Mathematical Morphology

A mathematical tool for the extraction and analysis of discrete quantized image structure.

- Does not change image representation. (It is a system of transformations from the space of discrete quantized images onto itself.)

- Implemented as set-theoretic operations with structuring elements.

- Fast algorithms (quasi-parallel processing), many applications, mainly microscopy image processing.
Binary Mathematical Morphology

Motivation example: pre-processing for hand-written character recognition


**Representation of image and the structuring element**

- image = a set of labeled vertices (pixels)
- a regular rectangular (or hexagonal) grid in the space of dimension $n$ (here $n = 2$)
- binary (integer) values

- we will assume zero values behind image edges
- binary morphology is based on set-theoretic operations with images
Mathematical Morphology: Notation

$X, Y$ – discrete quantized image

$B, E, L$ – structural element

$D(B)$ – domain of structural element $B$

$X^c$ – complement of set $X$

$X_h$ – translation of set $X$ by vector $h$

$X \oplus B$ – dilation (of $X$ by $B$)

$X \ominus B$ – erosion

$X \circ B$ – opening

$X \bullet B$ – closing

$\beta(X)$ – morphological gradient of $X$

$X \otimes B$ – Serra transform (hit-or-miss)

$X \oslash B$ – thinning
Translation $X_h$

\[ X_h(p) = X(p - h), \quad p \in X \subset \mathbb{Z}^2 \]

Example:

$h = (1, 2)$
Binary Dilation \( X \oplus B \)

\[
X \oplus B = \bigcup \{ y, B(y) = 1 \} X_y
\]

**Example:**

- Locus of all non-zero image pixels translated by the set of vectors defined by the structuring element.
Binary Dilation Example

\[ X \oplus B \]

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Dilation Properties

Observation: decomposability of element $B$

1. $X \oplus B = B \oplus X$

2. $X \ominus (B \oplus D) = (X \ominus B) \ominus D$ decomposable element

3. $X \ominus (B \cup D) = (X \ominus B) \cup (X \ominus D)$

4. $X \ominus B \not\supseteq X$ dilation is not ‘inflation’

5. \ldots
**Binary Erosion** \( X \ominus B \)

\[
X \ominus B = \bigcap \{y, B(y) = 1\} X_{-y}
\]

**Example:**

\[
X \ominus B = X_{(0,0)} \cap X_{(-1,0)}
\]

- Erosion: Locus of (non-zero) image pixels to which structuring element \( B \) can be inserted
Binary Erosion Example

\[ X \ominus B \]

\[ (X \ominus B) \oplus B \neq X \]
Erosion Properties

1. \( X \ominus B \neq B \ominus X \)

\[
\begin{array}{c}
\text{\includegraphics[width=2cm]{image1.png}} \ominus \text{\includegraphics[width=2cm]{image2.png}} = \text{\includegraphics[width=2cm]{image3.png}} \\
\text{\includegraphics[width=2cm]{image4.png}} \ominus \text{\includegraphics[width=2cm]{image5.png}} = \text{\includegraphics[width=2cm]{image6.png}}
\end{array}
\]

2. \( X \ominus (B \oplus D) = (X \ominus B) \ominus D \) decomposable element

3. \( X \ominus (B \cup D) = (B \ominus X) \cap (D \ominus Y) \)

4. If \((0, 0) \in B\) then \( X \ominus B \subseteq X \)

\[
\begin{array}{c}
\text{\includegraphics[width=2cm]{image7.png}} \ominus \text{\includegraphics[width=2cm]{image8.png}} = \text{\includegraphics[width=2cm]{image9.png}} \quad \text{counter-example}
\end{array}
\]

5. \( (X \oplus B) \ominus B \neq X \)

\[
\begin{array}{c}
\text{\includegraphics[width=2cm]{image10.png}} \oplus \text{\includegraphics[width=2cm]{image11.png}} = \text{\includegraphics[width=2cm]{image12.png}} \\
\text{\includegraphics[width=2cm]{image13.png}} \ominus \text{\includegraphics[width=2cm]{image14.png}} = \text{\includegraphics[width=2cm]{image15.png}} \neq \text{\includegraphics[width=2cm]{image16.png}}
\end{array}
\]
6. \((X \ominus B) \oplus B \neq X\)

\[
\begin{array}{c}
\begin{array}{c}
\mathbb{C} \ominus \mathbb{D}
\end{array}
\oplus
\begin{array}{c}
\mathbb{D}
\end{array}
= \\
\begin{array}{c}
\mathbb{C}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\mathbb{E} \ominus \mathbb{D}
\end{array}
\oplus
\begin{array}{c}
\mathbb{D}
\end{array}
= \\
\begin{array}{c}
\mathbb{F}
\end{array}
\neq \\
\begin{array}{c}
\mathbb{G}
\end{array}
\end{array}
\]

7. \ldots
Morphological Opening and Closing

\[ X \circ B = (X \ominus B) \oplus B \] opening

\[ X \bullet B = (X \oplus B) \ominus B \] closing

\[ B \]

Mathematical Morphology

R. Šára
Properties of Opening and Closing

1. idempotence

\[(X \circ B) \circ B = X \circ B\]
\[(X \bullet B) \bullet B = X \bullet B\]

2. antiextensivity of opening

\[X \circ B \subseteq X\]

3. extensivity of closing

\[X \subseteq X \bullet B\]

4. \ldots
Morphological Gradient $\beta(X)$

$$\beta(X) = X \setminus (X \ominus S_{3,3})$$

**Example:**

![Grids showing $X$, $X \ominus S_{3,3}$, and $\beta(X)$ with $S_{3,3}$]

**Remarks**

- interior boundary
- 4-connectivity
- exterior boundary: $\beta^*(X) = (X \oplus S_{3,3}) \setminus X$
Morphological Gradient Example

\[ X \]

\[ \beta(X) \]

\[ \beta^*(X) \]
Serra Transform (Hit-or-Miss)

\[ X \otimes B = (X \ominus B_1) \cap (X^c \ominus B_2) \]

correlation with two constraints

\[ B = \{B_1, B_2\}, \quad B_1 \cap B_2 = \emptyset \]

1. \( X \ominus B_1 \) – locus of object pixels similar to \( B_1 \)
2. \( X^c \ominus B_2 \) – locus of background pixels similar to \( B_2 \)

• not every pair \( B \) gives \( X \otimes B \neq \emptyset \)

**Example**: detection of “endpoints” from the left:

\[ B = \begin{array}{c|c|c} \times \times & \circ \circ & \times \times \\ \hline \times \times & \circ \circ & \times \times \\ \end{array} : \quad B_1 = \begin{array}{c|c|c} \circ \circ & \times \times \\ \times \times & \circ \circ \\ \hline \times \times & \circ \circ \\ \end{array}, \quad B_2 = \begin{array}{c|c|c} \times \times & \circ \circ \\ \times \times & \circ \circ \\ \hline \times \times & \circ \circ \\ \end{array} \]

\[ X \]

\[ X \otimes B \]
Sequential Thinning

\[ X \ominus B = X \setminus (X \otimes B) \]

\[ X \ominus \{B_i\}_{i=1}^n = X \ominus B_1 \ominus B_2 \ominus \cdots \ominus B_n \]

- the result is order-dependent!

**Example:**

\[ X \]

\[ L_1 \]

\[ X \ominus L_1 \]

Golay alphabet:

\[ L_1 \]

\[ L_2 \]

\[ L_3 \]

\[ L_4 \]

\[ \cdots \]

\[ L_8 \]

\[ E_1 \]

\[ E_2 \]

\[ E_3 \]

\[ E_4 \]

\[ \cdots \]
Sequential Thinning Example

image $I$

$X = (I < 245)$

$Y = X \circ S_{3,3}$

$Y \ominus \{L_i\}_{i=1}^{8}$, repeat till convergence

our goal: smoothed skeleton (see later)
Skeleton\textsuperscript{1} Smoothing

**Input:** skeleton $X$

1. shortening of endings $n$-times
   
   $X_1 = \left( X \ominus \{E_i\}_{i=1}^{4} \right)^{n}_{k=1}$

2. ending point detection
   
   $X_2 = \bigcup_{i=1}^{4} (X \otimes E_i)$

3. conditional dilation $n$-times
   
   $X_3 = \left( (X_2 \oplus S_{3,3}) \cap X \right)^{n}_{k=1}$

4. smoothed skeleton
   
   $Y = X_1 \cup X_3$

---

\textsuperscript{1}I will call the result of sequential thinning a skeleton even if this is incorrect.
## Analogy with Convolution and Correlation

\[(X \oplus B)(x) = \bigcup_{\{y, B(y)=1\}} X_y = \bigcup_{\{y, B(y)=1\}} X(x-y)\]

\[(X \ominus B)(x) = \bigcap_{\{y, B(y)=1\}} X_{-y} = \bigcap_{\{y, B(y)=1\}} X(x+y)\]

\[(X \oplus B)(x) = \max_{y \in D(B)} \left( X(x-y) + B(y) \right)\]

\[(X \ominus B)(x) = \min_{y \in D(B)} \left( X(x+y) - B(y) \right)\]

\[(f \ast g)(x) = \sum_{y \in D(f)} f(y) g(x-y)\]

\[(f \oslash g)(x) = \sum_{y \in D(f)} f^*(y) g(x+y)\]

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<tr>
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<th>convolution</th>
<th>binary dilation</th>
<th>gray-scale dilation</th>
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Gray-Scale Morphology

\[ X^C = C - X, \quad C - \text{maximum element} \text{ (e.g. 255)} \]

\[ X_h(p) = X(p - h) \]

\[ X \oplus B = \max_{y \in D(B)} X(x - y) + B(y) \]

\[ X \ominus B = \min_{y \in D(B)} X(x + y) - B(y) \]

\[ X \circ B = (X \ominus B) \oplus B \]

\[ X \bullet B = (X \oplus B) \ominus B \]

\[ \beta(X) = X - (X \ominus S_{3,3}) \]

\[ X \otimes B = \min(X \ominus B_1, X^c \ominus B_2) \]

\[ X \odot B = X - (X \otimes B) \]
Gray-Scale Morphology Examples

**Example 1:** Local maxima/minima detection in image: application of morphological gradient.

**Example 2:** Segmentation of cell boundaries in the images of human cornea: application of morphological watershed.
Example 3: 100% visual quality inspection of a maximal thermometer capillary application of top hat transform.
Example 4: Granulometry$^2$

$X$ largest square probes

granulometric spectre

$^2$Sorry for this example.