A0. Basic Geometric Entities and their Representations

Points

- in 2-space m
- in 3-space X

Lines

- in 2-space n
- in 3-space O

Planes in 3-space π

Associated vector representations

$$m, X, n, O$$

Associated homogeneous representations

$$\underline{m}, \underline{X}, \underline{n}, \underline{O}$$

Coordinate vectors (m_1, m_2, m_3)

(sometimes without transposition)

Image line

line in the plane

$$a\,u + b\,v + c = 0$$

corresponds to (homogeneous) vector

$$\boxed{\underline{n} = (a,\ b,\ c)}$$

equivalence class, $\lambda \in \mathbb{R},\ \lambda \neq 0$

$$(\lambda a,\ \lambda b,\ \lambda c) \sim (a,\ b,\ c)$$

The set of equivalence classes of vectors in $\mathbb{R}^3 \setminus (0,0,0)$ forms the projective space $\mathbb{P}^2$. (Set of rays.)

Ideal line (at infinity)

$$\underline{n}_\infty = (0,0,1)$$

N-vector

$$N(\underline{n}) = \pm \frac{\underline{n}}{\|\underline{n}\|}$$

Point (on a line)

Point $m = (u, v)$ is incident on the line $\underline{n} = (a, b, c)$ iff

$$a\,u + b\,v + c = 0$$

rewritten (with scalar product):

$$\big((u, v, 1), (a, b, c)\big) = (\underline{m}, \underline{n}) = 0$$

Point represented by (homogeneous) vector

$$\boxed{\underline{m} = (u, v, 1)}$$

equivalence class, $\lambda \in \mathbb{R}$, $\lambda \neq 0$

$$(m_1,\ m_2,\ m_3) = \lambda\,\underline{m} \sim \underline{m}$$

ideal point (at infinity),

$$(m_1, m_2, 0)$$

line at infinity

$$\underline{n} = (0, 0, 1)$$

Line Intersection and Point Join

The point of intersection m of image lines n and n', $n \not\sim n'$ is

$$\underline{m} = \underline{n} \times \underline{n}'.$$

The join n of two image points m and m', $m \not\sim m'$ is

$$\underline{n} = \underline{m} \times \underline{m}'.$$

Paralel lines intersect at line at infinity $\underline{n}_\infty = (0, 0, 1)$

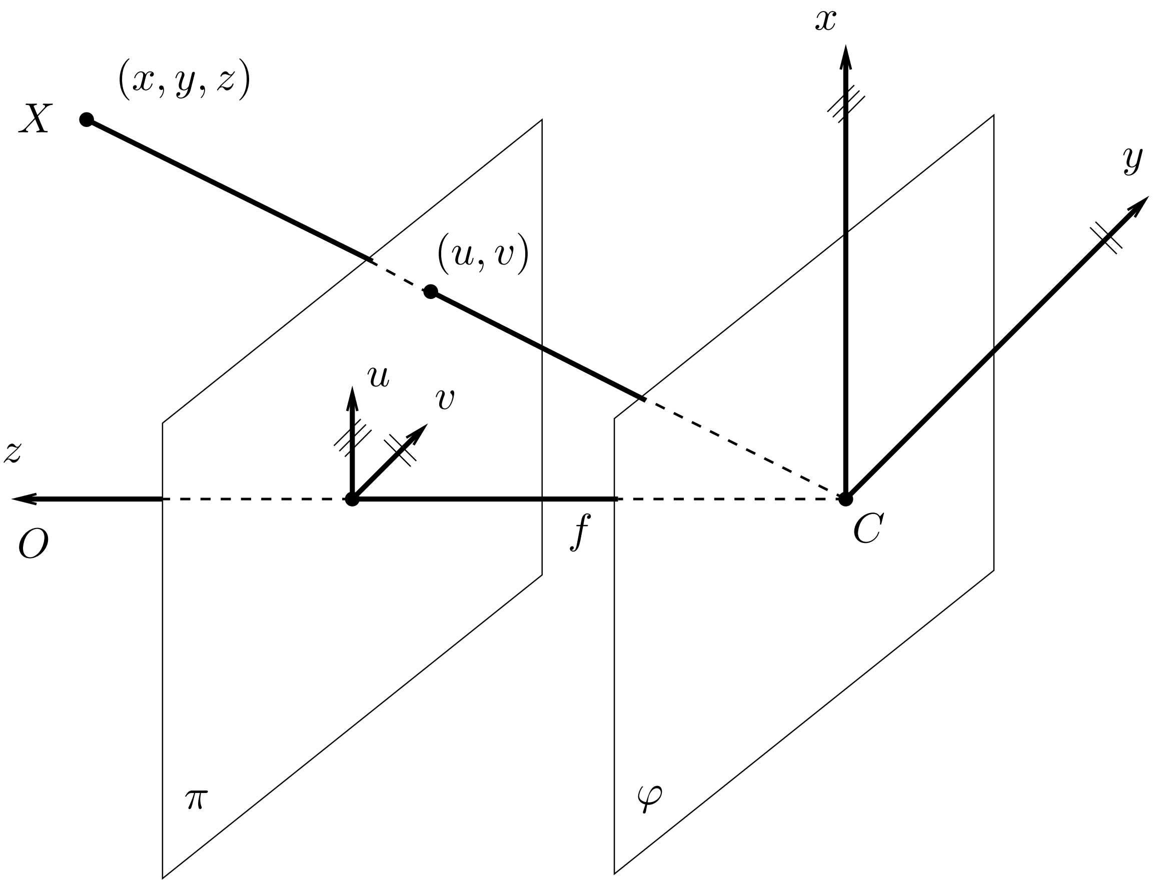
$$a\,u + b\,v + c = 0,$$

$$a\,u + b\,v + d = 0, \qquad\qquad d \neq c$$

$$(a, b, c) \times (a, b, d) \sim (b, -a, 0)$$

line at infinity $\sim$ a set of directions

A1. Perspective Camera



$$u = f\,\frac{x}{z}, \quad v = f\,\frac{y}{z}$$

coordinate choice $\sim$ normal camera

C – center of projection

O – optical axis

$O \cap \pi$ – principal point

π – image plane

φ – focal plane

Perspective Camera Representations: (I) By 3×4 projection matrix

$$u = f\,\frac{x}{z}, \quad v = f\,\frac{y}{z}$$

$$\begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} = \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$\underline{m} = P\underline{X}$$

$$u = \frac{m_1}{m_3}, \quad v = \frac{m_2}{m_3}$$

Will show: the set of projection matrices of finite projective cameras is isomorphic to the set of homogeneous 3×4 matrices for which the left hand 3×3 submatrix is non-singular.

1. changing image coordinate system

$$\underline{m} = K \left[E, \quad 0 \right] \underline{X}$$

$$K = \begin{bmatrix} f\,k_u & -f\,k_u \cot\theta & u_0 \\ 0 & f\,k_v \csc\theta & v_0 \\ 0 & 0 & 1 \end{bmatrix}$$ calibration mtx

$E \in \mathbb{R}^{3,3}$ unit matrix

2. changing world reference frame

$$X' = R\,(X - C), \qquad R^\top R = E$$

3. changing both

$$\underline{m} = K R \left[E, -C \right] \underline{X}$$

$$\boxed{P = K R \left[E, -C \right]}$$

general perspective camera, 11 DOF

Projection Matrix Decomposition

$$P = \begin{bmatrix} Q, & q \end{bmatrix} \quad \longrightarrow \quad KR \begin{bmatrix} E, & -C \end{bmatrix}$$

$Q \in \mathbb{R}^{3,3}$	full rank (if perspective cam.)
$q \in \mathbb{R}^{3,1}$	
$K \in \mathbb{R}^{3,3}$	upper triangular
$R \in \mathbb{R}^{3,3}$	$R^\top R = E$ unitary

1. optical center

$$C = -Q^{-1}q \qquad \text{(perspective)}$$

$$C = \operatorname{rnull}(Q) \qquad \text{(affine)}$$

$$\underline{C} = \operatorname{rnull}(P) \qquad \text{(general, see later)}$$

2. intrinsic parameter and rotation matrices

$$Q = KR$$

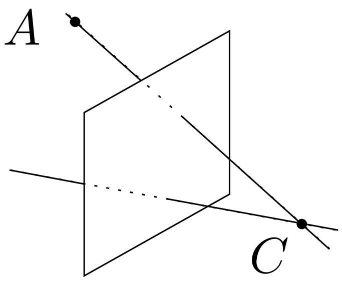
   a. RQ decomposition using Givens rotations
   b. RQ decomposition using Householder matrices
   c. QR-decomposition + post-processing

[Hartley & Zisserman 2000, p.552],
[Golub & van Loan 1989, sec. 5.2]

Optical Center

Obs: P has right null-space (rank 3 but 4 columns)

1. Let there be $\underline{C}$ s.t. $P\,\underline{C} = 0$.
   We show the $\underline{C}$ is the center of projection
2. Let there be line $A\,C$. In homogeneous coordinates we can write



$$\underline{X}(\lambda) = \lambda\,\underline{A} + (1-\lambda)\,\underline{C}$$

3. It images to

$$P\underline{X}(\lambda) = \lambda\,P\,\underline{A} + (1-\lambda)\,P\,\underline{C} = \lambda\,P\,\underline{A}$$

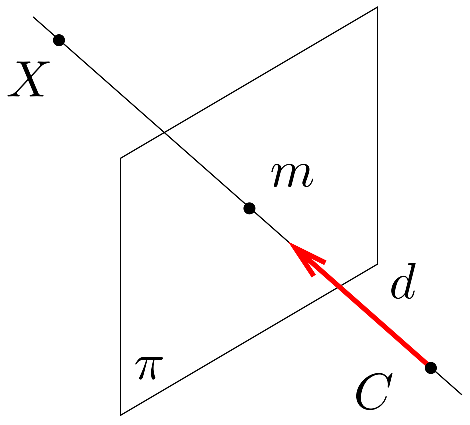
   a. whole line images to a single point $\Rightarrow$ it passes through the optical center
   b. must hold for all choices of $\underline{A}$ $\Rightarrow$ the only common point is the optical center

4. Hence

$$0 = P\,\underline{C} = \begin{bmatrix} Q, & q \end{bmatrix} \begin{bmatrix} C \\ 1 \end{bmatrix} = Q\,C + q \;\Rightarrow\; \boxed{C = -Q^{-1}q}$$

[H & Z, p. 144]

Optical Ray



1. Consider line (d line direction vector, $\lambda \in \mathbb{R}$)

$$X = C + \lambda\, d$$

2. The image of point X is

$$\underline{m} = \begin{bmatrix} Q, & q \end{bmatrix} \begin{bmatrix} X \\ 1 \end{bmatrix} = Q(C + \lambda d) + q = \lambda\, Q\, d =$$

$$= \lambda \begin{bmatrix} Q, & q \end{bmatrix} \begin{bmatrix} d \\ 0 \end{bmatrix}$$

- Line direction vector is represented by a point at infinity
- Optical ray direction vector corresponding to image point m is

$$\boxed{d = Q^{-1}\underline{m}}$$

Optical Axis

Obs: O.A. is the focal plane normal vector

1. Focal plane images to line at infinity:

$$\begin{bmatrix} u \\ v \\ 0 \end{bmatrix} = \begin{bmatrix} q_1^\top & q_{14} \\ q_2^\top & q_{24} \\ q_3^\top & q_{34} \end{bmatrix} \begin{bmatrix} X \\ 1 \end{bmatrix}$$

2. Point X in focal plane iff

$$q_3^\top X + q_{34} = 0$$

3. This is a plane with $\pm q_3$ as the normal vector

4. **O.A. Sign:** substitution $P \mapsto \lambda P$ must not change optical axis direction
5. We select (assuming $\det(R) > 0$)

$$\boxed{v = \det(Q)\, q_3}$$

since $\det(Q) \mapsto \lambda^3 \det(Q)$ and $q_3 \mapsto \lambda\, q_3$

[H & Z, p. 147]

Principal Point

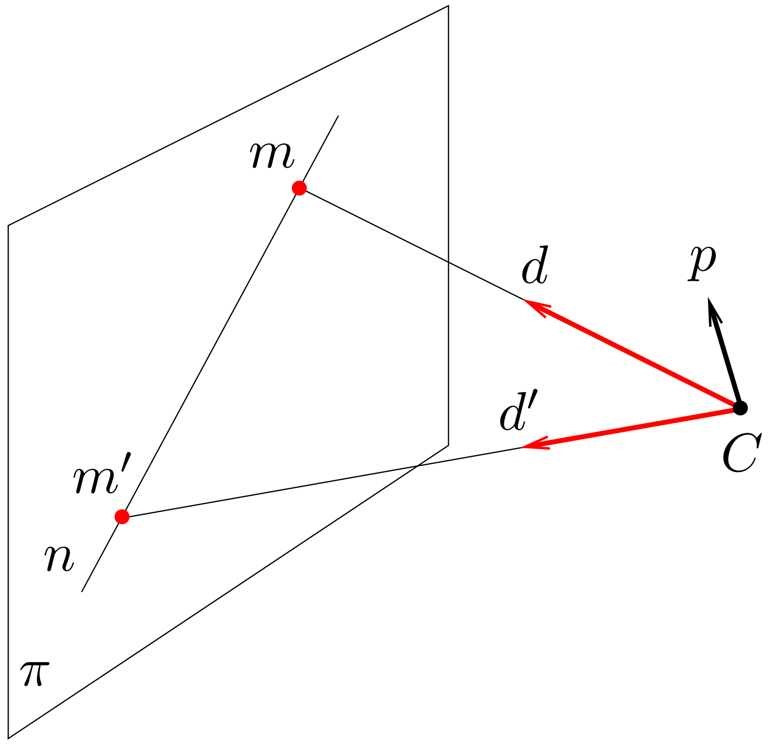
Obs: the intersection of image plane and optical axis

1. We take point at infinity on optical axis that projects to P.P. m_0
2. Then

$$\underline{m}_0 = \begin{bmatrix} Q, & q \end{bmatrix} \begin{bmatrix} q_3 \\ 0 \end{bmatrix} = Q\, q_3$$

Optical plane

A spatial plane with normal p passing through the optical center C and a given image line n.



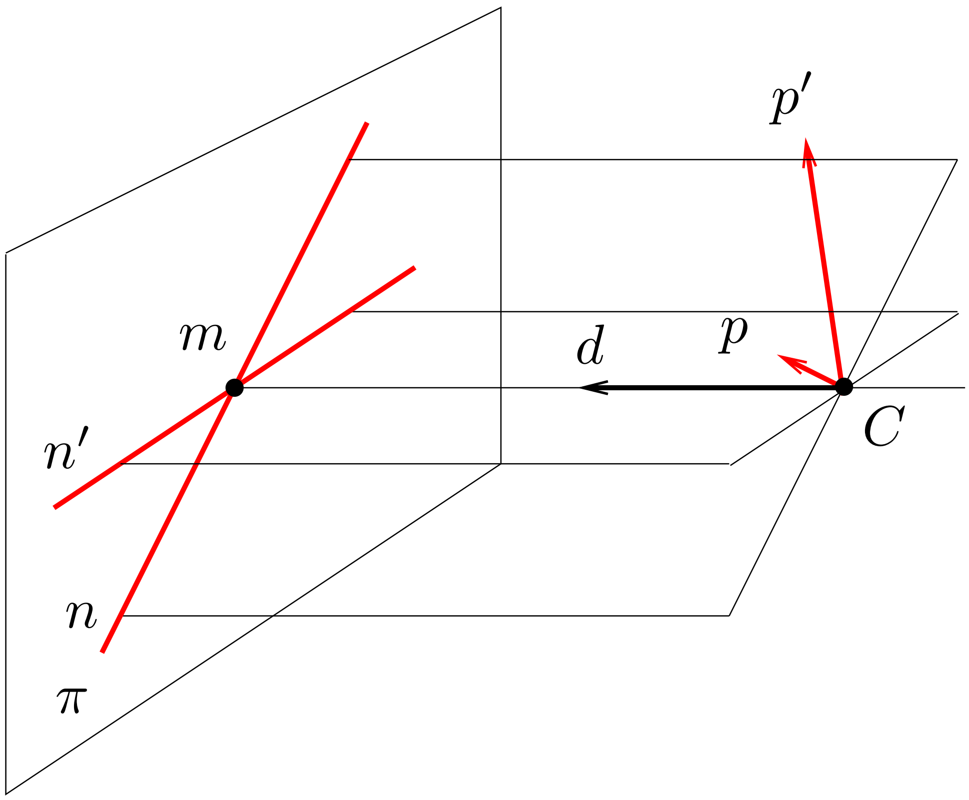
optical ray given by m $\qquad d = Q^{-1}\underline{m}$

optical ray given by m' $\qquad d' = Q^{-1}\underline{m}'$

$$p = d \times d' = Q^{-1}\underline{m} \times Q^{-1}\underline{m}' = Q^{\top}(\underline{m} \times \underline{m}') \stackrel{!}{=} Q^{\top}\underline{n}$$

optical plane given by n: $\boxed{p = Q^{\top}\underline{n}}$

Note 1: Optical Ray as Optical Plane Intersection



optical plane given by n $\qquad p = Q^\top \underline{n}$

optical plane given by n' $\qquad p' = Q^\top \underline{n}'$

$$d = p \times p' = Q^\top \underline{n} \times Q^\top \underline{n}' = Q^{-1}(\underline{n} \times \underline{n}') \overset{!}{=} Q^{-1}\underline{m}$$

optical ray given by m $\qquad d = Q^{-1}\underline{m}$

Summary: optical center, ray, axis, plane

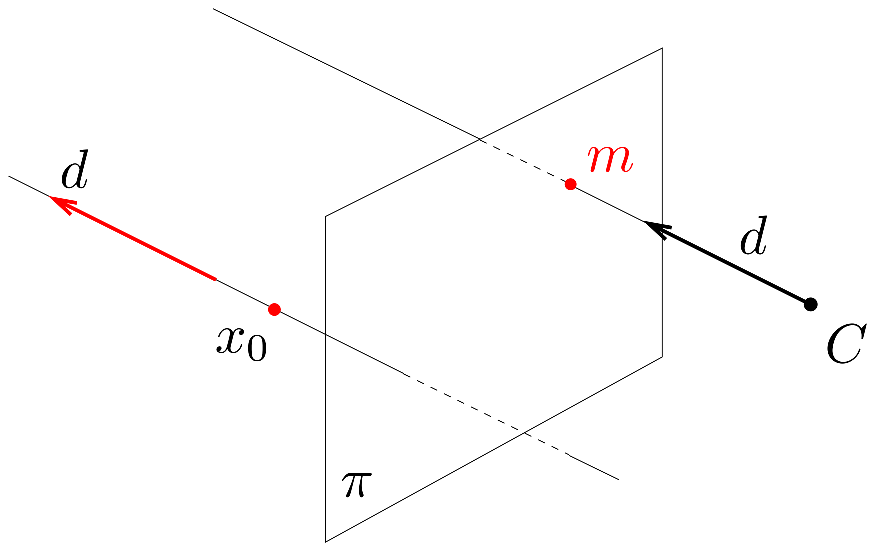
General camera

$$P = [Q,\, q] = \begin{bmatrix} q_1^\top & q_{14} \\ q_2^\top & q_{24} \\ q_3^\top & q_{34} \end{bmatrix}$$

$\underline{C} = \mathrm{rnull}(P)$	optical center
$d = Q^{-1}\,\underline{m}$	optical ray direction
$\det(Q)\, q_3$	optical axis (outward)
$Q\, q_3$	principal point
$p = Q^\top\, \underline{n}$	optical plane

A2. Vanishing Point

Vanishing point: the limit of the projection of a point that moves along a space line indefinitely in one direction. (the image of the point at infinity on the line)

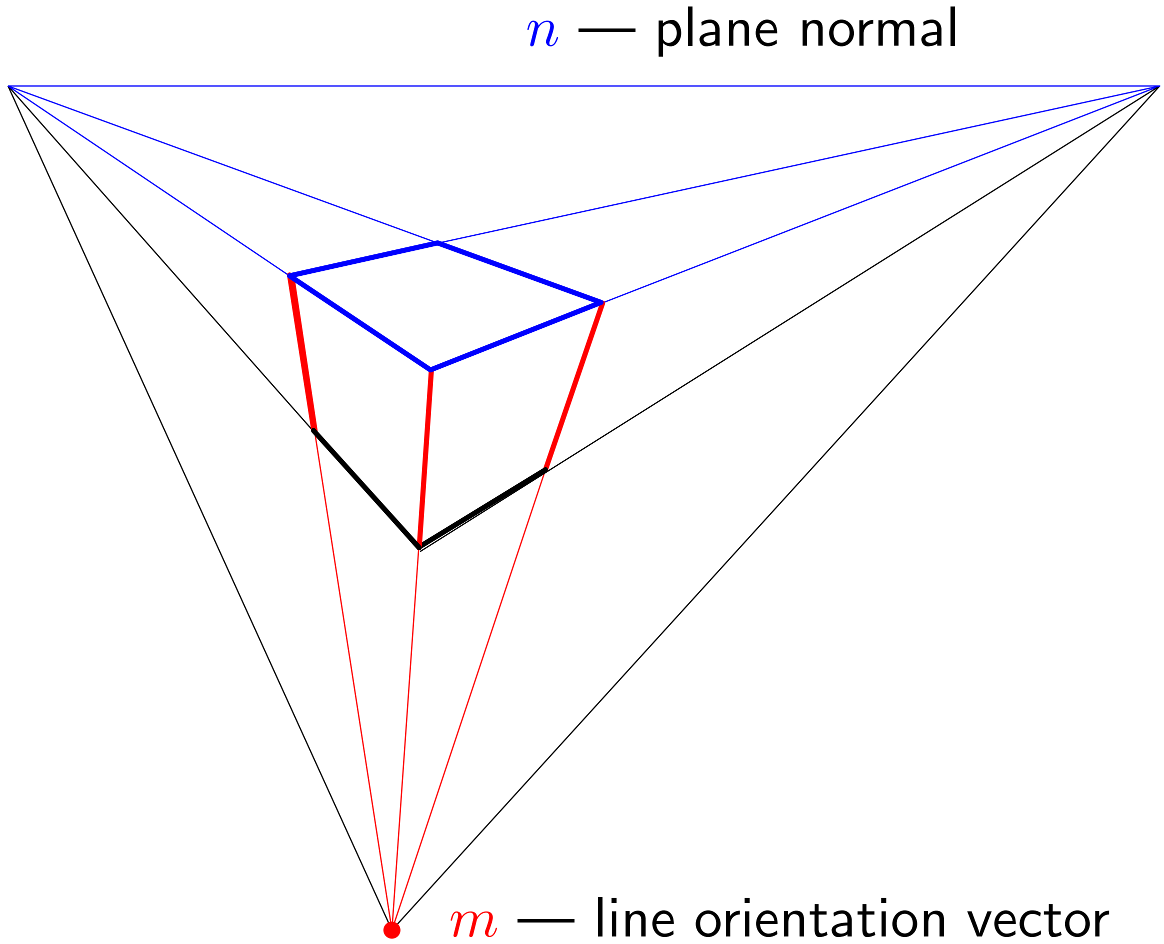


$$\underline{m}_\infty = \lim_{\lambda \to \pm\infty} P \begin{bmatrix} x_0 + \lambda d \\ 1 \end{bmatrix} = \cdots = Q\, d$$

- V.P. is independent on line position, depends on its orientation only
- A space line of direction vector d has a V.P. represented by $\underline{m} = Q\, d$
- V.P. m corresponds to spatial direction $d = Q^{-1}\underline{m}$ (optical ray through m)

- How high is the sun?

Vanishing Line



Vanishing line: the set of vanishing points of all lines in a plane (the image of the line at infinity in the plane)

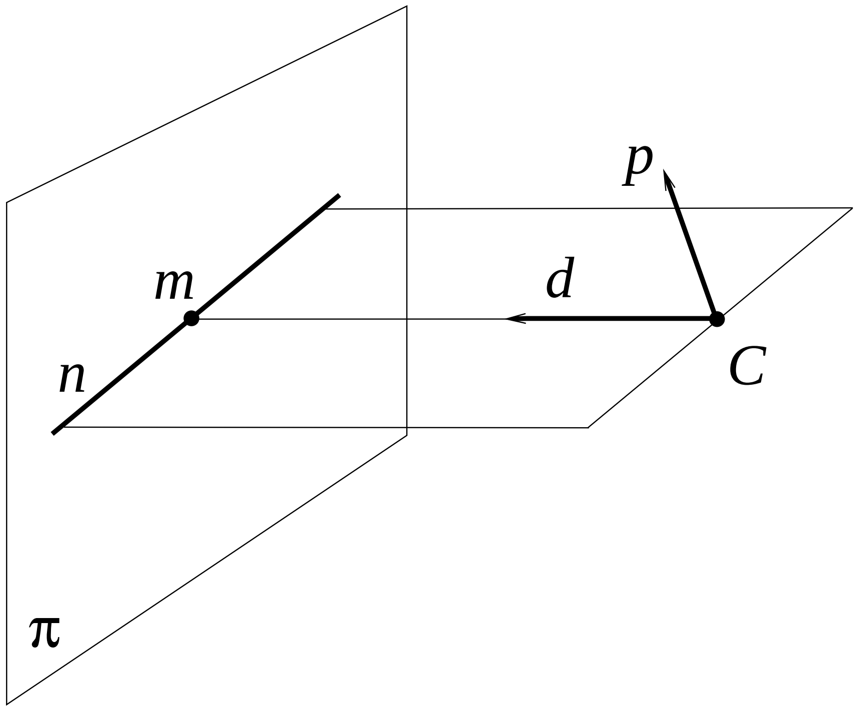
- A space plane of normal vector p has a V.L. represented by $\underline{n} = Q^{-\top} p$.
- V.L. n corresponds to space plane of normal vector $p = Q^{\top} \underline{n}$

- Fly above, keeping the altitude constant!
- Later: How long is the Nuselsky Bridge?

A4. Incidence, Duality

1. **Verify:** An image point m is *incident* on image line n iff

$$(\underline{m},\, \underline{n}) = 0$$



$$0 \equiv (d,\, p) = (Q^{-1}\underline{m},\, Q^{\top}\underline{n}) = (\underline{m},\, Q^{-\top}Q^{\top}\underline{n}) = (\underline{m},\, \underline{n})$$

2. The principle of duality in $\mathcal{P}^2$: "The role of image points and image lines in any theorem of projective geometry of the plane are interchangeable."

A5. Collinearity of Points

Image points m_i are collinear iff there are no more than two independent vectors among $\underline{m}_i$, $i = 1, \dots, k$, $k \geq 2$.

I.e.

$$\exists\, \underline{n},\ \|\underline{n}\| > 0\colon\ (\underline{m}_i,\, \underline{n}) = 0, \quad i = 1, \dots, k$$

We construct

$$0 = \sum_{i=1}^{k} (\underline{m}_i, \underline{n})^2 = \underline{n}^\top \left(\sum_{i=1}^{k} \underline{m}_i\, \underline{m}_i^\top \right) \underline{n} = \dots$$

- ... gives quadratic form $(\underline{n}, M\underline{n}) = 0$, where

$$M = \sum_{i=1}^{k} \underline{m}_i\, \underline{m}_i^\top$$

  is a symmetric positive definite *moment matrix*.

- The $\underline{n}$ is its null-space.
- Necessary and sufficient condition for collinearity $\operatorname{rank}(M) = 2$ $\quad \operatorname{rank}(M) = 1 \sim$ co-located pts
- The $\underline{n}$ represents the common line

Concurrency of Lines

Image lines n_i are concurrent iff there are two independent vectors among $\underline{n}_i$, $i = 1, \dots, l$, $l \geq 2$.

$$N = \sum_{i=1}^{l} \underline{n}_i \, \underline{n}_i^\top$$

- Intersection point $\underline{m}$: null-space of matrix N.

Numerical procedure:

1. Error $\varepsilon_i = (\underline{n}_i, \underline{m}) \sim$ distance of m from n_i

   use normalized $\underline{n}_i = (n_{i1}, n_{i2}, n_{i3})$ s.t. $n_1^2 + n_2^2 = 1$

2. $$J = \sum_{i=1}^{l} \varepsilon_i^2 = (\underline{m},\, N\underline{m}), \qquad \underline{m}^* = \arg\min_{\underline{m}} J$$

3. If

   $$N = \sigma_1 \, u_1 u_1^\top + \sigma_2 \, u_2 u_2^\top + \sigma_3 \, u_3 u_3^\top, \; \sigma_1 \geq \sigma_2 \geq \sigma_3 \geq 0$$

   is the SVD of N then $\underline{m}^* = u_3$ and $J^* = \sigma_3$

A6. Collineations

- Linear projective space $\mathbb{P}^2$
- **Collineation:** bijection acting on image points *and* image lines, such that
  1. collinear image points are mapped to collinear image points,
  2. concurrent image lines are mapped to concurrent image lines,
  3. incidence is preserved.
- A mapping $h : \mathbb{P}^2 \to \mathbb{P}^2$ is a collineation iff there exists a non-singular 3×3 matrix H such that

$$h(\underline{x}) = H\,\underline{x} \quad \text{for all } \underline{x} \in \mathbb{P}^2$$

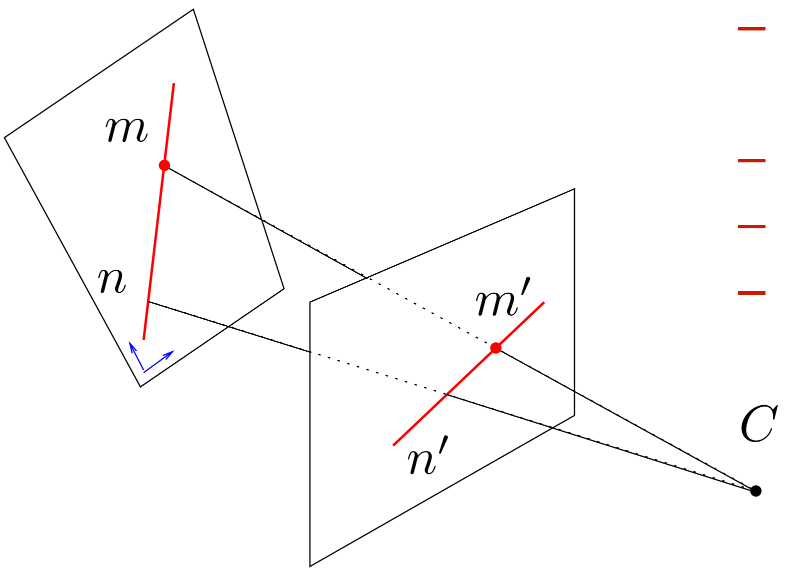
- 8 DOF, given by 4 correspondences (points,lines) in a general position
- Homogeneous matrix representant: $\det H = 1$
- Collineations form a group isomorphic to $SO(3)$

- Image point and image line map as follows:

$$\underline{m}' = A^\top \underline{m} \qquad \text{image point (contravariant)}$$
$$\underline{n}' = A^{-1} \underline{n} \qquad \text{image line (covariant)}$$

- Note: Collineation A maps points by $A^\top$

- perspective re-projection from a plane is a collineation:



  - local coordinate system in the plane
  - point $\mapsto$ point
  - line $\mapsto$ line
  - incidence preserved

- camera intrinsic parameter change is a collineation $(K_2\, K_1^{-1})^\top$
- camera rotation around optical center is a $(K R K^{-1})^\top$, in general $(K_2\, R_2\, R_1^{-1} K_1^{-1})^\top$.

Collineation Decomposition

Unique decomposition:

$$A = A_P\, A_A\, A_S$$

$$A_S = \begin{bmatrix} s\,R & t \\ 0^\top & 1 \end{bmatrix}^\top \qquad \text{similarity}$$

$$A_A = \begin{bmatrix} K & 0 \\ 0^\top & 1 \end{bmatrix}^\top \qquad \text{special affine}$$

$$A_P = \begin{bmatrix} E & 0 \\ v^\top & w \end{bmatrix}^\top \qquad \text{special projective (elation)}$$

K – upper triangular matrix, $\det K = 1$

R – orthogonal, $R^\top R = E$

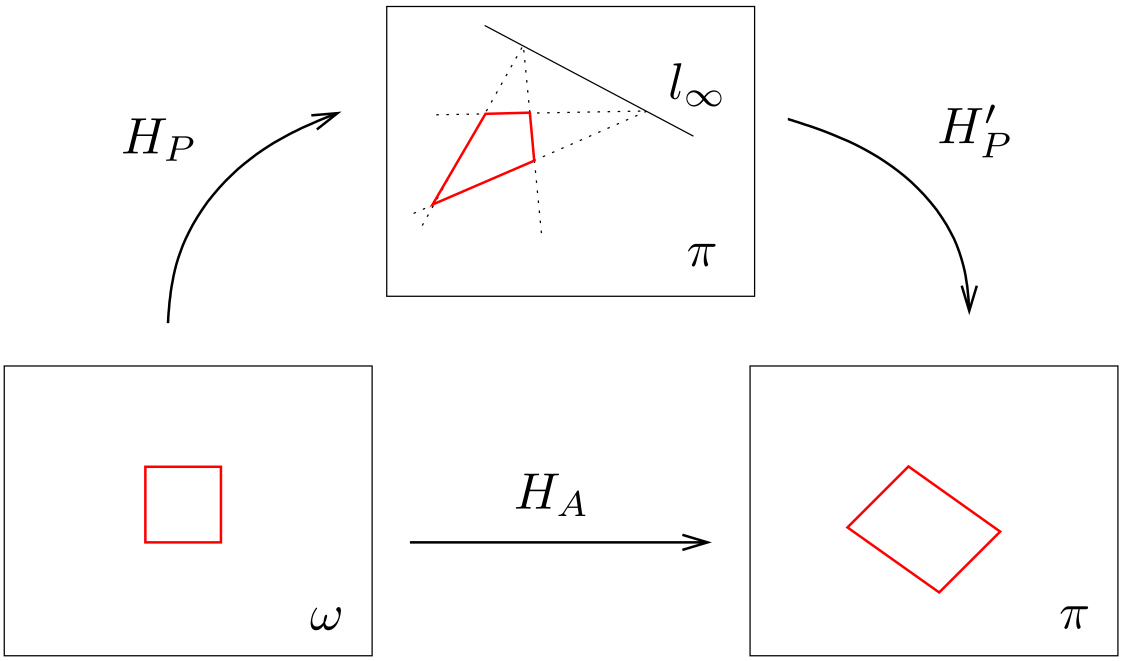
$s, w \in \mathbb{R}$, $s > 0$, $w \neq 0$

$$A^\top = \begin{bmatrix} sRK + t\,v^\top & w\,t \\ v^\top & w \end{bmatrix}$$

uniqueness of decomposition $\leftrightarrow$ uniqueness of QR decomposition

A7. Affine Rectification

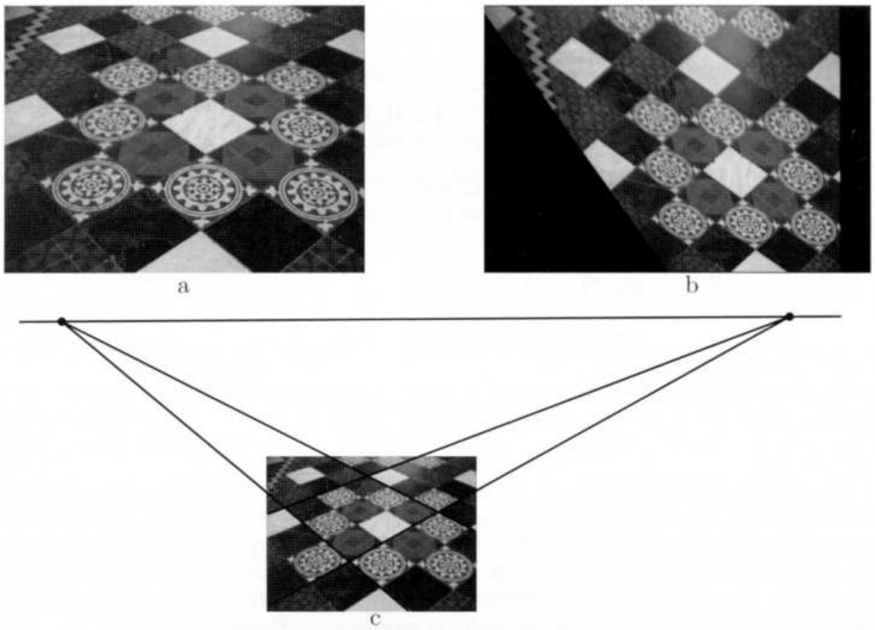
- Given a world plane ω and the image plane π
- π is the image of ω
- Remove 'perspective distortion' from the image



1. the image of the line at infinity in ω is the vanishing line $l_\infty = (l_1, l_2, l_3)$
2. we will send l_∞ to the line at infinity $l^*_\infty = (0, 0, 1)$ in π by H'_P
3. the combination $H_A = H'_P H_P$ preserves l^*_∞ in ω and π: $\quad l^*_\infty = H_A^{-1} l^*_\infty$

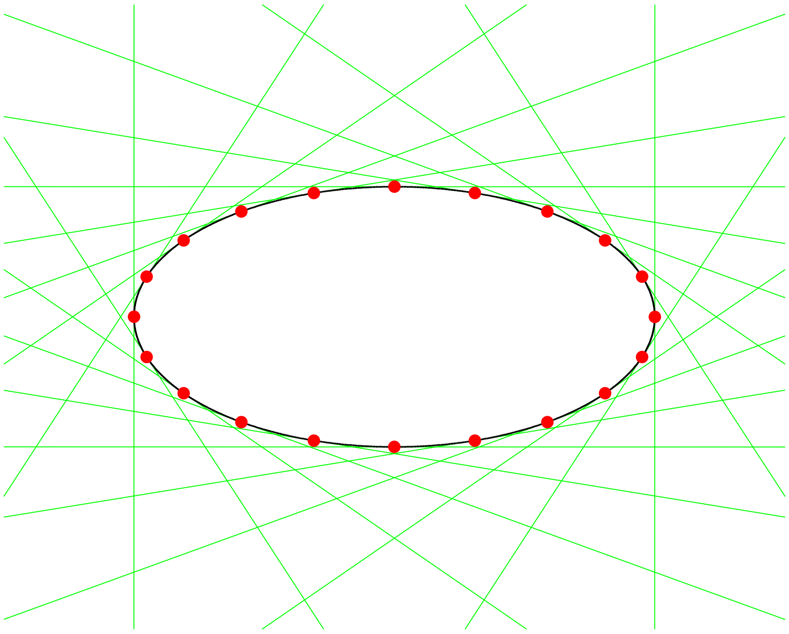
4. hence
$$H_A = \begin{bmatrix} \cdot & \cdot & 0 \\ \cdot & \cdot & 0 \\ \cdot & \cdot & 1 \end{bmatrix}$$
and the mapping H_A is affine

5. the colineation that maps vanishing line $l_\infty = (l_1,\, l_2,\, l_3)$ to the canonical representation $l^*_\infty = (0,\, 0,\, 1)$ is
$$H_A = \begin{bmatrix} 1 & 0 & l_1 \\ 0 & 1 & l_2 \\ 0 & 0 & l_3 \end{bmatrix} A_A\,,$$
where A_A is any affine mapping



Q: what about metric rectification? We need to know about invariants to similarity $\mapsto$ circular points

A8. Conics and Dual Conics



Point conic: $\underline{m}^\top C\,\underline{m} = 0$ point-conic incidence

C – homogeneous symmetric matrix

Ex: ellipse $x^2 + 2\,y^2 - 1 = 0$: $C = \mathrm{diag}\,(1, 2, -1)$

Tangent line n at point m: $\underline{m}^\top \underbrace{C\,\underline{m}}_{\text{line } \underline{n}} = 0$

Line conic: $\underline{n}^\top C^*\,\underline{n} = 0$ line-conic incidence

Transformation under collineation:

$$\underline{m} \mapsto A^\top \underline{m}, \quad C \mapsto A^{-1} C A^{-\top}, \quad C^* \mapsto A^\top C^* A$$

Conic Dual to Circular Points

Similarity

$$Y = A_S^\top X \qquad\qquad A_S^\top = \begin{bmatrix} s\,R & t \\ 0^\top & 1 \end{bmatrix}$$

- looking for points X invariant to A_S: $X = A_S^\top X$
- they exist: $I = A_S^\top I, \quad J = A_S^\top J,$

  where $I = (1,\, i,\, 0)^\top, \quad J = (1,\, -i,\, 0)^\top$
- they are called circular points

  Every circle intersects l_∞ at I, J:
  circle points: $x_1^2 + x_2^2 - x_3^2\, r^2 = 0, \quad l_\infty$ points: $x_3 = 0$
- there is a degenerate dual conic

$$C_\infty^* = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = I J^\top + J I^\top$$

- C_∞^* preserved under similarity A_S
- C_∞^*: the conic 'attached' to each plane in $\mathbb{P}^3$

A9. Angles in Projective Plane

Given two lines $\underline{l}$ and $\underline{n}$ in plane ρ, C^*_∞ associated with ρ:

$$\cos\Theta = \frac{\underline{l}^\top C^*_\infty \underline{n}}{\sqrt{(\underline{l}^\top C^*_\infty \underline{l})(\underline{n}^\top C^*_\infty \underline{n})}}$$

1. $\cos\Theta$ is invariant to all collineations

$$\begin{aligned} \underline{l} &\mapsto A^{-1}\underline{l} \\ C^*_\infty &\mapsto A^\top C^*_\infty A \\ \underline{l}^\top C^*_\infty \underline{n} &\mapsto \underline{l}^\top C^*_\infty \underline{n} \end{aligned}$$

(perspective projection from plane ρ to image is a collineation)

2. gauge for measuring angles in plane ρ, where

$$C^*_\infty = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

3. perpendicularity condition

$$l \perp n \text{ in } \rho \;\Leftrightarrow\; \text{in image plane: } \underline{l}^\top C^*_\infty \underline{n} = 0$$

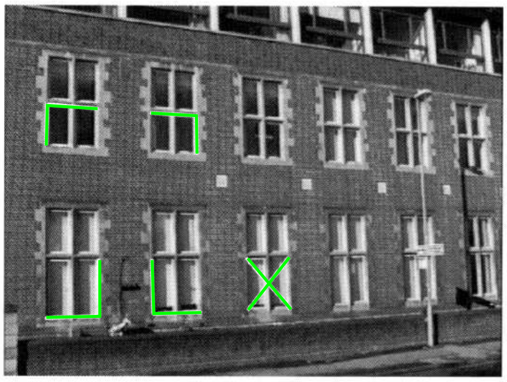
4. $C^*_\infty \sim$ 5 DOF (3×3, symmetric, homogeneous)

Direct Metric Rectification

1. under collineation: $C^*_\infty \mapsto A^\top C^*_\infty A$
2. C^*_∞ positive def. $\Rightarrow SVD(C^*_\infty) = U\,\mathrm{diag}(1,1,0)\,U^\top$
3. $U^\top$ is the rectifying homography (up to similarity)
4. Given an image from a perspective camera in which perpendicular pairs of lines can be identified
5. 5 line pairs orthogonal in ρ suffice to estimate C^*_∞:

$$\underline{n}_i^\top C^*_\infty \underline{n}_j = 0$$

6. homogeneous system of linear equations $Dc = 0$
7. Rectifying homography $U^\top$ from SVD decomposition of C^*_∞



a

b

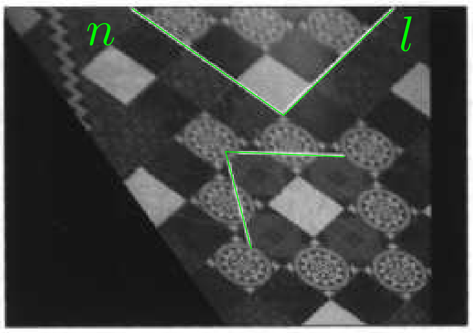
8. drawback: needs 5 pairs of orthogonal lines

Stratified Metric Rectification

1. the image of l^*_∞ in ρ gives affine rectification
2. the current image differs from the rectified image by a collineation $H = H_A\, H_S$ (special affine, similarity)
3. Since $H_P \sim E$, under H the C^*_∞ maps as follows:

$$C^*_\infty \ \mapsto \ H^\top C^*_\infty H = \begin{bmatrix} ZZ^T & 0 \\ 0^\top & 0 \end{bmatrix}, \qquad Z = sRK$$

   $ZZ^\top$ has 2 DOF + scale $\Rightarrow$ need 2 more constraints
4. $\underline{l}^\top C^*_\infty\, \underline{n} = 0$ gives the constraints, linear in $ZZ^\top$
5. Z from $Z\,Z^\top$ by Cholesky decomposition
6. Rectifying homography is then $\begin{bmatrix} Z & 0 \\ 0^\top & 1 \end{bmatrix}^\top$
7. A single parallelogram suffices for rectification



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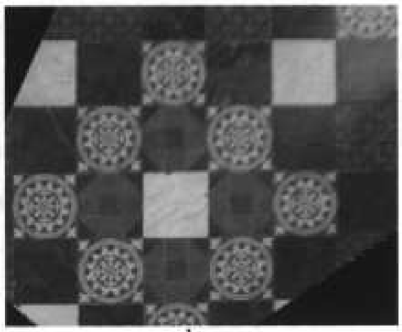
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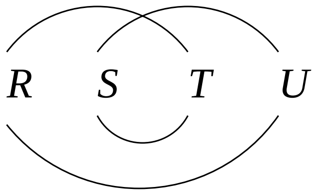
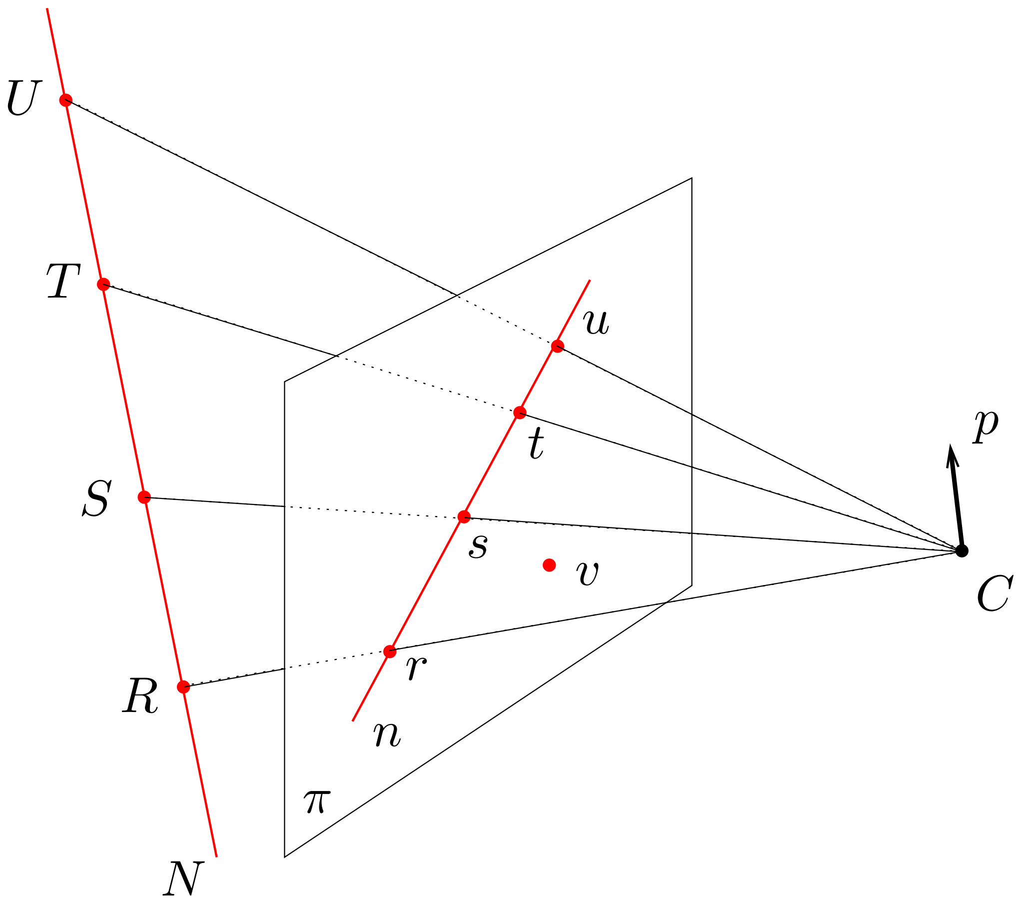
A10. Cross Ratio

Four collinear space points R, S, T, U define cross-ratio

$$[RSTU] = \frac{|RT|}{|RU|}\,\frac{|SU|}{|ST|}$$



signed distances w.r.t. to a fixed line orientation



Let $r, s, t, u \in n$ be images of spatial points $R, S, T, U \in N$, let $v \notin n$ be an image point. Then:

$$[RSTU] = \frac{|RT|}{|RU|}\,\frac{|SU|}{|ST|} = \frac{\begin{vmatrix} \underline{r} & \underline{t} & \underline{v} \end{vmatrix}}{\begin{vmatrix} \underline{r} & \underline{u} & \underline{v} \end{vmatrix}}\,\frac{\begin{vmatrix} \underline{s} & \underline{u} & \underline{v} \end{vmatrix}}{\begin{vmatrix} \underline{s} & \underline{t} & \underline{v} \end{vmatrix}}$$

note: $\begin{vmatrix} \underline{r} & \underline{t} & \underline{v} \end{vmatrix} = (\underline{r} \times \underline{t}, \underline{v})$

Corollaries:

1. The cross-ratio is invariant under collineations.

2. The cross-ratio is invariant to perspective projection.

$$[RSTU] = [r\,s\,t\,u]$$

Given four collinear points, any perspective camera will “see” the same cross-ratio of their images.

A11. Projective Coordinates

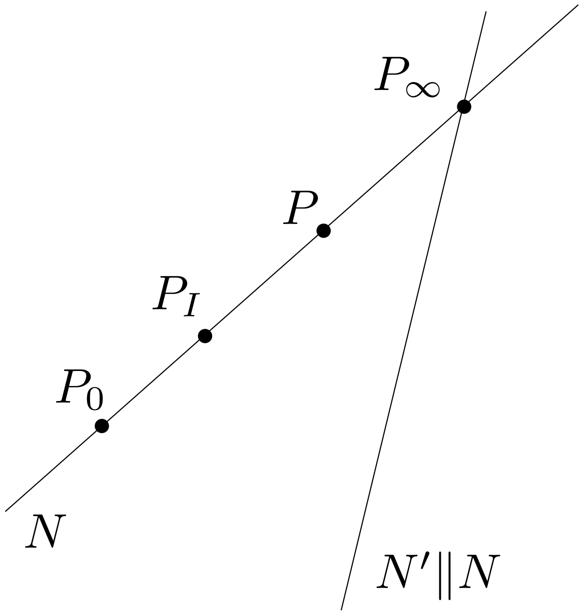
The 1-D projective coordinate of a point P:

$$[P] = [P_\infty\, P_0\, P_I\, P] = \frac{|P_\infty P_I|}{|P_0 P_I|}\,\frac{|P_0 P|}{|P_\infty P|}$$

P_0 – the origin $\quad [P_0] = 0$

P_I – the unit point $\quad [P_I] = 1$

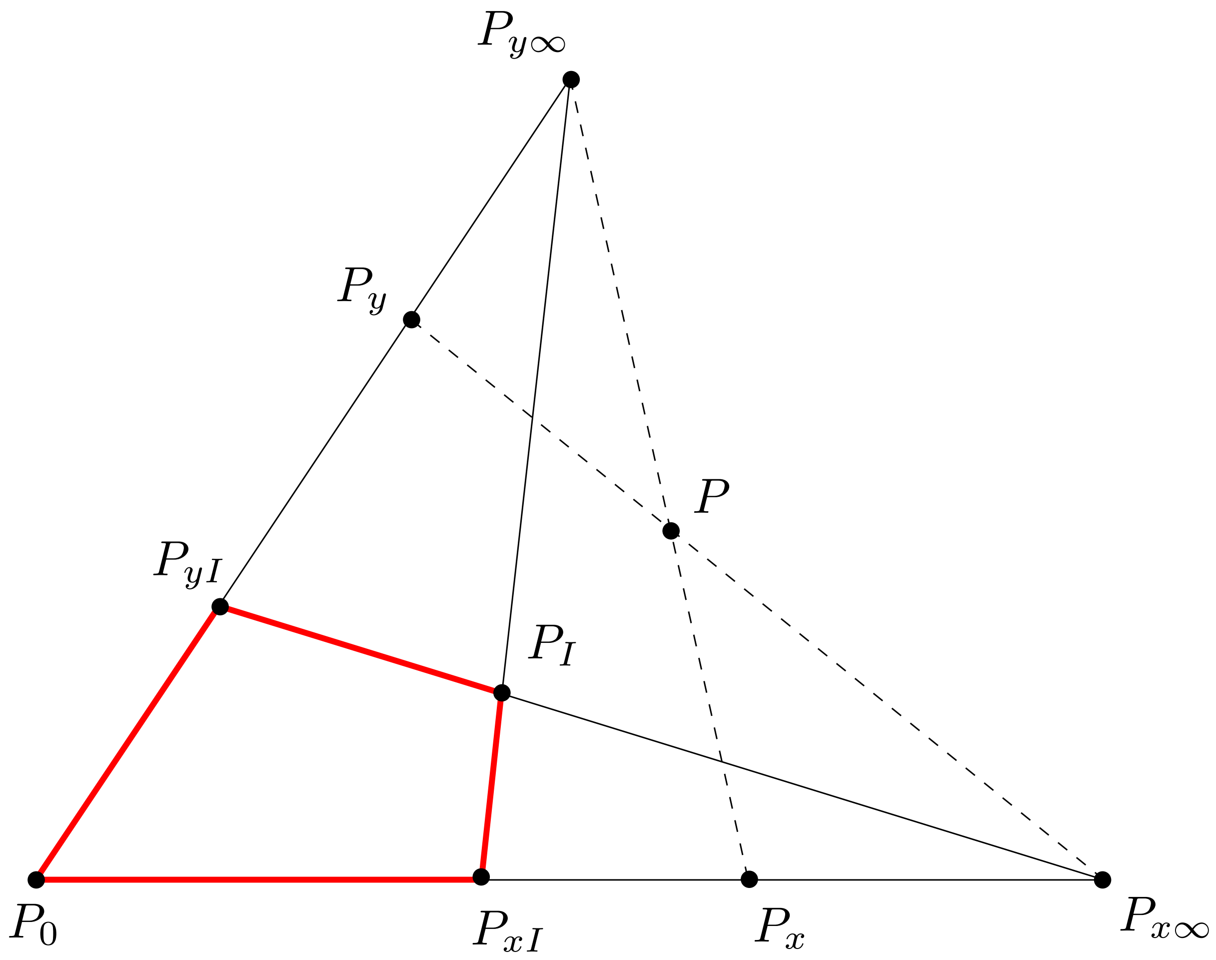
P_∞ – the supporting point $\quad [P_\infty] = \pm\infty$



Application

Given the image of a line N, the origin, the unit point, and the vanishing point, then the Euclidean coordinate of any point $P \in N$ can be determined.

- 2-D projective coordinates of a point P

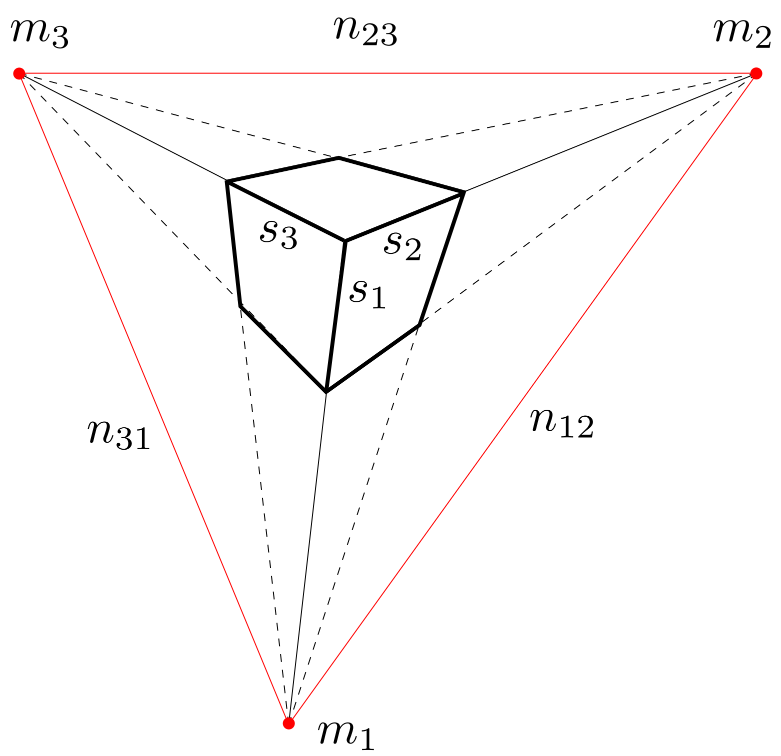


$$[P_x] = [P_{x\infty}\ P_0\ P_{xI}\ P_x]$$
$$[P_y] = [P_{y\infty}\ P_0\ P_{yI}\ P_y]$$

Little Spy Calibrating His Camera. . .

A12. Conjugacy and Orthogonality

$$P = Q\,[E, -C]$$
$$Q = KR$$
$$N_{ij} \mapsto n_{ij}, \quad i \neq j$$
$$S_i \mapsto s_i$$
$$d_i = Q^{-1}\underline{m}_i$$
$$p_{ij} = Q^\top \underline{n}_{ij}$$
$$\Omega = \left(KK^\top\right)^{-1}$$



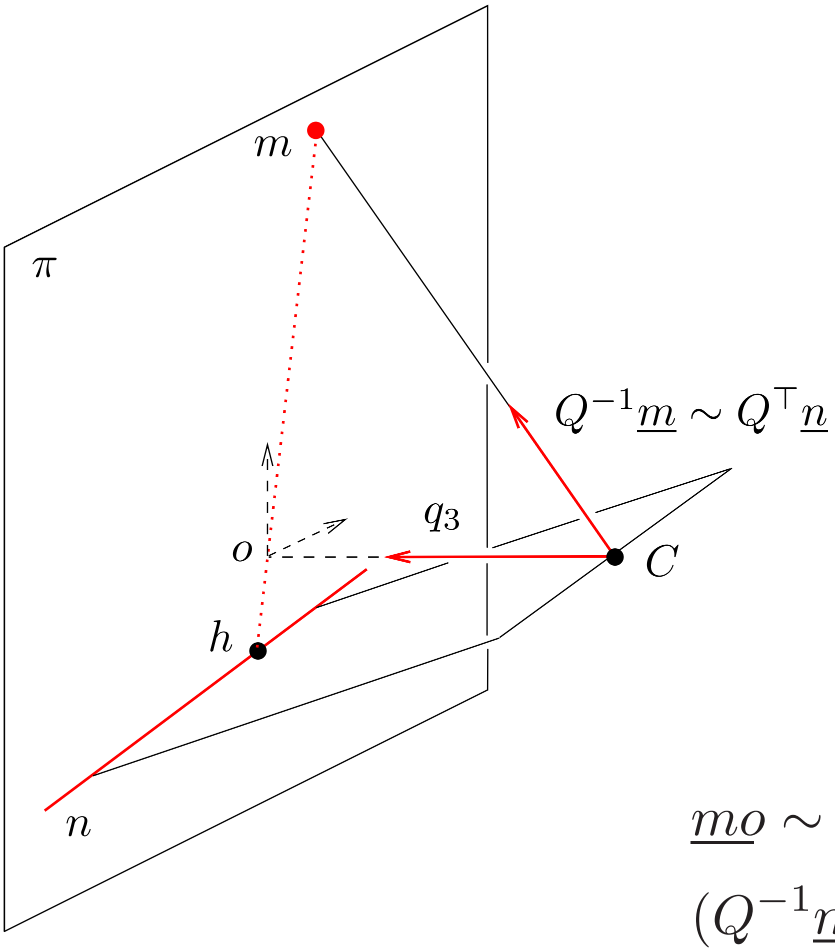
Observations $(i \neq j \neq k)$:

$$S_i \perp S_j \Rightarrow (\underline{m}_i,\, \Omega^\top\, \underline{m}_j) = 0 \qquad \text{conjugate points}$$
$$N_{ij} \perp N_{ik} \Rightarrow (\underline{n}_{ij},\, \Omega^{-1}\underline{n}_{ik}) = 0 \qquad \text{conjugate lines}$$
$$N_{ij} \parallel d_k \Rightarrow \underline{n}_{ij} = \Omega^\top \underline{m}_k \qquad \text{a polar and a pole}$$

Triangle (m_1, m_2, m_3) of such properties is called the self-polar triangle.

Pole, Polar

- The pole of an image line n is the image point m, $\underline{m} = \Omega^{-1}\underline{n}$.
- The polar of an image point m is the image line n, $\underline{n} = \Omega^{T}\underline{m}$.



$$\underline{mo} \sim Q^{-1}\underline{m} \times q_3,$$

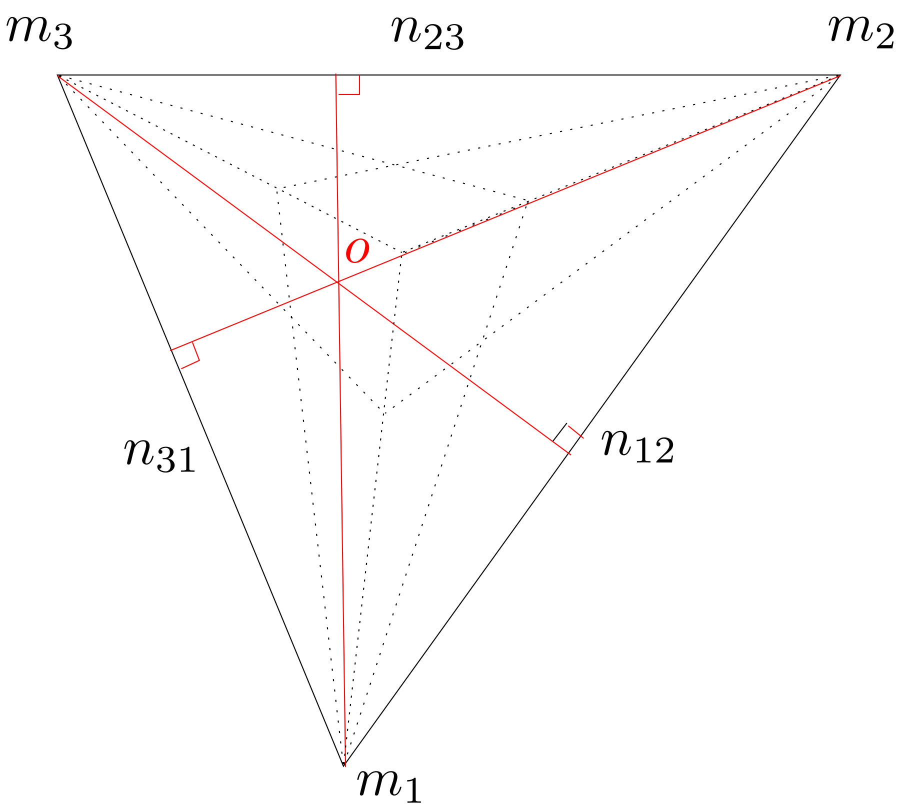
$$(Q^{-1}\underline{m} \times q_3,\ Q^{\top}\underline{n}) = 0$$

Observation: line $mo \perp n$ at point h.

A13. Perspective Camera Calibration

Image Center (u_0, v_0)

The orthocentre of a self-polar triangle is the image center.



Focal Length and Aspect Ratio Calibration

Conjugated image points m_i, m_j:

$$\underline{m}_i = \begin{bmatrix} u_i & v_i & 1 \end{bmatrix}^\top \qquad i = 1, 2, 3$$

$$\underline{o} = \begin{bmatrix} u_0 & v_0 & 1 \end{bmatrix}^\top \qquad \text{image center}$$

$$\Omega = \left(K K^\top \right)^{-1} \qquad \text{symmetric matrix}$$

Image points m_i, m_j conjugate iff $(\underline{m}_i, \Omega^\top \underline{m}_j) = 0$.

After expansion:

$$\boxed{\frac{(u_i - u_0)(u_j - u_0)}{k_u^2} + \frac{(v_i - v_0)(v_j - v_0)}{k_v^2} + f^2 = 0}$$

Focal Length and Aspect Ratio ($i \neq j \neq k$)

$$(k_u f)^2 = \frac{u_i - u_0}{v_j - v_k} \begin{vmatrix} \underline{m}_j & \underline{m}_k & \underline{o} \end{vmatrix}$$

$$(k_v f)^2 = \frac{v_0 - v_i}{u_j - u_k} \begin{vmatrix} \underline{m}_j & \underline{m}_k & \underline{o} \end{vmatrix}$$

Problem: numerical stability. Better: Cholesky decomp. of Ω.

Orientation Matrix and the Center of Projection

Orientation Matrix R

$$R = K^{-1} \left[\underline{m}_1 \quad \underline{m}_2 \quad \underline{m}_3\right]$$

Center of Projection

from the image m_o of the coordinate origin O

$$C = -\lambda\, (KR)^{-1}\underline{m}_o, \quad \lambda \neq 0$$

λ from a point X in known distance from O

[Kanatani93, p.62]

Summary:

1. Self-polar triangle calibrates:
   - image center (u_0, v_0)
   - focal length $\pm k_u f$ and aspect ratio $\frac{k_u}{k_v}$
   - camera orientation matrix R
2. Additional length calibrates:
   - center of projection C.

Any 6 vertices of a box of unknown side suffice.