Histogram equalization — derivation

Input: histogram $H(p)$ of the $N \times N$ image with gray leveles $p = \langle p_0, p_k \rangle$.

Aim: find a monotonic pixel brightness transformation $q = \mathcal{T}(p)$, such that the desired output histogram $G(q)$ is uniform over the whole output brightness scale $q = \langle q_0, q_k \rangle$.

The monotonicity of the transformation implies:

$$\sum_{i=0}^{k} G(q_i) = \sum_{i=0}^{k} H(p_i) \,.$$

Equalized histogram $\approx$ uniform density

$$G(q) = \frac{N^2}{q_k - q_0} \,.$$

Histogram equalization — derivation II

The exactly uniform histogram may be obtained only in continuous space.

$$\int_{q_0}^{q} G(s)\, ds \;=\; \int_{p_0}^{p} H(s)\, ds\,.$$

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$$N^2(q - q_0) = (q_k - q_0) \int_{p_0}^{p} H(s)\, ds\,.$$

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$$q = \mathcal{T}(p)$$

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$$
\begin{aligned}
\int_{q_0}^{q} G(s)\, ds &= \int_{p_0}^{p} H(s)\, ds \,. \\
\int_{q_0}^{q} \frac{N^2}{q_k - q_0}\, ds &= \int_{p_0}^{p} H(s)\, ds \,. \\
\frac{N^2(q - q_0)}{q_k - q_0} &= \int_{p_0}^{p} H(s)\, ds \,. \\
N^2(q - q_0) &= (q_k - q_0) \int_{p_0}^{p} H(s)\, ds \,. \\
q = \mathcal{T}(p) &= \frac{q_k - q_0}{N^2} \int_{p_0}^{p} H(s)\, ds + q_0 \,.
\end{aligned}
$$

Histogram equalization — derivation III

Continous space distribution function

$$q = \mathcal{T}(p) = \frac{q_k - q_0}{N^2} \int_{p_0}^{p} H(s)\, ds + q_0 \,.$$

Dicrete space cumulative histogram

$$q = \mathcal{T}(p) = \frac{q_k - q_0}{N^2} \sum_{i=p_0}^{p} H(i) \; + q_0 \,.$$