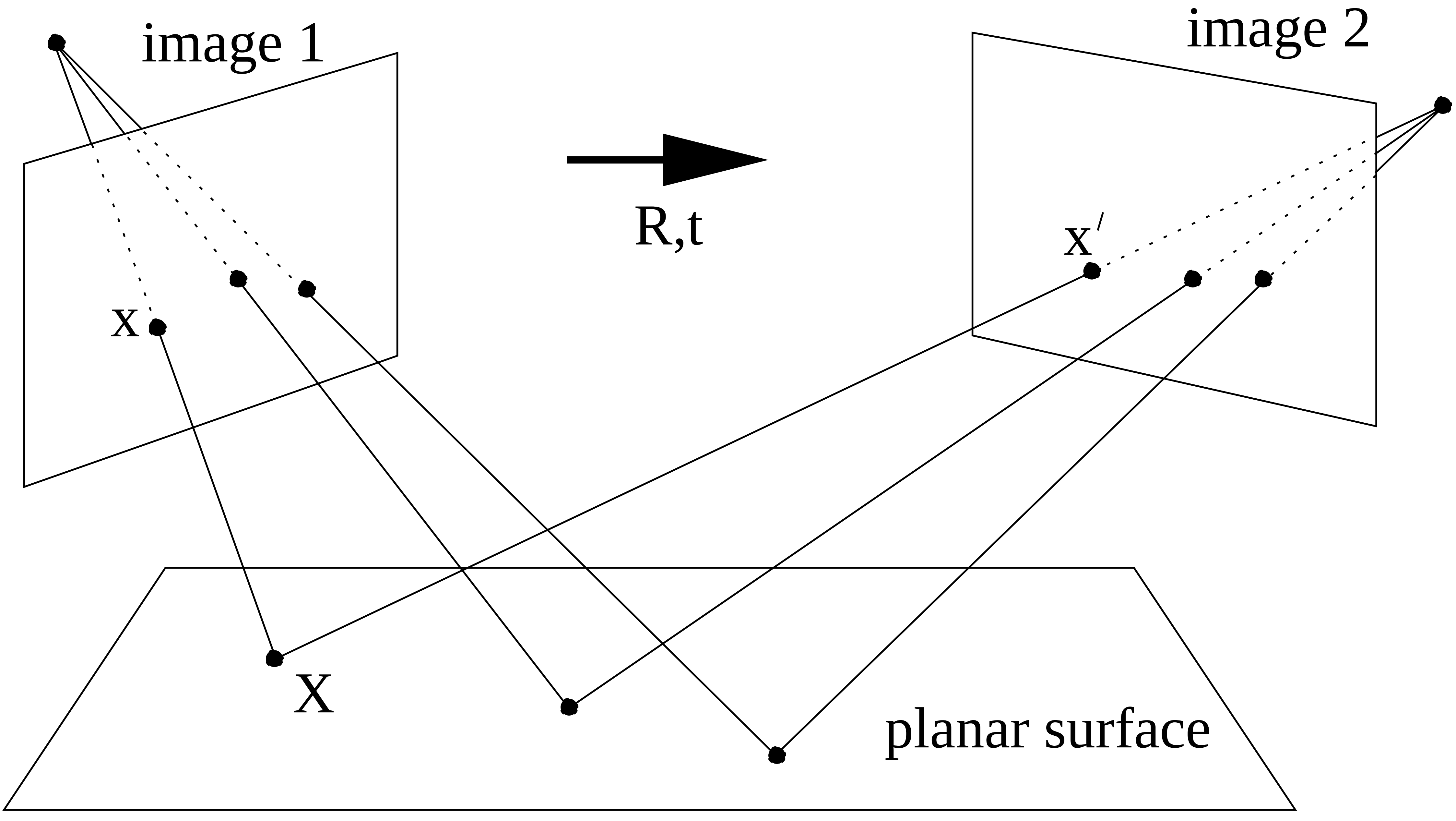
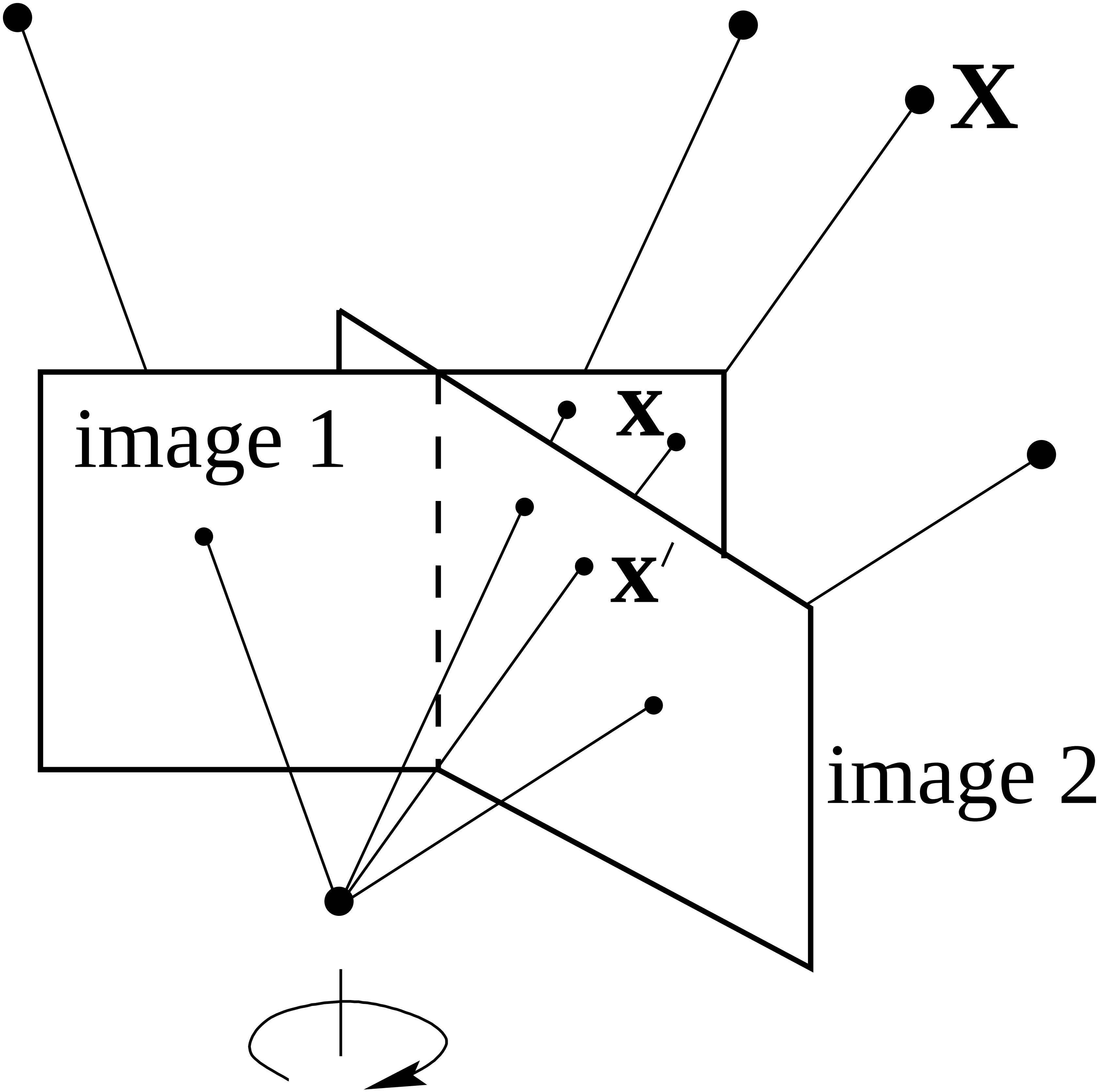


image 1
image 2
R,t
x
x'
X
planar surface



X
x
image 1
x′
image 2