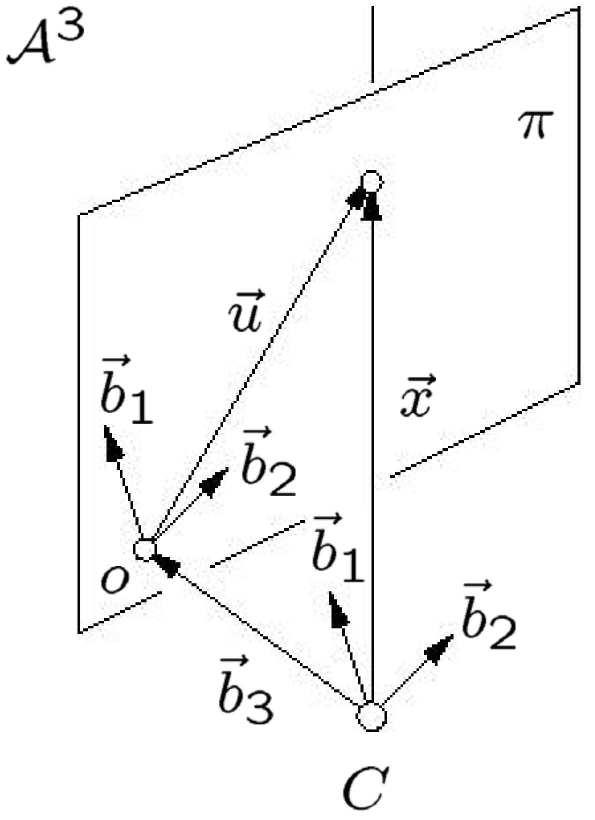
CAMERA COORDINATE SYSTEM

Direction vector of projection ray



Miracle: The coordinates od the direction vector of a projection ray can be constructed by a adding "1" to image coordinates:

We measure in image

$$\vec{u} = u\,\vec{b}_1 + v\,\vec{b}_2 \sim \mathbf{u}_{(\vec{b}_1,\vec{b}_2)} = \begin{pmatrix} u \\ v \end{pmatrix}$$

Coordinate system with origin C

$$\begin{aligned} \beta &= (\vec{b}_1, \vec{b}_2, \vec{b}_3) \\ S &= (C, \beta) \\ \vec{b}_3 &= \varphi(C, o) \end{aligned}$$

Triangle equality

$$\vec{x} = \vec{u} + \vec{b}_3$$

$$\vec{x} = u\,\vec{b}_1 + v\,\vec{b}_2 + 1\,\vec{b}_3 \sim \mathbf{x}_\beta = \begin{pmatrix} u \\ v \\ 1 \end{pmatrix}$$

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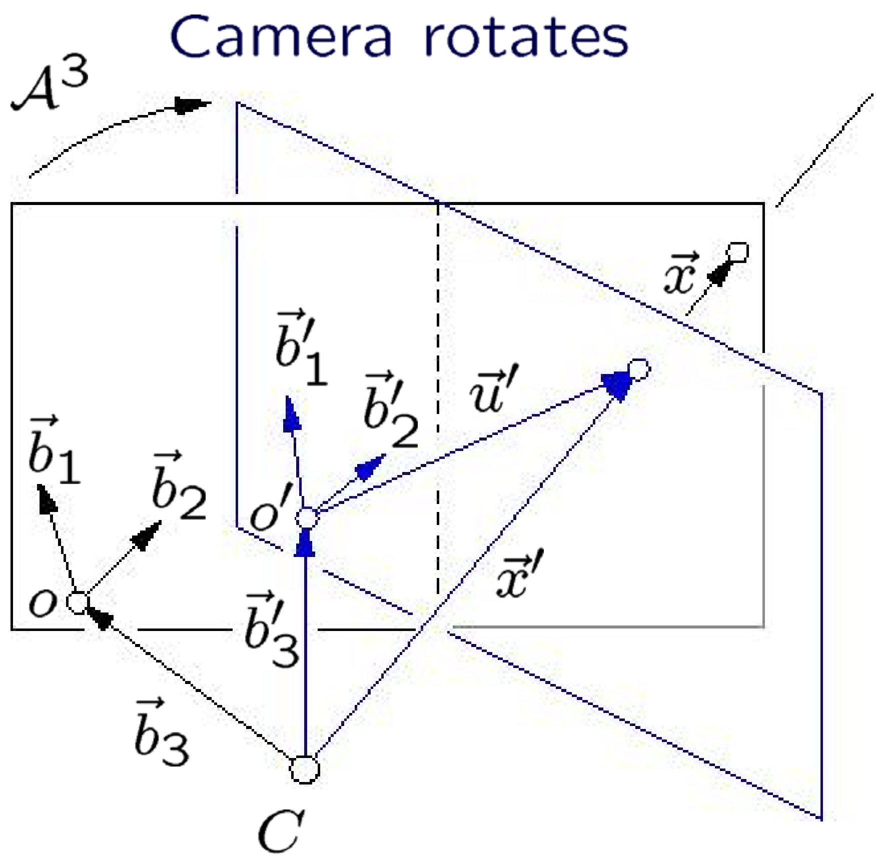
ROTATING CAMERA



Two coordinate systems with the same origin C

$$\beta = (\vec{b}_1, \vec{b}_2, \vec{b}_3)$$
$$S = (C, \beta)$$

$$\beta' = (\vec{b}'_1, \vec{b}'_2, \vec{b}'_3)$$
$$S' = (C, \beta')$$

Direction vectors of a ray

$\exists \alpha \in \mathbb{R}: \alpha \vec{x}' = \vec{x}$

We measure

$$\vec{u} = u\,\vec{b}_1 + v\,\vec{b}_2 \sim \mathbf{u}_{(\vec{b}_1,\vec{b}_2)} = \begin{pmatrix} u \\ v \end{pmatrix}$$

$$\vec{u}' = u'\,\vec{b}_1 + v'\,\vec{b}'_2 \sim \mathbf{u}'_{(\vec{b}'_1,\vec{b}'_2)} = \begin{pmatrix} u' \\ v' \end{pmatrix}$$

Triangle equality

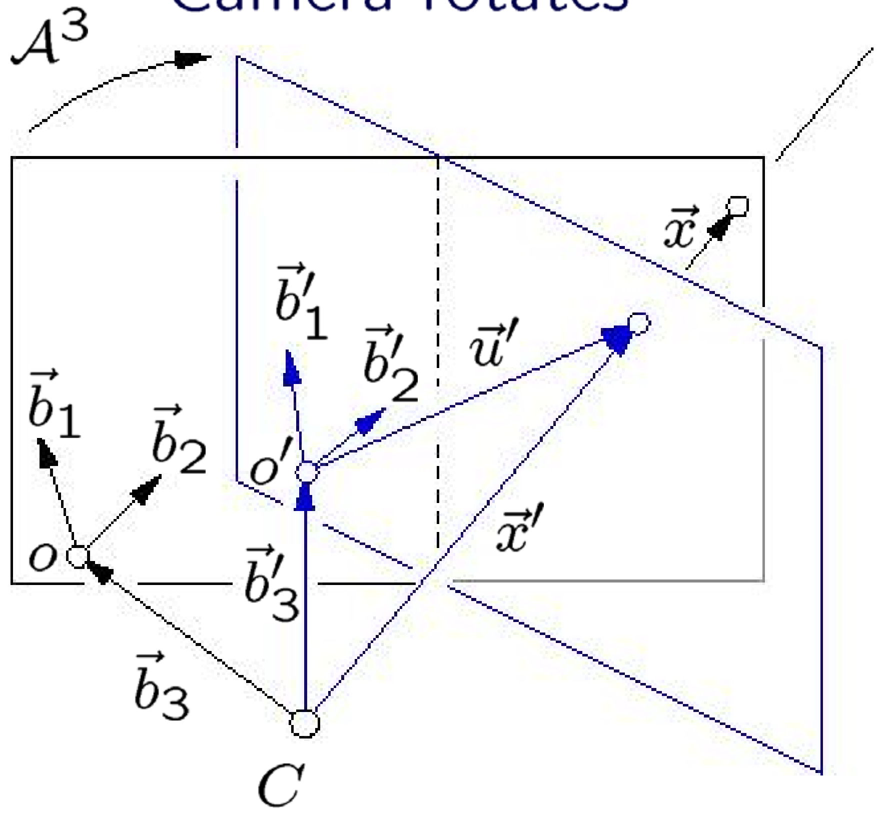
$$\vec{x} = \vec{u} + \vec{b}_3$$

$$\vec{x}' = \vec{u}' + \vec{b}'_3$$

$$\vec{x} = u\,\vec{b}_1 + v\,\vec{b}_2 + 1\,\vec{b}_3 \sim \mathbf{x}_\beta = \begin{pmatrix} u \\ v \\ 1 \end{pmatrix}$$

$$\vec{x}' = u'\,\vec{b}_1 + v'\,\vec{b}'_2 + 1\,\vec{b}'_3 \sim \mathbf{x}'_{\beta'} = \begin{pmatrix} u' \\ v' \\ 1 \end{pmatrix}$$

Camera rotates



$\exists$ matrix $\mathtt{H} \in \mathbb{R}^{3\times 3}$, rank $\mathtt{H} = 3$, so that coordinates of the vector $\vec{x}$ in β, β'

$$\mathbf{x}_{\beta'} = \mathtt{H}\,\mathbf{x}_\beta$$

Coordinates of the vector $\vec{x}$, $\vec{x}'$ in β':

$$\alpha\,\mathbf{x}'_{\beta'} = \mathbf{x}_{\beta'}$$

and therefore

$$\alpha\,\mathbf{x}'_{\beta'} = \mathtt{H}\,\mathbf{x}_\beta$$

Columns of $\mathtt{H}$ are coordinates of basic vectors of β, $\vec{b}_1, \vec{b}_2, \vec{b}_3$, in basis β'

$$\mathtt{H} = \begin{pmatrix} | & | & | \\ \mathbf{b}_{1\beta'} & \mathbf{b}_{2\beta'} & \mathbf{b}_{3\beta'} \\ | & | & | \end{pmatrix}$$

Wrapping up:

$\exists\,\mathtt{H} \in \mathbb{R}^{3\times 3}$, rank $\mathtt{H} = 3$, so that

$\forall\,(u,v) \overset{corr}{\longleftrightarrow} (u',v')\ \exists \alpha \in \mathbb{R}:$

$$\alpha \begin{pmatrix} u' \\ v' \\ 1 \end{pmatrix} = \mathtt{H} \begin{pmatrix} u \\ v \\ 1 \end{pmatrix}$$

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COMPUTING THE HOMOGRAPHY
from 4 correspondences

Computing the homography

$\exists\, \mathtt{H} \in \mathbb{R}^{3\times 3}$, rank $\mathtt{H} = 3$, so that $\forall\, (u,v) \overset{corr}{\leftrightarrow} (u',v')\ \exists \alpha \in \mathbb{R}$:

$$\alpha \begin{pmatrix} u' \\ v' \\ 1 \end{pmatrix} = \mathtt{H} \begin{pmatrix} u \\ v \\ 1 \end{pmatrix}$$

Introduce symbols for rows of homography $\mathtt{H}$

$$\mathtt{H} = \begin{pmatrix} \mathbf{h}_1^\top \\ \mathbf{h}_2^\top \\ \mathbf{h}_3^\top \end{pmatrix}$$

and rewrite the above matrix equation as

$$\begin{aligned} \alpha\, u' &= \mathbf{h}_1^\top \mathbf{x} \\ \alpha\, v' &= \mathbf{h}_2^\top \mathbf{x} \\ \alpha &= \mathbf{h}_3^\top \mathbf{x} \end{aligned}$$

Computing the homography

Eliminate α from the first two equations using the third one

$$\begin{aligned}(\mathbf{h}_3^\top \mathbf{x})\, u' &= \mathbf{h}_1^\top \mathbf{x} \\ (\mathbf{h}_3^\top \mathbf{x})\, v' &= \mathbf{h}_2^\top \mathbf{x}\end{aligned}$$

move all to the left hand side and reshape it using $\mathbf{x}^\top \mathbf{y} = \mathbf{y}^\top \mathbf{x}$

$$\begin{aligned}\mathbf{x}^\top \mathbf{h}_1 - (u' \mathbf{x}^\top)\, \mathbf{h}_3 &= 0 \\ \mathbf{x}^\top \mathbf{h}_2 - (v' \mathbf{x}^\top)\, \mathbf{h}_3 &= 0\end{aligned}$$

Introduce notation

$$\mathbf{h} = \begin{pmatrix} \mathbf{h}_1^\top & \mathbf{h}_2^\top & \mathbf{h}_3^\top \end{pmatrix}^\top$$

and express the above two equations in a matrix form

$$\begin{pmatrix} u & v & 1 & 0 & 0 & 0 & -u'u & -u'v & -u' \\ 0 & 0 & 0 & u & v & 1 & -v'u & -v'v & -v' \end{pmatrix} \mathbf{h} = 0$$

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Computing the homography

Every correspondence $\overset{corr}{\leftrightarrow} (u', v')$ brings two rows to a matrix

$$\underbrace{\begin{pmatrix} u & v & 1 & 0 & 0 & 0 & -u'u & -u'v & -u' \\ 0 & 0 & 0 & u & v & 1 & -v'u & -v'v & -v' \\ & & & & \vdots & & & & \end{pmatrix}}_{\mathtt{A}} \mathbf{h} = 0$$

$$\mathtt{A} \qquad \mathbf{h} = 0$$

If $\mathtt{G} = \lambda \mathtt{H}$, $\lambda \neq 0$ then they both determine the same homography since

$$\exists \alpha : \alpha \mathbf{y} = \mathtt{G}\mathbf{x} \Rightarrow \exists \beta : \beta \mathbf{y} = \mathtt{H}\mathbf{x}$$

where $\beta = \frac{\alpha}{\lambda}$

We are therefore looking for one-dimensional subspaces of 3×3 matrices of $rank$ 3. Each such subspace determines one homography. Also note that the zero matrix, $\mathtt{0}$, does not represent an interesting mapping.

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Computing the homography

We need therefore at least 4 correspondences in general position to obtain 8 rows in

$$\begin{pmatrix} u & v & 1 & 0 & 0 & 0 & -u'u & -u'v & -u' \\ 0 & 0 & 0 & u & v & 1 & -v'u & -v'v & -v' \\ & & & & \vdots & & & & \end{pmatrix} \mathbf{h} = 0$$

$$\mathtt{A} \quad \mathbf{h} = 0$$

By general position we mean that the matrix $\mathtt{A}$ must have $rank$ 8 to provide a single one-dimensional subspace of its solutions.

The general positions, i.e. $rank\,\mathtt{A} = 8$, are those when no 3 out of the 4 points are on the same line.

Computing the homography

Notice that $\mathtt{A}$ can be written in the form

$$\mathtt{A} = \begin{pmatrix} u_1 & v_1 & 1 & 0 & 0 & 0 & -u'_1 u_1 & -u'_1 v_1 & -u'_1 \\ u_2 & v_2 & 1 & 0 & 0 & 0 & -u'_2 u_2 & -u'_2 v_2 & -u'_2 \\ & & & & \vdots & & & & \\ 0 & 0 & 0 & u_1 & v_1 & 1 & -v'_1 u_1 & -v'_1 v_1 & -v'_1 \\ 0 & 0 & 0 & u_2 & v_2 & 1 & -v'_2 u_2 & -v'_2 v_2 & -v'_2 \\ & & & & \vdots & & & & \end{pmatrix}$$

which can be rewritten more concisely as

$$\mathtt{A} = \begin{pmatrix} \mathbf{x}_1^\top & \mathbf{0} & -u'_1\, \mathbf{x}_1^\top \\ \mathbf{x}_2^\top & \mathbf{0} & -u'_2\, \mathbf{x}_2^\top \\ & \vdots & \\ \mathbf{0} & \mathbf{x}_1^\top & -v'_1\, \mathbf{x}_1^\top \\ \mathbf{0} & \mathbf{x}_2^\top & -v'_2\, \mathbf{x}_2^\top \\ & \vdots & \end{pmatrix}$$

Computing the homography from 4 points on 2 lines in Matlab

```
% 4 points

>>x = [0 0 1;1 0 1;0 1 1;1 1 1]';
>>y = [1 1 1;1 0 1;0 1 1;0 0 1]';

% the 2-line algorithm

>>A       = [[x' zeros(size(x')) [-y(1,:)'*ones(1,3)].*(x')]
             [zeros(size(x')) x' [-y(2,:)'*ones(1,3)].*(x')]];
>>H       = reshape(null(A),3,3)';

% verification

>> e = y - (H*x)./[[1;1;1]*(H(3,:)*x)]

e =

  1.0e-015 *

         0    0.0481   -0.2220   -0.4441
         0    0.2220    0.0961   -0.2220
         0         0         0         0
```