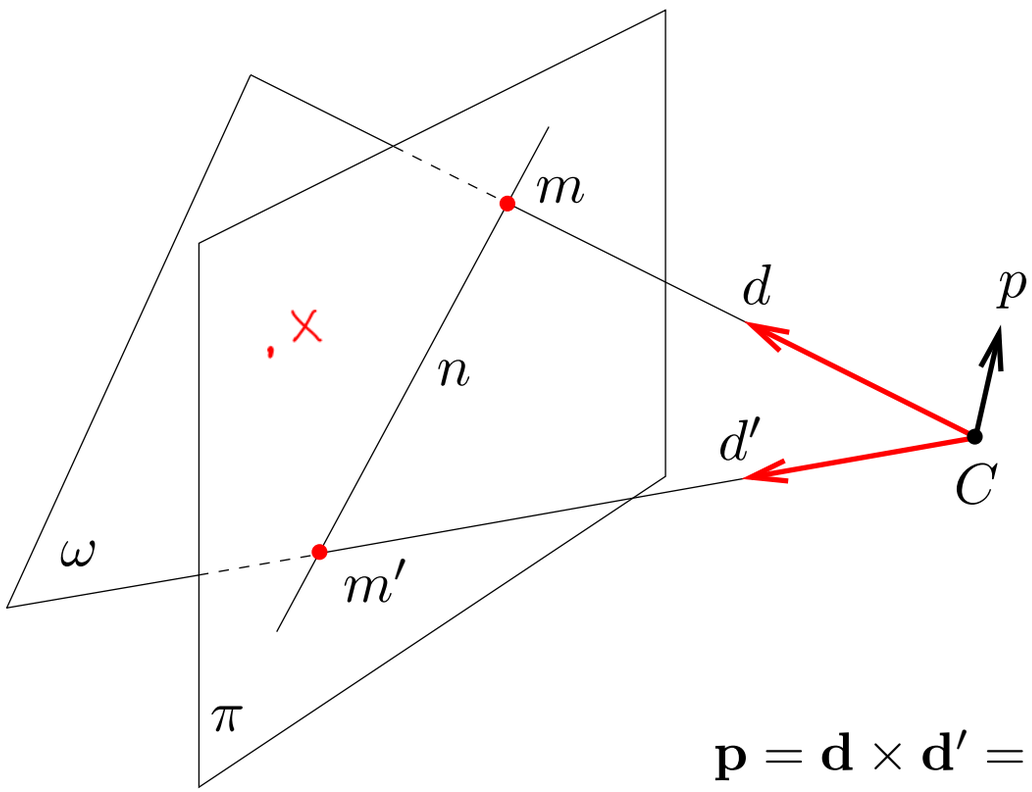
Optical Plane

A spatial plane with normal p passing through the optical center C and a given image line n.



optical ray given by m $\quad \mathbf{d} = \mathbf{Q}^{-1}\underline{\mathbf{m}}$

optical ray given by m' $\quad \mathbf{d}' = \mathbf{Q}^{-1}\underline{\mathbf{m}}'$

$$\mathbf{p} = \mathbf{d} \times \mathbf{d}' = (\mathbf{Q}^{-1}\underline{\mathbf{m}}) \times (\mathbf{Q}^{-1}\underline{\mathbf{m}}') = \mathbf{Q}^\top(\underline{\mathbf{m}} \times \underline{\mathbf{m}}') = \mathbf{Q}^\top\underline{\mathbf{n}}$$

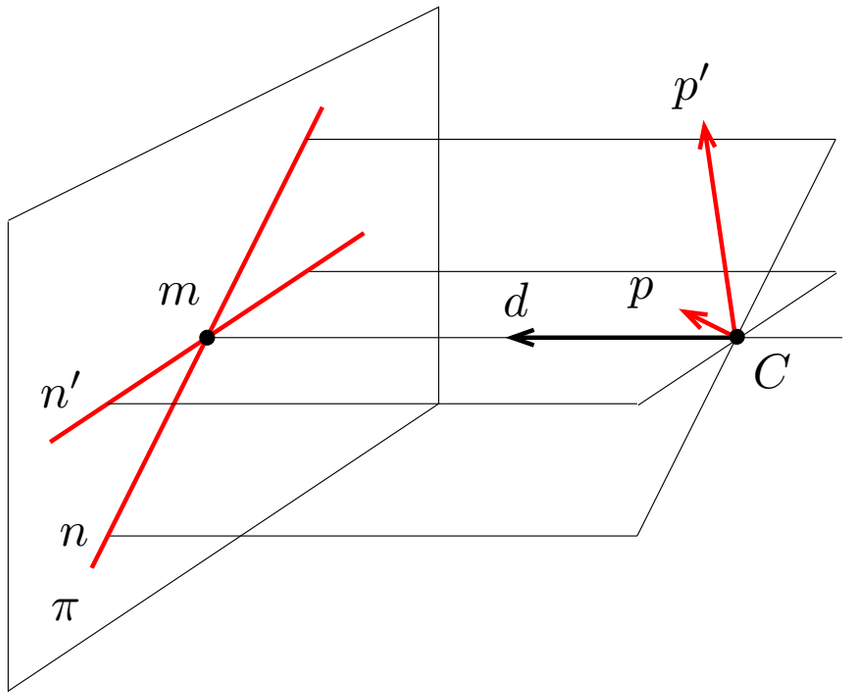
Hence, $0 = \mathbf{p}^\top(\mathbf{X} - \mathbf{C}) = \underline{\mathbf{n}}^\top\mathbf{Q}(\mathbf{X} - \mathbf{C}) = \underline{\mathbf{n}}^\top\mathbf{P}\underline{\mathbf{X}} = (\mathbf{P}^\top\underline{\mathbf{n}})^\top\underline{\mathbf{X}}$ for every X in the plane ω.

$\omega^\top X = 0$

optical plane given by n: $\boxed{\omega \simeq \mathbf{P}^\top\underline{\mathbf{n}}}$

$\omega_1\, x + \omega_2\, y + \omega_3\, z + \omega_4 = 0$

Note: Optical Ray as Optical Plane Intersection



optical plane normal given by n $\qquad \mathbf{p} = \mathbf{Q}^\top \underline{\mathbf{n}}$

optical plane normal given by n' $\qquad \mathbf{p}' = \mathbf{Q}^\top \underline{\mathbf{n}}'$

$$\mathbf{d} = \mathbf{p} \times \mathbf{p}' = (\mathbf{Q}^\top \underline{\mathbf{n}}) \times (\mathbf{Q}^\top \underline{\mathbf{n}}') = \mathbf{Q}^{-1}(\underline{\mathbf{n}} \times \underline{\mathbf{n}}') = \mathbf{Q}^{-1}\underline{\mathbf{m}}$$

Summary: optical center, ray, axis, plane

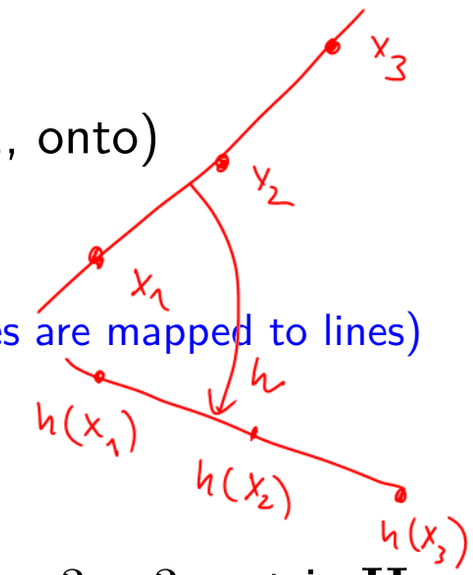
General camera

$$\mathbf{P} = \left[\mathbf{Q},\, \mathbf{q}\right] = \begin{bmatrix} \mathbf{q}_1^\top & q_{14} \\ \mathbf{q}_2^\top & q_{24} \\ \mathbf{q}_3^\top & q_{34} \end{bmatrix} = \mathbf{K}\,\mathbf{R}\left[\mathbf{I}, -\mathbf{C}\right]$$

$\underline{\mathbf{C}} = \mathrm{rnull}(\mathbf{P})$	optical center
$\mathbf{d} = \mathbf{Q}^{-1}\,\underline{\mathbf{m}}$	optical ray direction
$\det(\mathbf{Q})\,\mathbf{q}_3$	optical axis (outward)
$\mathbf{Q}\,\mathbf{q}_3$	principal point
$\mathbf{p} = \mathbf{Q}^\top\,\underline{\mathbf{n}}$	optical plane normal
$\mathbf{K} = \begin{bmatrix} f\,k_u & -f\,k_u\,\cot\theta & u_0 \\ 0 & f\,k_v/\sin\theta & v_0 \\ 0 & 0 & 1 \end{bmatrix}$	camera (calibration) matrix
$\mathbf{R}$	camera orientation matrix

Collineations

- Linear projective space $\mathbb{P}^2$: Linear space of all 3-vectors (including points at infinity and the line at infinity) excluding the zero vector.



- **Collineation:** Let $\underline{\mathbf{x}}_1$, $\underline{\mathbf{x}}_2$, $\underline{\mathbf{x}}_3$ be collinear points in $\mathbb{P}^2$. Bijection (1:1, onto) $h\colon \mathbb{P}^2 \mapsto \mathbb{P}^2$ is a collineation iff $h(\underline{\mathbf{x}}_1)$, $h(\underline{\mathbf{x}}_2)$, $h(\underline{\mathbf{x}}_3)$ are collinear.

  i.e.
  - collinear image points are mapped to collinear image points (lines are mapped to lines)
  - concurrent image lines are mapped to concurrent image lines
  - point-line incidence is preserved

- A mapping $h : \mathbb{P}^2 \to \mathbb{P}^2$ is a collineation <u>iff</u> there exists a non-singular 3×3 matrix $\mathbf{H}$ such that

$$h(\underline{\mathbf{x}}) \simeq \mathbf{H}\,\underline{\mathbf{x}} \quad \text{for all } \underline{\mathbf{x}} \in \mathbb{P}^2 \qquad \lambda H \quad \lambda \neq 0$$

- Homogeneous matrix representant: $\det \mathbf{H} = 1$

- Collineations form a group isomorphic to $SO(3)$

  group of 3×3 matrices with unit determinant and with matrix multiplication

- Collineation has **8 DOF**.

- It is given by 4 correspondences (points,lines) in a general position.

Mapping Points and Lines by Collineation

$$\underline{\mathbf{m}}' \simeq \mathbf{H}\,\underline{\mathbf{m}} \qquad \text{image point}$$

$$\underline{\mathbf{n}}' \simeq \mathbf{H}^{-\top}\underline{\mathbf{n}} \qquad \text{image line}$$

Notes:

- Incidence is preserved: $(\underline{\mathbf{m}}')^\top \underline{\mathbf{n}}' \simeq \underline{\mathbf{m}}^\top \mathbf{H}^\top \mathbf{H}^{-\top} \underline{\mathbf{n}} = \underline{\mathbf{m}}^\top \underline{\mathbf{n}} = 0$

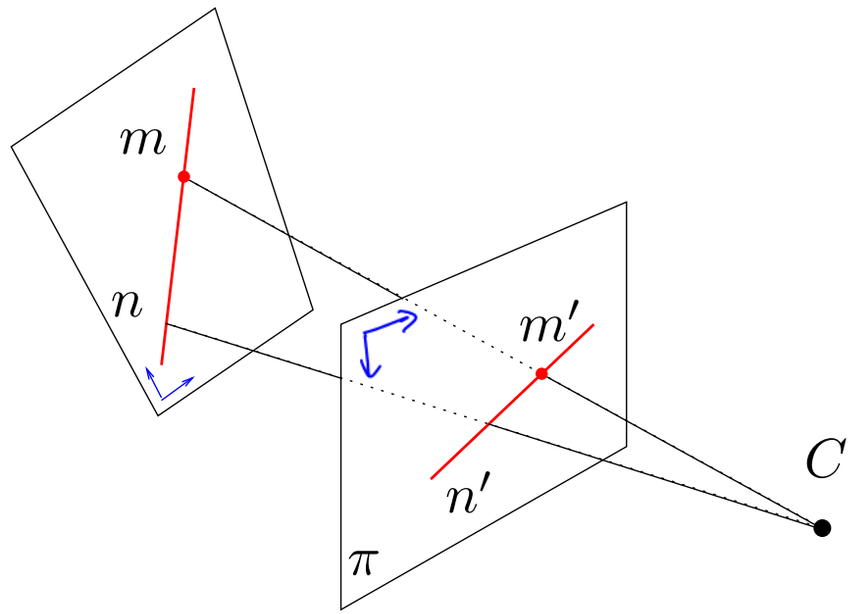
- Extending pixel coordinates to homogeneous coordinates $\underline{\mathbf{m}} = [\,u, v, 1\,]^\top$
- Conversion of the result $\underline{\mathbf{m}}' = (m_1', m_2', m_3')^\top$ to pixels:

$$u' = \frac{m_1'}{m_3'}, \qquad v' = \frac{m_2'}{m_3'}$$

- Can use the unity for the homogeneous coordinate on one side of the equation only!

Homography: Perspective Projection from Plane to Image Plane

Perspective re-projection from a plane is a collineation:



- local coordinate system in the plane
- collinear points map to collinear points
  - concurrent lines map to concurrent lines
  - incidence preserved

I will often use the word 'homography' to mean 'collineation'

Homographies Everywhere

- images of planar objects differ from reality by a homography
- flat-bed scanner subjects the image to a homography
- camera intrinsic parameter change is $\mathbf{K}_2\,\mathbf{K}_1^{-1}$
- zooming (focal length change) is $\mathbf{K}_2\mathbf{K}_1^{-1}$
- camera rotation around optical center is $\mathbf{K}\mathbf{R}\mathbf{K}^{-1}$, in general $\mathbf{K}_2\,\mathbf{R}_2\,\mathbf{R}_1^{-1}\mathbf{K}_1^{-1}$.

$P = KR[I_1 - c]$

$K' K^{-1} P$

Collineation Decomposition

Unique decompositions: $\mathbf{A} = \mathbf{A}_S\,\mathbf{A}_A\,\mathbf{A}_P \quad (= \mathbf{A}'_P\,\mathbf{A}'_A\,\mathbf{A}'_S)$

$$\mathbf{A}_S = \begin{bmatrix} s\,\mathbf{R} & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix}$$ similarity

$$\mathbf{A}_A = \begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0}^\top & 1 \end{bmatrix}$$ special affine

$K = \begin{bmatrix} 1 & a \\ 0 & b \end{bmatrix}$ $b > 0$

$$\mathbf{A}_P = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{v}^\top & w \end{bmatrix}$$ special projective (elation)

$\mathbf{K}$ – upper triangular matrix, $\det \mathbf{K} = 1$

$\mathbf{R}$ – orthogonal, $\mathbf{R}^\top \mathbf{R} = \mathbf{I}$, $\det \mathbf{R} = 1$

$s, w \in \mathbb{R}$, $s > 0$, $w \neq 0$

$$\mathbf{A} = \begin{bmatrix} s\mathbf{R}\mathbf{K} + \mathbf{t}\,\mathbf{v}^\top & w\,\mathbf{t} \\ \mathbf{v}^\top & w \end{bmatrix}$$

- uniqueness: 'skinny' QR decomposition is unique [Golub & van Loan]
- $\mathbf{A}_S$, $\mathbf{A}_A$, $\mathbf{A}_P$ are collineation subgroups
  (eg. $\mathbf{K} = \mathbf{K}_1\mathbf{K}_2$, $\mathbf{K}^{-1}$, $\mathbf{I}$ are all upper triangular with unit determinant, associativity holds)

Collineation Subgroups

group	DOF	matrix	invariant properties
projective	8	$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$	incidence, concurrency, colinearity, cross-ratio, convex hull, order of contact (intersection, tangency, inflection), tangent discontinuities and cusps.
affine	6	$\begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$	all above plus: parallelism, ratio of areas, ratio of lengths on parallel lines, linear combinations of vectors (e.g. midpoints), line at infinity $\mathbf{l}_\infty$ (not pointwise) $= [0,0,1]^\top$
similarity	4	$\begin{bmatrix} s\,\cos\phi & s\,\sin\phi & t_x \\ -s\,\sin\phi & s\,\cos\phi & t_y \\ 0 & 0 & 1 \end{bmatrix}$	all above plus: ratio of lengths, angle, the circular points $I = [1, i, 0]^\top$, $J = [1, -i, 0]^\top$.
Euclidean	3	$\begin{bmatrix} \cos\phi & \sin\phi & t_x \\ -\sin\phi & \cos\phi & t_y \\ 0 & 0 & 1 \end{bmatrix}$	all above plus: length, area

[H&Z, R2, p. 44]

Vanishing Point

úběžník

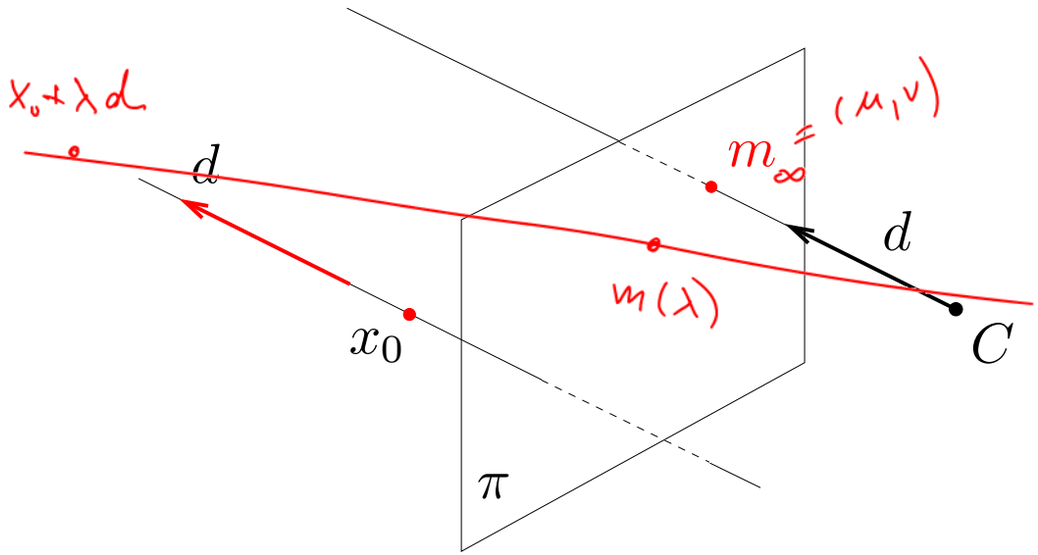
How far is the engine?

distance between sleepers 0.806m but we cannot count them, resolution is too low

We will review some life-saving theory. . .

Vanishing Point

Vanishing point: the limit of the projection of a point that moves along a space line infinitely in one direction. (the image of the point at infinity on the line)



$$u = \frac{q_1^T \cdot (x_0 + \lambda d) + q_{14}}{q_3^T (x_0 + \lambda d) + q_{34}} \to \frac{q_1^T d}{q_3^T d}$$

$$v = \frac{q_2^T (\quad) + q_{24}}{q_3^T (\quad) + q_{34}} \to \frac{q_2^T d}{q_3^T d}$$

$P = [Q, q]$

$$\underline{\mathbf{m}}_\infty = \lim_{\lambda \to \pm\infty} \mathbf{P} \begin{bmatrix} \mathbf{x}_0 + \lambda \mathbf{d} \\ 1 \end{bmatrix} = \cdots = \mathbf{Q}\,\mathbf{d}$$

- V.P. is independent on line position, depends on its orientation only
- A space line of direction vector $\mathbf{d}$ has a V.P. represented by $\underline{\mathbf{m}} = \mathbf{Q}\,\mathbf{d}$
- V.P. m corresponds to spatial direction $\mathbf{d} = \mathbf{Q}^{-1}\underline{\mathbf{m}}$ (optical ray through m)

Vanishing Line

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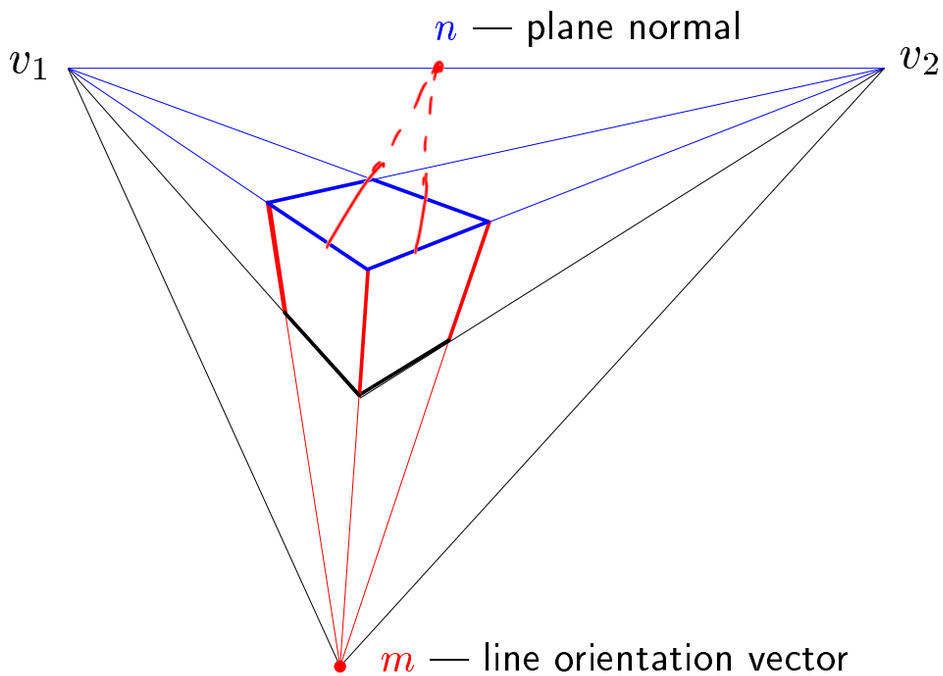


$$\mathbf{d}_1 = \mathbf{Q}^{-1}\underline{\mathbf{v}}_1, \quad \mathbf{d}_2 = \mathbf{Q}^{-1}\underline{\mathbf{v}}_2$$

$$\mathbf{p} = \mathbf{d}_1 \times \mathbf{d}_2 = \mathbf{Q}^\top(\underline{\mathbf{v}}_1 \times \underline{\mathbf{v}}_2) = \mathbf{Q}^\top\underline{\mathbf{n}}$$

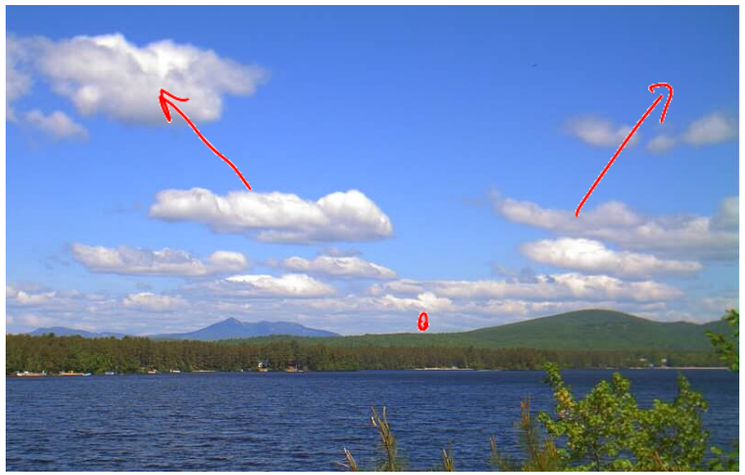
Vanishing line: the set of vanishing points of all lines in a plane

(the image of the line at infinity in the plane)

- A space plane of normal vector $\mathbf{p}$ has a V.L. represented by $\underline{\mathbf{n}} = \mathbf{Q}^{-\top}\mathbf{p}$.
- V.L. n corresponds to space plane of normal vector $\mathbf{p} = \mathbf{Q}^\top\underline{\mathbf{n}}$

Some Vanishing Point Applications

where is the sun?



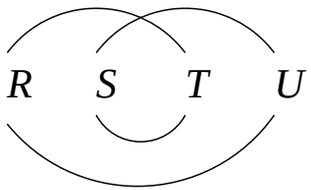
what is the wind direction?
(must have video)

fly above,
keeping constant altitude!

Cross Ratio

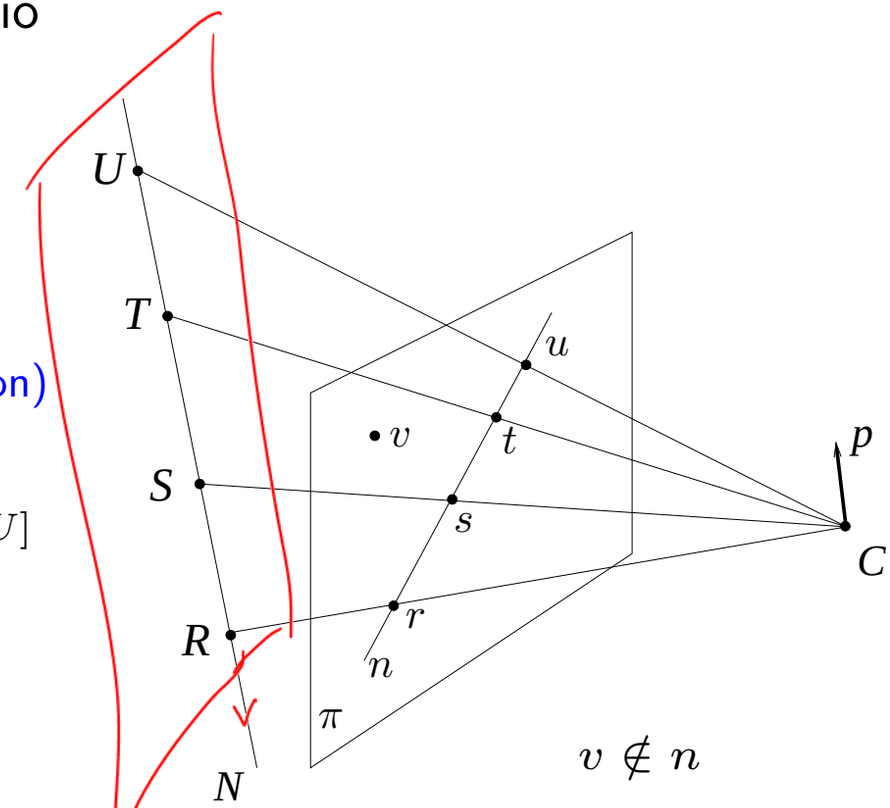
Four collinear space points R, S, T, U define cross-ratio

$$[RSTU] = \frac{|RT|}{|RU|}\frac{|SU|}{|ST|}$$



$|RT|$ – signed distance from R to T (w.r.t. a fixed line orientation)

$$[SRUT] = [RSTU],\ [RSUT] = \frac{1}{[RSTU]},\ [RTSU] = 1 - [RSTU]$$



Observation: $[RSTU] = \dfrac{|\underline{\mathbf{r}},\ \underline{\mathbf{t}},\ \underline{\mathbf{v}}|}{|\underline{\mathbf{r}},\ \underline{\mathbf{u}},\ \underline{\mathbf{v}}|} \cdot \dfrac{|\underline{\mathbf{s}},\ \underline{\mathbf{u}},\ \underline{\mathbf{v}}|}{|\underline{\mathbf{s}},\ \underline{\mathbf{t}},\ \underline{\mathbf{v}}|}$, $\quad |\underline{\mathbf{r}},\ \underline{\mathbf{t}},\ \underline{\mathbf{v}}| = \det[\underline{\mathbf{r}},\ \underline{\mathbf{t}},\ \underline{\mathbf{v}}] = (\underline{\mathbf{r}} \times \underline{\mathbf{t}})^\top \underline{\mathbf{v}}$

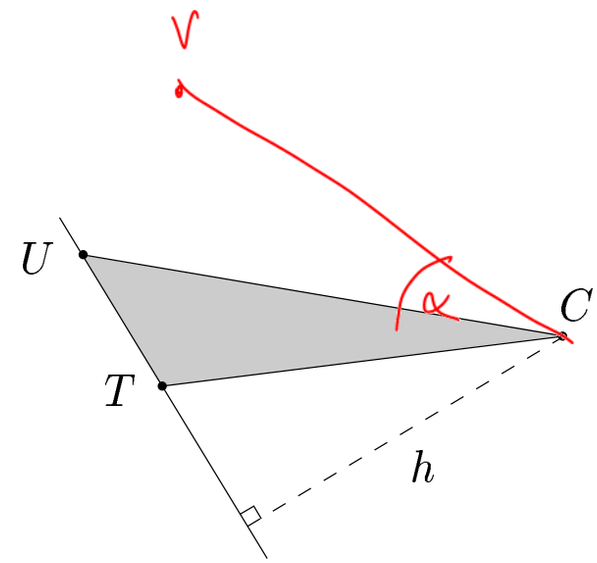
Corollaries:

- Cross ratio is invariant under collineations $\underline{\mathbf{x}}' = \mathbf{A}\underline{\mathbf{x}}$.
- Cross ratio is invariant under perspective projection: $[RSTU] = [r\,s\,t\,u]$

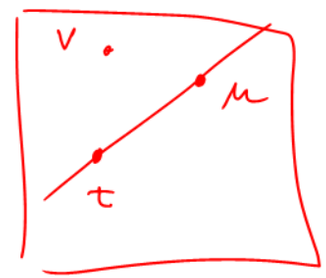
- Given four collinear points, any perspective camera will "see" the same cross-ratio of their images.
- We measure the same cross-ratio in image as on the world line

Let n be the image of spatial line N, let there be image point v such that $\underline{\mathbf{n}}^\top \underline{\mathbf{v}} \neq 0$. Then

$$[RSTU] = \frac{(\underline{\mathbf{r}} \times \underline{\mathbf{t}})^\top \underline{\mathbf{v}}}{(\underline{\mathbf{r}} \times \underline{\mathbf{u}})^\top \underline{\mathbf{v}}} \cdot \frac{(\underline{\mathbf{s}} \times \underline{\mathbf{u}})^\top \underline{\mathbf{v}}}{(\underline{\mathbf{s}} \times \underline{\mathbf{t}})^\top \underline{\mathbf{v}}}.$$



- triangle area: $\frac{1}{2}|UT| \cdot h = \frac{1}{2}\|CU \times CT\|$



$$(CU \times CT)^\top CV = (\lambda_U \mathbf{Q}^{-1}\underline{\mathbf{u}} \times \lambda_T \mathbf{Q}^{-1}\underline{\mathbf{t}})^\top \lambda_V \mathbf{Q}^{-1}\underline{\mathbf{v}} = \lambda_U \lambda_T \lambda_V \left(\mathbf{Q}^\top(\underline{\mathbf{u}} \times \underline{\mathbf{t}})\right)^\top \mathbf{Q}^{-1}\underline{\mathbf{v}} =$$

$$= \lambda_U \lambda_T \lambda_V (\underline{\mathbf{u}} \times \underline{\mathbf{t}})^\top \underline{\mathbf{v}} =$$

$$= \|CU \times CT\| \cdot |CV| \cdot \cos\alpha = |UT| \cdot h \cdot |CV| \cdot \cos\alpha$$

$$|UT| = \lambda_U \lambda_T \lambda'_V (\underline{\mathbf{u}} \times \underline{\mathbf{t}})^\top \underline{\mathbf{v}}$$

$$\frac{|RT|}{|RU|} \cdot \frac{|SU|}{|ST|} = \frac{\cancel{\lambda_R}\cancel{\lambda_T}\cancel{\lambda'_V}\,(\underline{\mathbf{r}} \times \underline{\mathbf{t}})^\top \underline{\mathbf{v}}}{\cancel{\lambda_R}\cancel{\lambda_U}\cancel{\lambda'_V}\,(\underline{\mathbf{r}} \times \underline{\mathbf{u}})^\top \underline{\mathbf{v}}} \cdot \frac{\cancel{\lambda_S}\cancel{\lambda_U}\cancel{\lambda'_V}\,(\underline{\mathbf{s}} \times \underline{\mathbf{u}})^\top \underline{\mathbf{v}}}{\cancel{\lambda_S}\cancel{\lambda_T}\cancel{\lambda'_V}\,(\underline{\mathbf{s}} \times \underline{\mathbf{t}})^\top \underline{\mathbf{v}}}$$

Invariance to Collineation

$\underline{\mathbf{r}}' = \lambda_r \mathbf{A}\underline{\mathbf{r}}$, $\underline{\mathbf{s}}' = \lambda_s \mathbf{A}\underline{\mathbf{s}}$, etc.

$$[\,r'\,s'\,t'\,u'\,] = \frac{|\,\underline{\mathbf{r}}',\,\underline{\mathbf{t}}',\,\underline{\mathbf{v}}'\,|}{|\,\underline{\mathbf{r}}',\,\underline{\mathbf{u}}',\,\underline{\mathbf{v}}'\,|} \cdot \frac{|\,\underline{\mathbf{s}}',\,\underline{\mathbf{u}}',\,\underline{\mathbf{v}}'\,|}{|\,\underline{\mathbf{s}}',\,\underline{\mathbf{t}}',\,\underline{\mathbf{v}}'\,|}$$

$$|\underline{\mathbf{r}}',\,\underline{\mathbf{t}}',\,\underline{\mathbf{v}}'| = |\lambda_r \mathbf{A}\underline{\mathbf{r}},\,\lambda_t \mathbf{A}\underline{\mathbf{t}},\,\lambda_v \mathbf{A}\underline{\mathbf{v}}| = \lambda_r\lambda_t\lambda_v|\mathbf{A}(\underline{\mathbf{r}},\underline{\mathbf{t}},\underline{\mathbf{v}})| = \lambda_r\lambda_t\lambda_v|\mathbf{A}| \cdot |\underline{\mathbf{r}},\underline{\mathbf{t}},\underline{\mathbf{v}}|$$

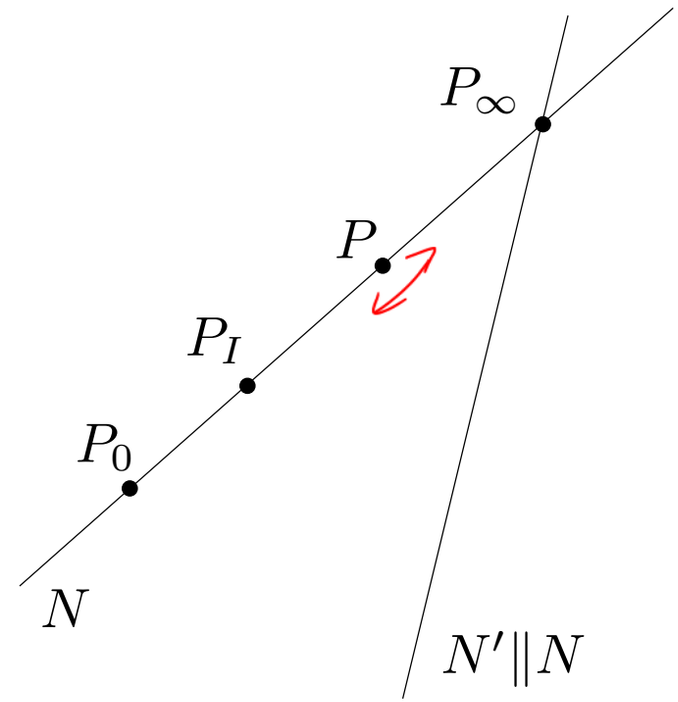
$$[\,r'\,s'\,t'\,u'\,] = \frac{\lambda_r\lambda_t\lambda_v|\mathbf{A}| \cdot |\underline{\mathbf{r}},\,\underline{\mathbf{t}},\,\underline{\mathbf{v}}|}{\lambda_r\lambda_u\lambda_v|\mathbf{A}| \cdot |\underline{\mathbf{r}},\,\underline{\mathbf{u}},\,\underline{\mathbf{v}}|} \cdot \frac{\lambda_s\lambda_u\lambda_v|\mathbf{A}| \cdot |\underline{\mathbf{s}},\,\underline{\mathbf{u}},\,\underline{\mathbf{v}}|}{\lambda_s\lambda_t\lambda_v|\mathbf{A}| \cdot |\underline{\mathbf{s}},\,\underline{\mathbf{t}},\,\underline{\mathbf{v}}|} = [\,r\,s\,t\,u\,]$$

$(\underline{\mathbf{n}}')^\top \underline{\mathbf{v}}' \simeq (\mathbf{A}^{-\top}\underline{\mathbf{n}})^\top \mathbf{A}\underline{\mathbf{v}} = \underline{\mathbf{n}}^\top \underline{\mathbf{v}}$: non-incidence of v and n is preserved

1D Projective Coordinates

The 1-D projective coordinate of a point P:

$$[P] = [P_\infty \, P_0 \, P_I \, P] = \frac{|P_\infty P_I|}{|P_0 P_I|} \frac{|P_0 P|}{|P_\infty P|}$$



P_0 – the origin $[P_0] = 0$

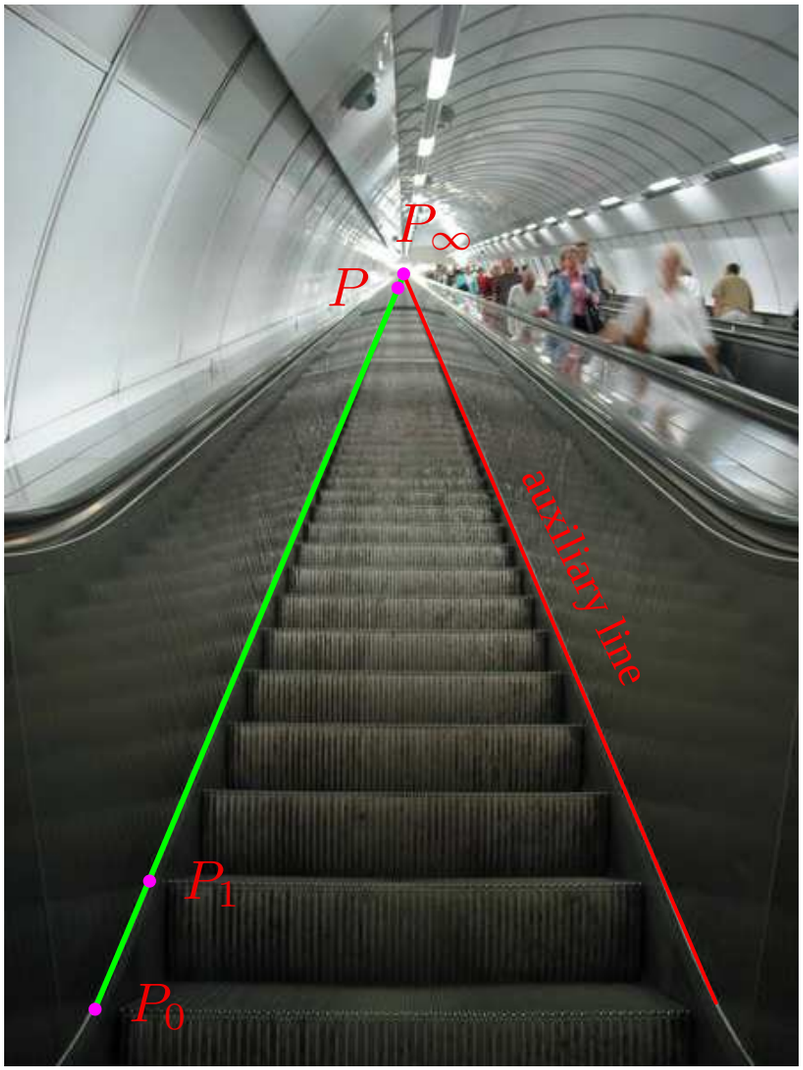
P_I – the unit point $[P_I] = 1$

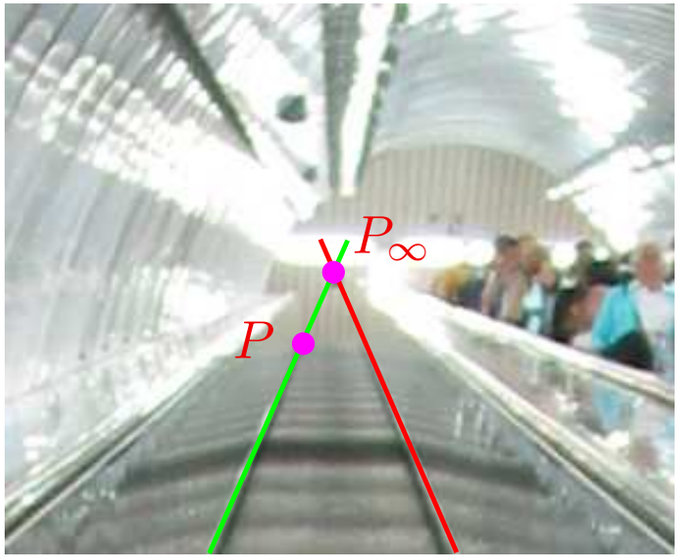
P_∞ – the supporting point $[P_\infty] = \pm\infty$

Application

Given the image of a line N, the origin, the unit point, and the vanishing point, then the Euclidean coordinate of any point $P \in N$ can be determined.

Application: Counting Steps

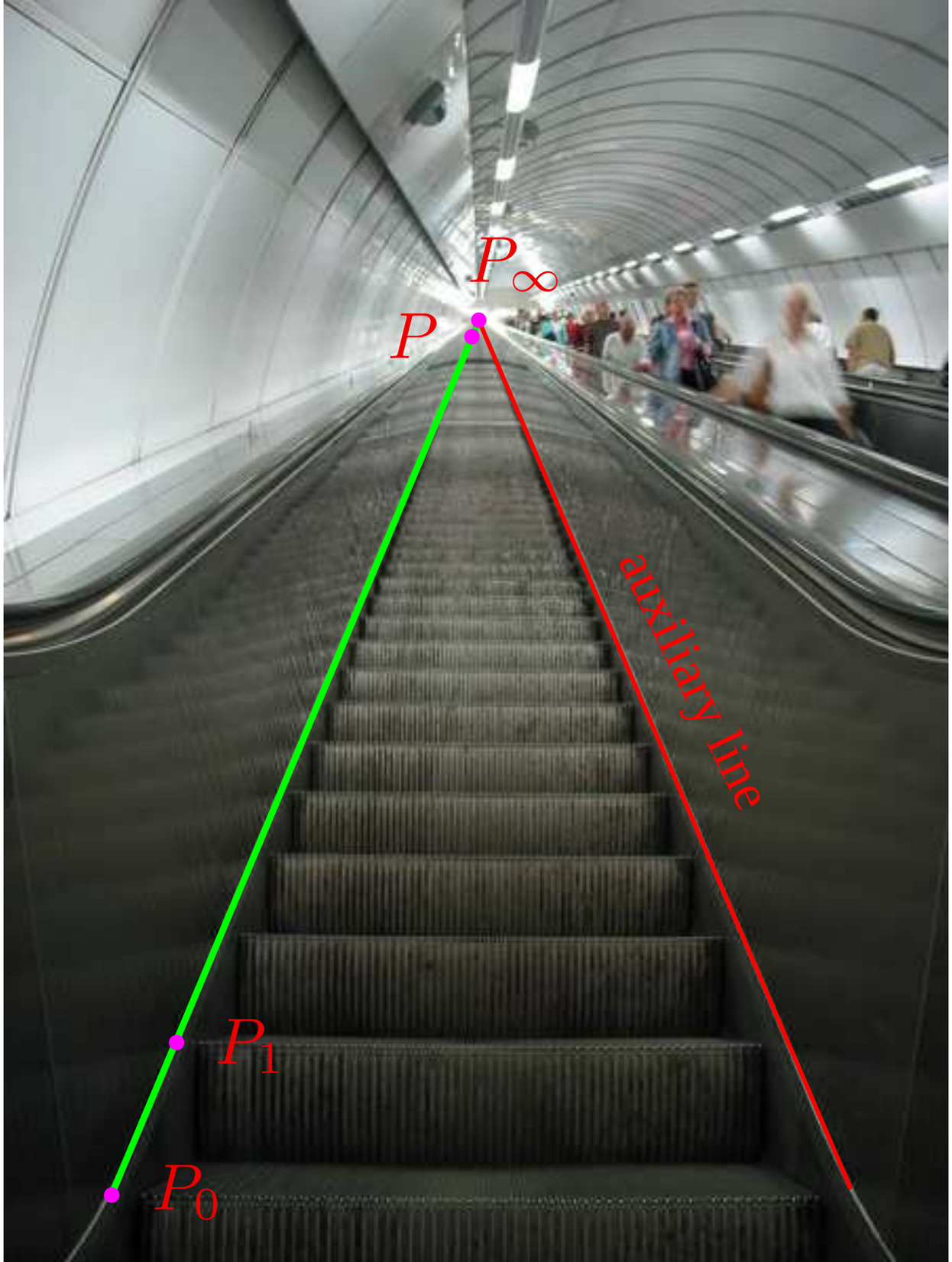


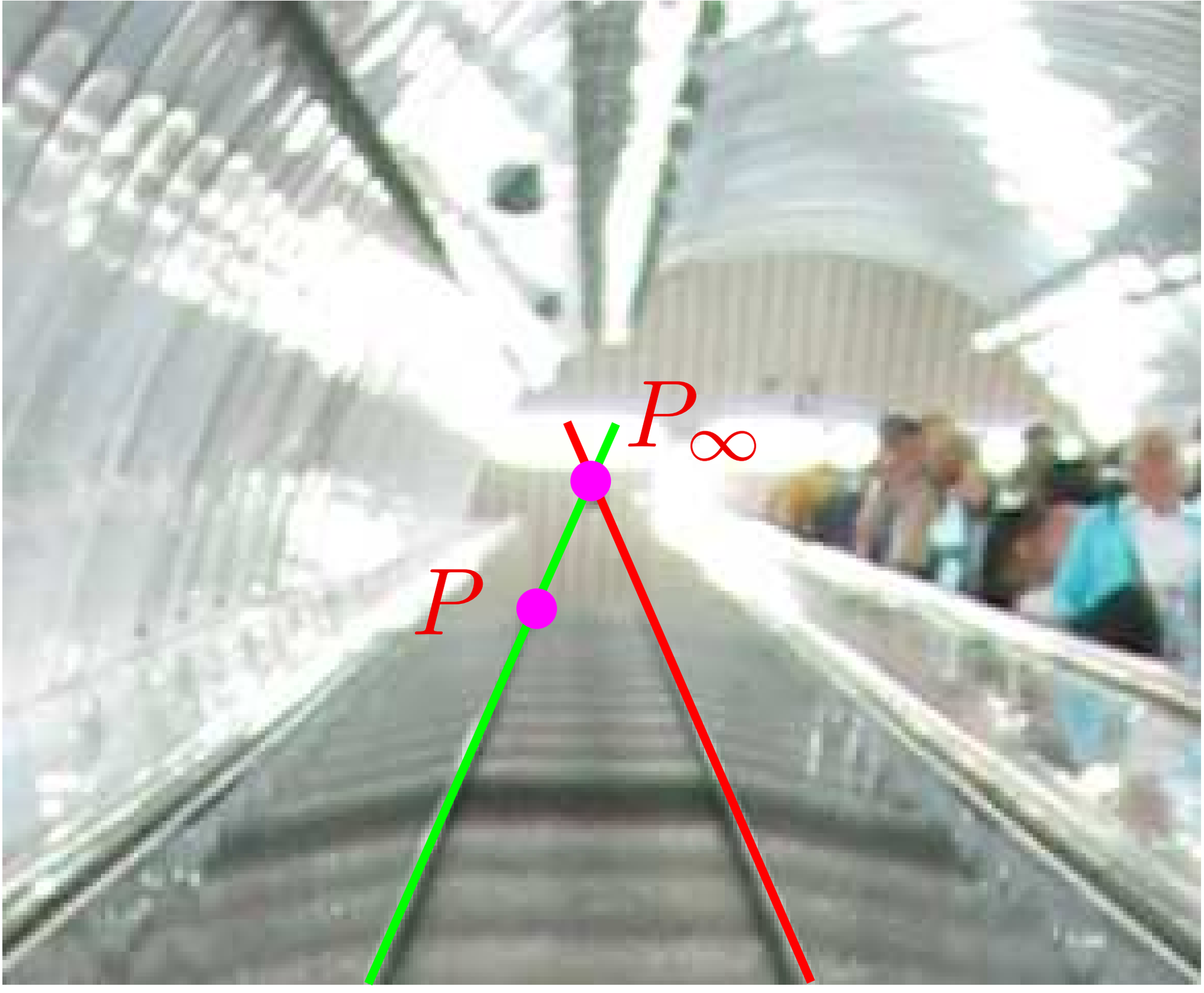


detail around the vanishing point

Result: $[P] = 214$ steps (correct answer is 216 steps)

The End


P_∞
P
auxiliary line
P_1
P_0


P_∞
P