Czech Pattern Recognition Workshop 2000, Tomáš Svoboda (Ed.)
Peršlák, Czech Republic, February 2–4, 2000
Czech Pattern Recognition Society

Relinking of Graph Pyramids by Means of a New Representation

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Abstract *In this paper we propose a new method to relink graph pyramids by local relinking operations. The relinking of graph pyramids by modifying the father-son links requires connectivity checks for the modified receptive fields. Some of these checks may become very complex, if the relinking is to be performed in parallel. Our method avoids these checks. By representing graph pyramids as bases of valuated matroids, the goal of the relinking is expressed by a valuation on the corresponding matroid. This valuation guides the local relinking operations. The valuation attains its maximal value if none of the local relinking operations yields higher values. The new method is used for an adaption of graph pyramids towards having a given receptive field.*

1 Introduction

To perceive an image is to transform it [Ser82]. In order to allow a clear distinction between transformations of image structure and transformations of image contents, we first represent the image as an attributed graph forming the base level of a graph pyramid. A common way to construct the base level graph is to create a vertex for each pixel and to let the edges represent the 4-connectivity of the pixel array. The attributes of the vertices, edges and faces are derived from the gray values or colors of the pixels. The other levels of the pyramid are formed by subsequent dual graph contractions [Kro95a] controlled by application defined models. A local function, the so called *reduction function*, derives the attributes of the current level from the level below. In all levels the attributes represent the image contents, while the structure of the image is given by the graph without the attributes. The graphs on the higher levels of the pyramid yield more and more abstract descriptions of the underlying image. However, the construction of the graph pyramid should not be restricted to a bottom-up procedure. The alternatives as given by a model usually induce constraints on neighborhoods in the graph pyramid. Holding to the separation of structure and contents we extend the influence of the model by allowing

1. relinking of the pyramid without adjusting the contents,
2. contents adjustments, classification without relinking.

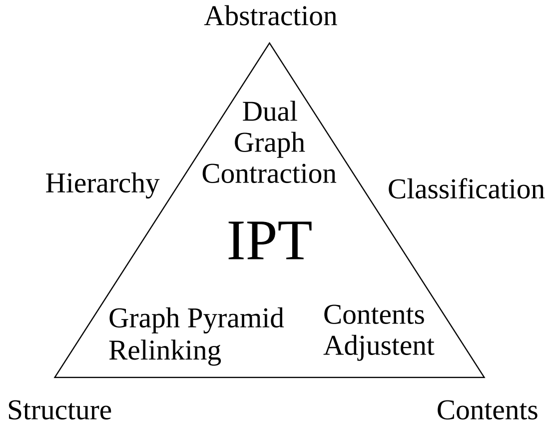


Figure 1: Iterative parallel transformations on graph pyramids.

These transformations are also utilized to increase the robustness of the pyramid. This paper is devoted to efficiently perform the relinking by iterated parallel transformations (IPT) [Sch97]. A variable linking of regular pyramids was first described in [BHR81]. An extension to irregular hierarchies of graphs is shown in [Nac95]. IPT for contents adjustment and classification, i.e. *relaxation*, has been applied to hierarchies of graphs in [WH96]. Since dual graph contraction is an IPT towards abstraction, the IPT considered so far can be organized in the triangle depicted in Fig. 1.

The paper is organized as follows. Section 2 is devoted to the construction of graph pyramids by dual graph contraction. Section 3 points out the necessity of connectivity checks when modifying the father-son links of a graph pyramid. In Section 4 we demonstrate how to code the construction of graph pyramids in the base level of the pyramid. As shown in Section 5, this coding gives rise to a new representation of graph pyramids in terms of matroid bases. By means of the new representation we arrive at a definition of local relinking operations on graph pyramids which do not require connectivity checks. Section 6 introduces valuations on matroids. The valuations are utilized to guide the local relinking operations. In Section 7 we apply the relinking to the adaption of graph pyramids towards having a given receptive field. We conclude in Section 8.

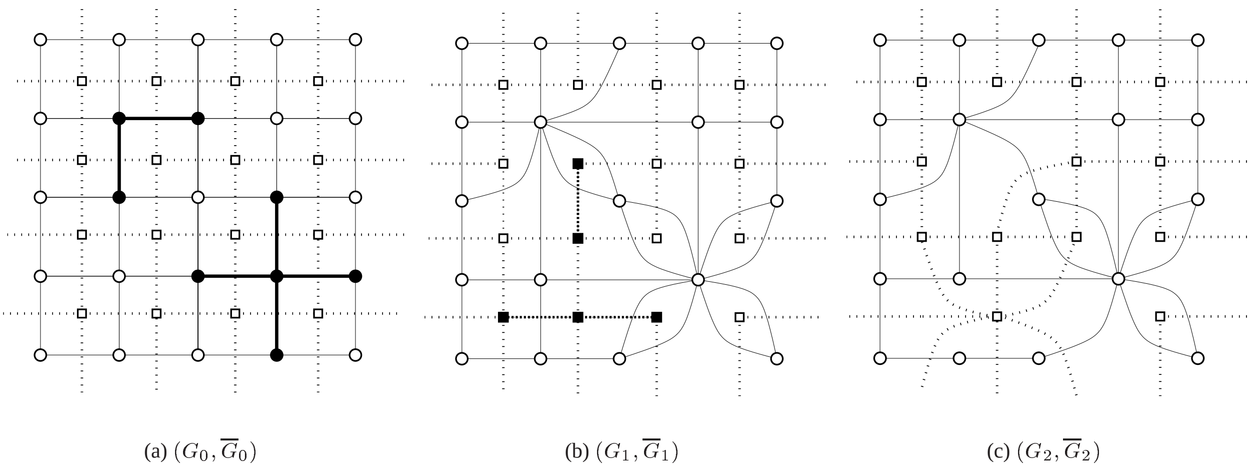


Figure 2: Dual graph contraction.

2 Dual Graph Contraction

The construction of graph pyramids by *dual graph contraction* (see Fig. 2) is described in [Kro95a]. Let $G_0 = (V_0, E_0)$ and $\overline{G_0} = (\overline{V_0}, \overline{E_0})$ denote a pair of plane graphs, where $\overline{G_0}$ is the dual of G_0. Dual graph contraction consists of two steps: *dual edge contraction* and *dual face contraction*. The dual edge contraction is specified by a spanning forest F_0 of E_0, the trees of which are referred to as *contraction kernels*. In Fig. 2a the non-trivial contraction kernels of G_0 are emphasized. Each contraction kernel T_0 of F_0 is contracted to one vertex v_1 of the graph $G_1 = (V_1, E_1)$ on the next level of the graph pyramid. For each vertex v_0 of T_0 the vertex v_1 is called *father* of v_0 and v_0 is called the *son* of v_1. Each edge of E_1 corresponds to exactly one edge in E_0, which does not belong to a contraction kernel. Let $\overline{F_0}$ denote the set of edges in $\overline{E_0}$, which are dual to the edges in F_0. Set $\overline{E_1} := \overline{E_0} \setminus \overline{F_0}$ and $\overline{G_1} := (\overline{V_0}, \overline{E_1})$. Note that G_1 and $\overline{G_1}$ form a dual pair of plane graphs.

The second step, called *dual face contraction*, is specified by contraction kernels $\overline{F_1}$ in $\overline{G_1}$. In Fig. 2b the contraction kernels of $\overline{G_1}$ are emphasized. Analogous to dual edge contraction, we generate $\overline{G_2}$ and set $G_2 := (V_1, E_2)$ with $E_2 := E_1 \setminus F_1$. Each vertex in G_2 has exactly one son in G_1, i.e. the vertex itself. The graphs G_2 and $\overline{G_2}$ form another dual pair of plane graphs. In [Kro95a] the role of dual face contraction is confined to the removal of faces bounded by less than three edges. In the following we will drop this restriction in order to apply the theory of matroids in a general way. Subsequent parallel edge [face] contraction steps may be summarized by a single edge [face] contraction step.

Each vertex in the graph pyramid represents a connected set of base level vertices, the so called *receptive field*. The receptive field of a base level vertex contains exactly the vertex itself. For each vertex v_k on the level $k \geq 1$ the *receptive field* $RF(v_k)$ is defined by all vertices in the base level of the pyramid which lead to v_k by climbing the pyramid from sons to fathers:

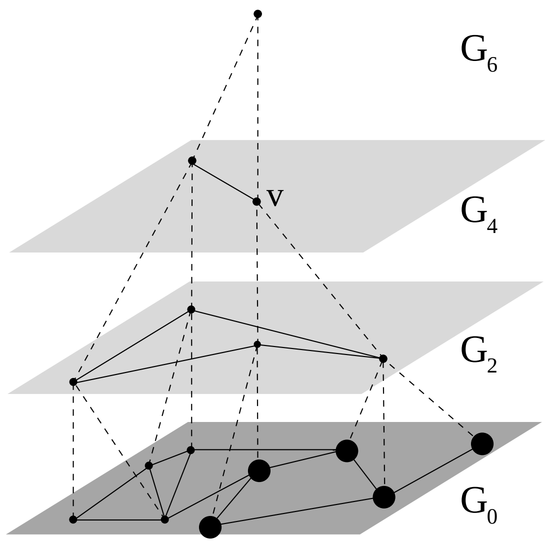


Figure 3: The vertices forming the receptive field of v are enlarged.

$$RF(v_0) = \{v_0\} \text{ for } v_0 \in V_0,$$
$$RF(v_k) = \bigcup(RF(v_{k-1}) \mid v_{k-1} \text{ is son of } v_k), \quad k > 0.$$

In Fig. 3 the odd levels are omitted. Note that the receptive fields in the graph pyramid do not overlap, since all vertices (except the apex) have exactly one father.

3 Modifying Father-Son Links

How can a graph pyramid be rebuilt by local relinking operations? The father-son links can all be represented between subsequent even levels of the graph pyramid (Fig. 3). In [Nac95] the relinking was performed by assigning new parents to sons. The new father must always be adjacent to the old father in the level of the fathers. This condition, however, is not sufficient for the new father-son links to represent a valid graph pyramid. Figs. 4a and 4b show that

- the sons of the new father are not necessarily connected in the level of the sons and that
- the remaining sons of the old father may be disrupted by the assignment of one son to a new father.

The conditions for a relinking operation to yield one of the above cases are checked easily. This checking can also be combined with a parallel application of the relinking operations [Nac95].

More severe problems occur, when levels higher than the level of the fathers are present in the graph pyramid. Fig. 5 illustrates that the receptive fields of vertices above the level of the fathers may also be disrupted. Checking whether a relinking operation causes this kind of disruption involves multiple pyramid levels. Furthermore, complex algorithms are required to check, whether multiple relinking operations may be performed in parallel. This is due to the fact, that the compatibility of relinking operations cannot be described locally [Nac95].

4 A new Coding for the Construction of Graph Pyramids

In the following we will assign labels to the edges in the base level of a graph pyramid, which specify the construction of the graph pyramid.

Let G_0 and $\overline{G}_0$ denote a pair of plane graphs and assume $\mathcal{P} = (G_0, G_1, \ldots, G_{2n})$ and $\overline{\mathcal{P}} = (\overline{G}_0, \overline{G}_1, \ldots, \overline{G}_{2n})$ to be graph pyramids constructed on top of the pair $(G_0, \overline{G_0})$ by dual graph contractions. We also assume that the apex G_{2n} is a graph with one vertex and zero edges. Let $G_i = (V_i, E_i)$ for all $0 \leq i \leq 2n$. The domain of all graph pyramids with the above properties is denoted by $\mathcal{D}(G_0, 2n)$. For each edge $e \in E_0$ let $l(e)$ denote the maximal level of $\mathcal{P}$ which contains e, i.e.

$$l(e) := max\{j \mid e \in E_j \setminus E_{j+1}\}. \quad (1)$$

The construction of the graph pyramid is determined by the above assignment of labels from $L := \{0, 1, 2, \ldots\}$ to the edges in E_0 (similar to [Kro95b]). The assignments are expressed by subsets of $E_0 \times L$. Let B denote a subset of $E_0 \times L$. We set

$$E^0(B) := \{e \in E_0 \mid \exists j \text{ with } (e, j) \in B \text{ and } j \equiv 0 \bmod 2\}. \quad (2)$$

If $B = \{(e, l(e)) \mid e \in E_0\}$, where $l(\cdot)$ from (1) refers to the construction of a graph pyramid, then B is called *buildup plan*. For a buildup plan B the following holds:

B1 $\forall e \in E_0\ \exists 1\ j \in L$ with $(e, j) \in B$ and

B2 $E^0(B)$ forms a spanning tree of G_0.

Conversely, let $B \subset E_0 \times L$. If B fulfills conditions **B1** and **B2**, then it is a buildup plan. This follows from the fact, that $E^0(B)$ forms a spanning tree in E_0 if and only if

$$E^1(B) := \{e \in E_0 \mid \exists j \text{ with } (e, j) \in B \text{ and } j \equiv 1 \bmod 2\} \quad (3)$$

forms a maximal edge set in E_0, whose removal does not destroy the connectivity of G_0. Hence, the edges in $\overline{G_0}$, which are dual to $E^1(B)$ form a spanning tree $\overline{E^1(B)}$ in $\overline{G_0}$ [TS92]. The contraction kernels for the dual edge contraction and the dual face contraction may be derived from the sets

$$\{(b, j) \in B \mid b \in E^0(B)\}$$

and

$$\{(b, j) \in B \mid b \in E^1(B)\}$$

respectively.

5 Representation of Graph Pyramids as Bases of Matroids

Let $\mathcal{B}$ denote the collection of all buildup plans for pyramids in $\mathcal{D}(G_0, 2n)$. Note that the elements of $\mathcal{B}$ are subsets of $E_0 \times L$. We require the edge set E_0 to be non-empty. Hence $\mathcal{B}$ is non-empty. The following theorem states an exchange property for sets in $\mathcal{B}$.

Theorem 5.1 *Let $B, B' \in \mathcal{B}$. For each $b \in B \setminus B'$ there exists $b' \in B' \setminus B$ such that $B \setminus \{b\} \cup \{b'\} \in \mathcal{B}$.*

Proof: It suffices to show that $E^0(B \setminus \{b\} \cup \{b'\})$ forms a spanning tree of G_0 or that $E^1(B \setminus \{b\} \cup \{b'\})$ forms a maximal set of edges from E_0, whose removal does not of destroy the connectivity of G_0.

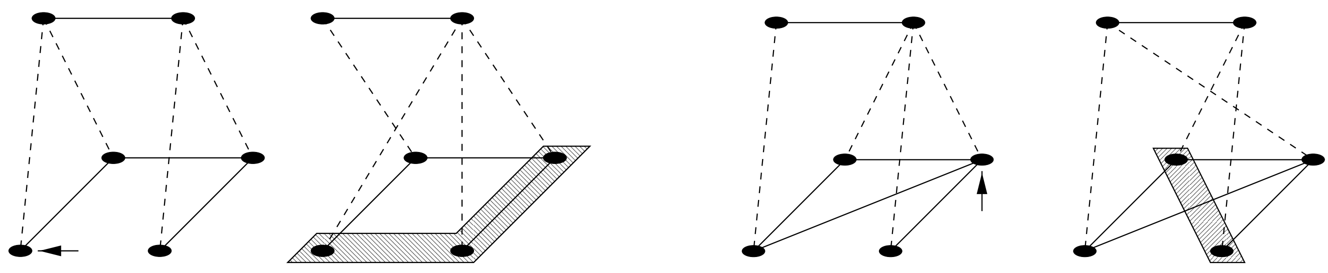
• Case $b = (e, l)$ with $l > 0$: In the unique cycle of $E^0(B') \cup \{e\}$ there exists $e' \notin E^0(B)$ (since $E^0(B)$ contains no cycles). Let $l' \in L$ denote the unique number with $(e', l') \in B'$ and set $b' := (e', l')$. Since $e' \notin E^0(B)$, it follows that $e' \neq e$. This implies $b' \neq b$ and (because of $e' \in E^0(B') \cup \{e\}$) $e' \in E^0(B')$, i.e. $l' > 0$. Since e and e' belong to the same cycle of $E^0(B') \cup \{e\}$ and have positive labels, it follows that $E^0(B \setminus \{b\} \cup \{b'\})$ forms a spanning tree of E_0.

• Case $b = (e, l)$ with $l < 0$: Removing the set $E^1(B') \cup \{e\}$ from E_0 would destroy the connectivity of G_0. However, the removal of $E^1(B)$ does not destroy the connectivity of G_0. Hence, there exists $e' \in E^1(B') \cup \{e\}, e' \notin E^1(B)$. Let $l' \in L$ denote the unique number with $(e', l') \in B'$ and set $b' := (e', l')$. Since $e' \notin E^1(B)$, it follows that $e' \neq e$. This implies $b' \neq b$ and (because of $e' \in E^1(B') \cup \{e\}$) $e' \in E^1(B')$, i.e. $l' < 0$. Since $e, e' \in E^1(B) \cup \{e'\}$ and $E^1(B) \cup \{e'\}$ is a minimal set of edges, whose removal destroys the connectivity of G_0, it follows that $E^1(B \setminus \{b\} \cup \{b'\})$ is a maximal set of edges from E_0, whose removal does not destroy the connectivity of G_0. □

Definition 5.2 *For $B \in \mathcal{B}, b \in B, b' \notin B$ the mapping modif$(B, b, b') := B \setminus \{b\} \cup \{b'\}$ is called local modification of B, if modif$(B, b, b') \in \mathcal{B}$.*

The sets in $\mathcal{B}$ determine an infinite *matroid* $\mathcal{M} := (E_0 \times L, \mathcal{I})$ on $E_0 \times L$, where

$$\mathcal{I} := \{I \subset B \mid B \in \mathcal{B}\} \quad (4)$$



(a) Sons of new father disconnected.

(b) Sons of old father disconnected.

Figure 4: Two cases of forbidden relinking (adapted from [Nac95]).

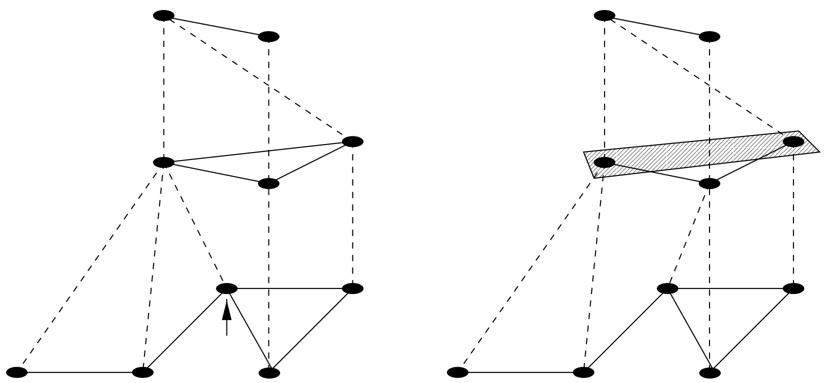


Figure 5: Relinking yields loss of connectivity in a higher pyramid level (adapted from [Nac95]).

[Oxl92]. The sets in $\mathcal{I}$ are referred to as *independent sets* and the elements of $\mathcal{B}$ are referred to as *bases*. Bases are maximal independent sets as in the theory of vector spaces, where *independent* means *linearly independent*. Since a matroid is completely determined by its bases, we may write $\mathcal{M} = \mathcal{M}(\mathcal{B})$. In [Bru69] the exchange property of Theorem 5.1 is extended:

Theorem 5.3 *Let $\mathcal{B}$ denote the collection of bases of a matroid and let $B, B' \in \mathcal{B}$. For each $b \in B \setminus B'$ there exists $b' \in B' \setminus B$ such that*
• $B \setminus \{b\} \cup \{b'\} \in \mathcal{B}$ *and*
• $B' \setminus \{b'\} \cup \{b\} \in \mathcal{B}$.

Theorem 5.3 implies that any $B \in \mathcal{B}$ can be adapted to any other $B' \in \mathcal{B}$ by local modifications only. The local modifications can always be chosen such that an element $b_1 \in B_1 \setminus B_2$ is exchanged by an element $b_2 \in B_2 \setminus B_1$.

Note that the number $\mid \{B_2 \setminus B_1\} \mid$ defines an edit distance between matroid bases and thus between graph pyramids with the same base level. Unlike [Bun99], the computation of the edit distance is enormously facilitated by a one-to-one correspondence between the edges in one base level and the edges in the other base level.

If the construction of $\mathcal{P}$ is determined by a matroid base B, each local modification of B induces an operation on $\mathcal{P}$. We define:

Definition 5.4 *An operation on a graph pyramid $\mathcal{P}$ is called local relinking operation, if it is induced by a local modification on a matroid base that describes the construction of $\mathcal{P}$.*

6 Valuated Matroids

In order to utilize local relinking operations for the adaption of a graph pyramid, the choice of the operations has to be determined by the goal of the adaption. We represent graph pyramids as bases of matroids and use a definition in [DW90], where R denotes, for example, the set of reals or the set of integers.

Definition 6.1 (Valuation on a Matroid) *A valuation on a matroid $\mathcal{M} = \mathcal{M}(\mathcal{B})$ is a function ω: $\mathcal{B} \to R$ which has the following exchange property. For $B, B' \in \mathcal{B}$ and $b \in B \setminus B'$ there exists $b' \in B' \setminus B$ such that*

- $B \setminus \{b\} \cup \{b'\} \in \mathcal{B}$,
- $B' \setminus \{b'\} \cup \{b\} \in \mathcal{B}$,
- $\omega(B) + \omega(B') \leq \omega(B \setminus \{b\} \cup \{b'\}) + \omega(B' \setminus \{b'\} \cup \{b\})$.

A matroid equipped with a valuation is called valuated matroid.

The following theorem [DW90] implies that valuations on matroids can be maximized by local modifications.

Theorem 6.2 *Let $B \in \mathcal{B}$ and let ω be a valuation on the matroid $\mathcal{M} = \mathcal{M}(\mathcal{B})$. Then $\omega(B)$ is maximal, if $\omega(B_m) \leq \omega(B)$ for all local modifications B_m of B.*

In order to utilize Theorem 6.2 for the adaption of graph pyramids by local relinking operations, we have to find a valuation on the corresponding matroid, which is maximal if and only if the goal of the adaption is reached. Then we apply a local relinking operation whenever it increases the valuation.

7 Adaption of Graph Pyramids

In this section we use valuated matroids to adapt a graph pyramid $\mathcal{P}$ towards having a receptive field equal to a given connected set T of vertices from the base level of $\mathcal{P}$. If there is no receptive field equal to T, we may still ask: How well does T fit into the pyramid $\mathcal{P}$? This question has a narrow metric and a wider structural aspect: If there exists a receptive field RF in $\mathcal{P}$ with a small distance (Hausdorff-distance for example) to T, we say that T fits well into $\mathcal{P}$. The wider structural aspect is the following: Can a good fit of T into $\mathcal{P}$ be achieved by only a few (including zero) local relinking operations on $\mathcal{P}$? This case is illustrated in Fig. 6b, where splitting off the receptive field G from RF yields T.

In the following, we will apply local relinking operations to the graph pyramid $\mathcal{P}$, such that one of its receptive fields becomes equal to T. In Fig. 7a and 5d the pyramid $\mathcal{P}$ and the adapted pyramid $\mathcal{P}'$ are illustrated by their receptive fields. The set T is given by the filled circles.

Since T is contained in the receptive field of the apex of $\mathcal{P}$, there exists a smallest receptive field of $\mathcal{P}$ which covers T completely. In particular, there exists a vertex v_T^{cov} in $\mathcal{P}$ such that $T \subset RF(v_T^{cov})$ and $T \not\subset RF(v)$ for all sons v of v_T^{cov}. If $T = RF(v_T^{cov})$ no adaption of $\mathcal{P}$ is needed. Otherwise structural modifications are needed only in the subpyramid of $\mathcal{P}$, whose apex is v_T^{cov}.

As explained in Section 5, we may describe the adaption of $\mathcal{P}$ by local modifications on the corresponding matroid base B. The set E_0 of edges in the base level of $\mathcal{P}$ is partitioned by the edge sets $E^0(B)$ and $E^1(B)$, as defined in (2) and (3). The edge sets $E^0(B)$ and $E^1(B)$, in turn, are partitioned with respect to T into three classes each. For $i \in \{0, 1\}$ we set:

- $E_1^i(B) := \{e = (u, v) \in E^i(B) \mid \{u, v\} \subset T\}$,
- $E_2^i(B) := \{e = (u, v) \in E^i(B) \mid \{u, v\} \cap T = \emptyset\}$,
- $E_3^i(B) := E^i(B) \setminus (E_1^i(B) \cup E_2^i(B))$.

Adapting $\mathcal{P}$ towards containing T as a receptive field, we focus on the following edges in $E^0(B)$:

Definition 7.1 *An edge $e = (w, z) \in E^0(B)$* **conflicts with** *T, if*

- *$e \in E_3^0(B)$ and*
- *one end point of e is contained in $RF(v_T^{cov}) \setminus T$.*

Theorem 7.2 *The graph pyramid $\mathcal{P}$ has no receptive field equal to $T \Leftrightarrow \mathcal{P}$ has edges conflicting with T.*

Proof of Theorem 7.2:
$\Rightarrow$: Assume that no receptive field of $\mathcal{P}$ equals T, i.e. $RF(v_T^{cov}) \supset T$ and $RF(v_T^{cov}) \neq T$. The set of all edges from $E^0(B)$ with both end vertices in $RF(v_T^{cov})$ forms a spanning tree of $RF(v_T^{cov})$ and thus contains an edge $e = (w, z)$ with $w \in T$ and $z \in RF(v_T^{cov}) \setminus T$. The edge e conflicts with T.
$\Leftarrow$: Let $e = (w, z)$ be an edge conflicting with T. Without loss of generality we assume $z \in RF(v_T^{cov}) \setminus T$. It follows that $RF(v_T^{cov}) \neq T$. If there was a receptive field in $\mathcal{P}$ equal to T, $RF(v_T^{cov})$ would equal T, a contradiction. □

7.1 Algorithm for the Adaption

The adaption of $\mathcal{P}$ towards containing T is done in three steps, all of which reduce the number of edges, which conflict with T:

1. The number of edges in $E_1^0(B)$ is increased without affecting edges in $E_2^0(B)$.
2. The number of edges in $E_2^0(B)$ is increased without affecting edges in $E_1^0(B)$.
3. The labels of the remaining edges conflicting with T are raised.

In order to perform the first two steps, we define valuations ω_1 and ω_2. The matroid base B is a subset of $E_0 \times L$. An element x of B can be written as $x = (e_x, l(e_x))$. For $e_x \notin E^0(B)$ let $C(B, e_x)$ denote the unique cycle in $E^0(B) \cup \{e_x\}$ and set

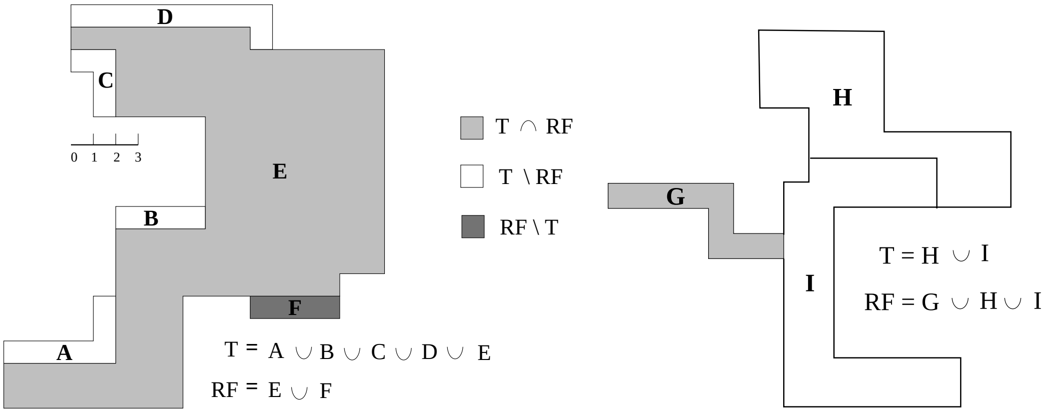
$$l_B(e_x) := max\{l(e) \mid e \in C(B, e_x), e \neq e_x\}. \quad (5)$$

Let $e_y \in C(B, e_x)$ with $e_y \neq e_x, l(e_y) = l_B(e_x)$. In [Kro95b] it is shown that the graph pyramid defined by $B \setminus \{(e_y, l(e_y))\} \cup \{(e_x, l_B(e_x))\}$ equals the graph pyramid defined by B. For $i \in \{1, 2\}$ we set $\omega_i(B) := \sum_{x \in B} val_i(x)$ with

$$val_i(x) := \begin{cases} 1 & : \; e_x \in E_i^1(B), l(e_x) = l_B(e_x) \\ 1 & : \; e_x \in E_1^0(B) \cup E_2^0(B) \\ -1 & : \; e_x \in E_i^0(B) \cup E_i^1(B), l(e_x) \neq l_B(e_x) \\ 0 & : \; \text{otherwise} \end{cases} \quad (6)$$

Consider the case $i = 1$ first. The value 1 is given for labeled edges, which we want to insert between vertices of T. The same value is given for labeled edges that we do not want to change anymore. The valuation $\omega_1(B)$ is maximal only if the edges in $E_1^0(B)$ form a spanning tree of T. In the case $i = 2$ the roles of T and the complement of T are reversed. Finally, the levels of the remaining edges conflicting with T are raised to the highest even label l_{acc} an edge between vertices of $RF(v_T^{cov})$ can have. These local relinking operations are guided by the valuation $\omega_3(B) := \sum_{x \in B} val_3(x)$ with

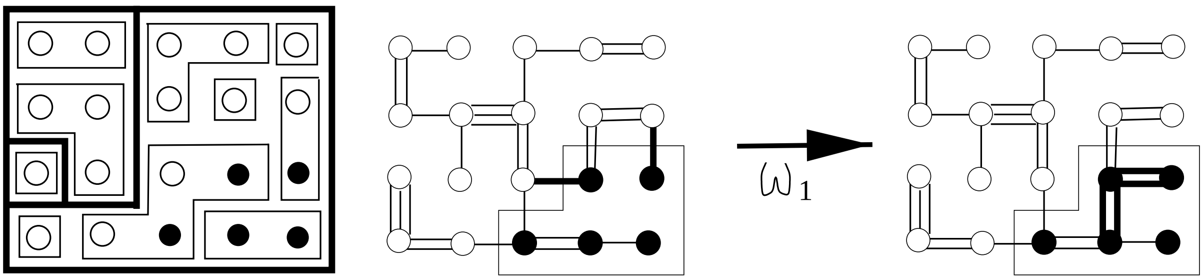
$$val_3(x) := \begin{cases} 1 & : \; e_x \in E_1^0(B) \cup E_2^0(B) \\ -1 & : \; e_x \in E_3^0(B), l(e_x) \neq l_{acc} \\ 0 & : \; \text{otherwise.} \end{cases} \quad (7)$$



(a) Two compact sets T and RF with Hausdorff distance 1.

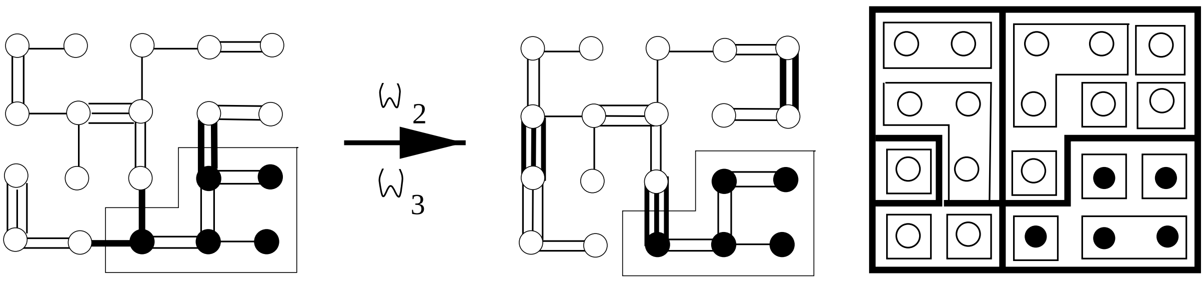
(b) Structural similarity of T and RF

Figure 6: Metric and structural comparison of receptive fields.



(a) $\mathcal{P}$

(b) Adaption guided by ω_1



(c) Adaption guided by ω_2 and ω_3

(d) $\mathcal{P}'$

Figure 7: Relinking towards a given receptive field. Modified edges are highlighted.

Note that each local modification (guided by ω_1, ω_2 or ω_3) reduces the number of edges conflicting with T by exactly one and raises the valuation by exactly one. The effect on the receptive fields can each time be described as detaching a part of $RF(v_T^{cov})$. These parts are fully determined by the edges conflicting with T.

7.2 Example

Fig. 7b shows that there are exactly two local modifications which raise the valuation ω_1. The total increase of ω_1 thus amounts to 2. Fig. 7c shows that ω_2 and ω_3 can be raised by 2 and 1 respectively. The comparison of Fig. 7a and Fig. 7d yields that none of the receptive fields completely contained in T or completely contained in the complement of T have been modified.

8 Conclusion

The new representation of graph pyramids by matroid bases allows to define a set of local modifications, such that

- each base can be transformed into each other base by local modifications,
- the set of bases is closed with respect to the local modifications,
- the local modifications have the potential to be applied in parallel,
- the local modifications may be guided by global objective functions.

Hence, the new representation allows to relink graph pyramids in an iterated parallel way. We suggest the new method for tracking and motion analysis. In connection with dual graph contraction and contents adjustment it is also suggested for graph based object recognition. Future work will focus on efficient algorithms for the parallel relinking of graph pyramids.

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