Stable Radial Distortion Calibration by Polynomial Matrix Inequalities Programming

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April 9, 2015


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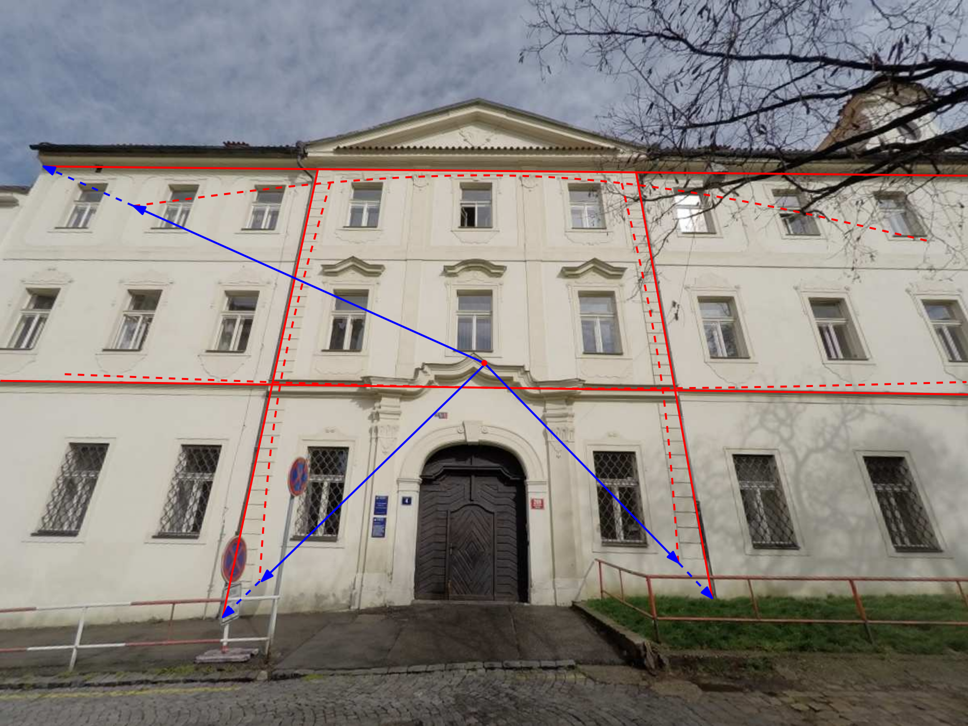
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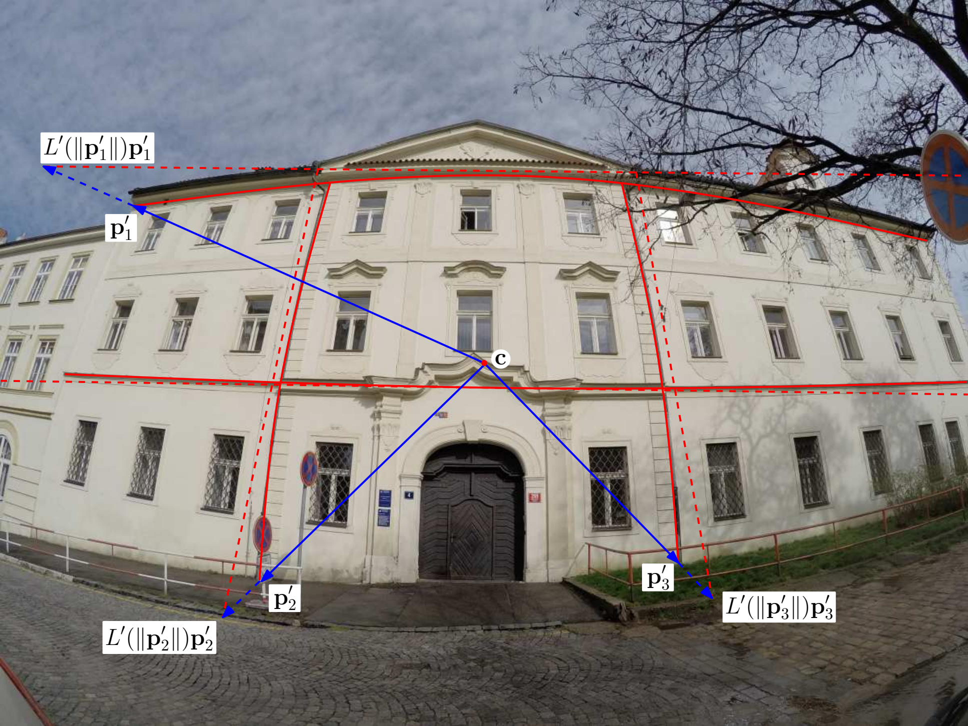
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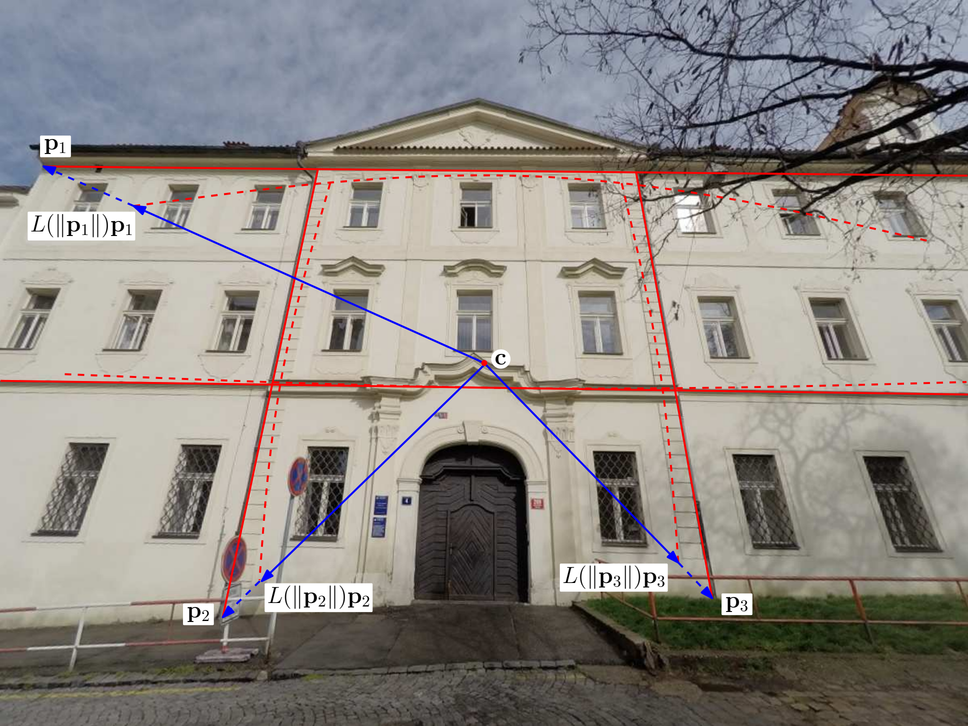
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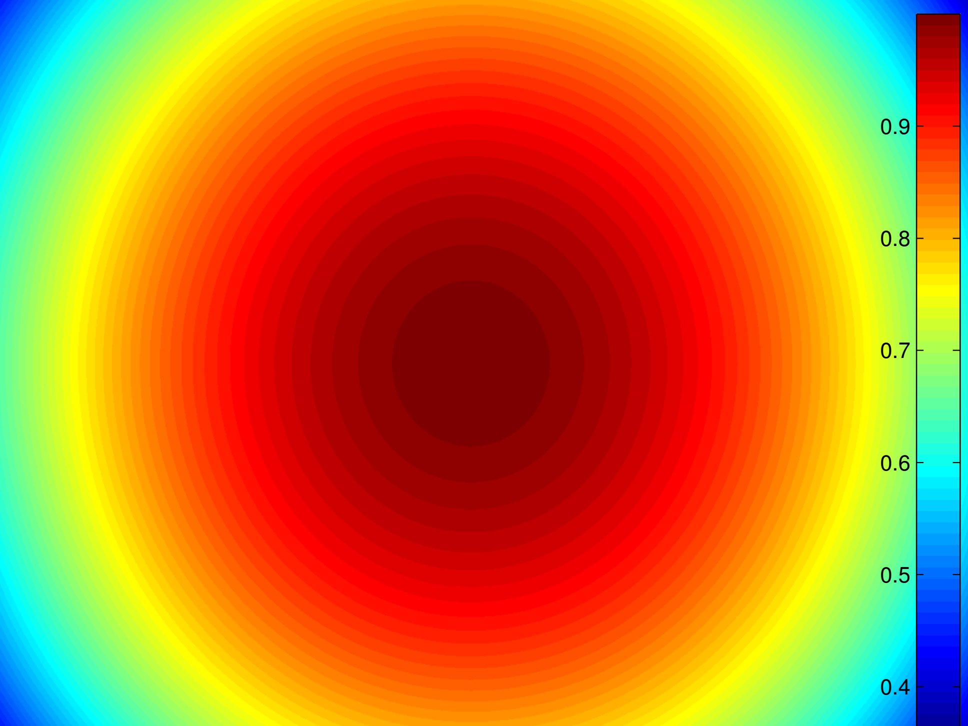
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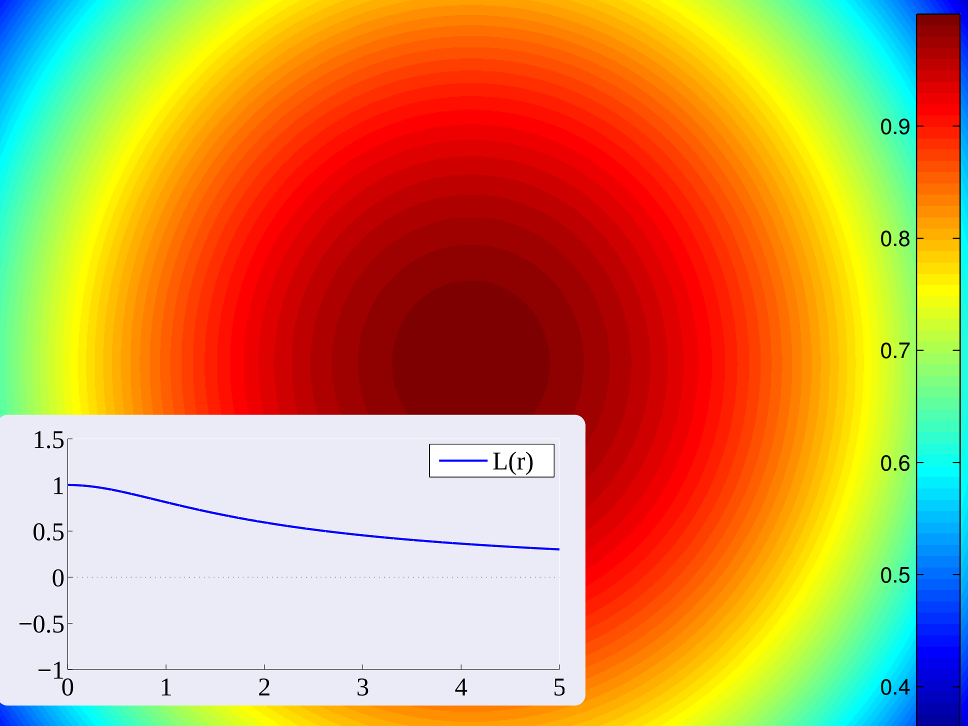

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L(r)

Properties of Radial Distortion

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$L'(r)$ as function of undistortion

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Radial distortion calibration

Radial distortion calibration is the estimation of $\mathbf{c}$ and L, assuming other camera properties stay the same.

Radial Distortion Calibration

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Distortion model $L(r)$

Polular choice: polynomial and rational functions (OpenCV, ...)

$$L(r, \mathbf{k}) = \frac{f(r, \mathbf{k})}{g(r, \mathbf{k})} = \frac{1 + k_1 r + k_2 r^2 + k_3 r^3}{1 + k_4 r + k_5 k^2 + k_6 r^3},\ \mathbf{k} \in \mathbb{R}^6.$$

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Radial distortion calibration as a part of camera calibration

$$\left.\begin{aligned} \mathbf{q}_{ij} &= \mathtt{R}_i \mathbf{X}_j + \mathbf{t}_i, \\ \mathbf{p}_{ij} &= (\mathbf{q}_{ij}^x, \mathbf{q}_{ij}^y)^\top / \mathbf{q}_{ij}^z, \\ \mathbf{p}'_{ij} &= L(\|\mathbf{p}_{ij}\|, \mathbf{k})\mathbf{p}_{ij}, \\ \mathbf{e}_{ij} &= (\mathbf{u}_{ij}, 1)^\top - \mathtt{K}(\mathbf{p}'_{ij}, 1)^\top. \end{aligned}\right\} \Rightarrow$$

$\min \mathcal{C}(\mathtt{K}, \mathtt{R}_i, \mathbf{t}_i, \mathbf{k}) = \sum_{i,j} \|\mathbf{e}_{ij}\|^2$

1. Initial parameter estimation
2. Local optimization (L-M)

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Radial distortion calibration as a part of homography estimation

$$\left.\begin{aligned} \mathbf{q}_i &= \mathtt{H}\mathbf{x}_i, \\ \mathbf{p}_i &= (\mathbf{q}_i^x, \mathbf{q}_i^y)^\top / \mathbf{q}_i^z, \\ \mathbf{p}'_i &= L(\|\mathbf{p}_i\|, \mathbf{k})\mathbf{p}_i, \\ \mathbf{e}_i &= \mathbf{u}_i - (\mathbf{p}'_i + \mathbf{c}). \end{aligned}\right\} \Rightarrow$$

$\min \mathcal{H}(\mathtt{H}, \mathbf{c}, \mathbf{k}) = \sum_i \|\mathbf{e}_i\|^2$

1. Initial parameter estimation
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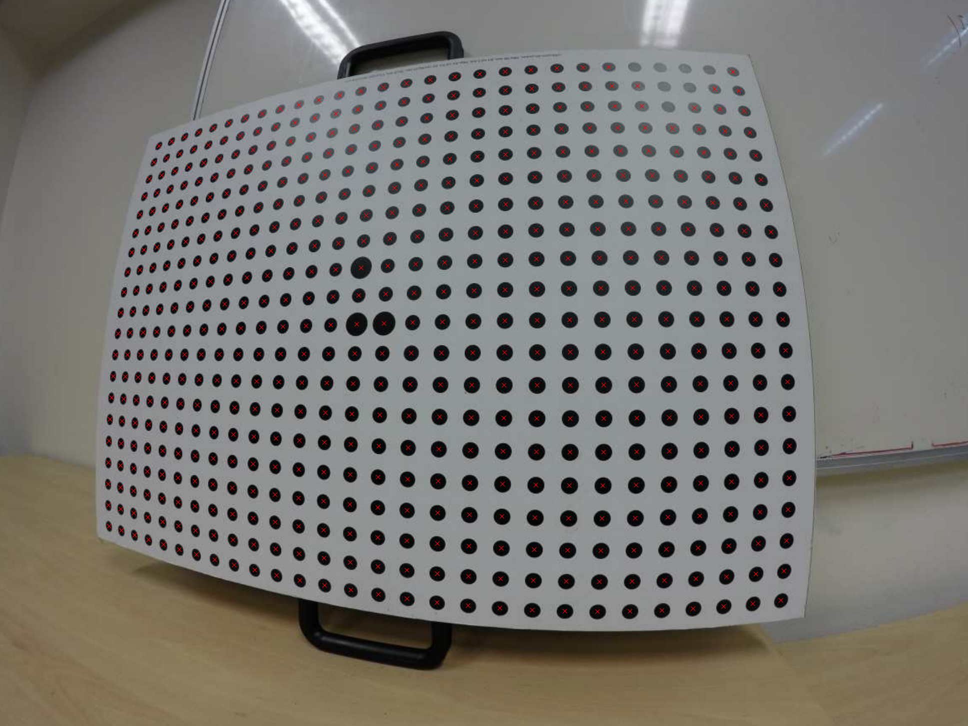
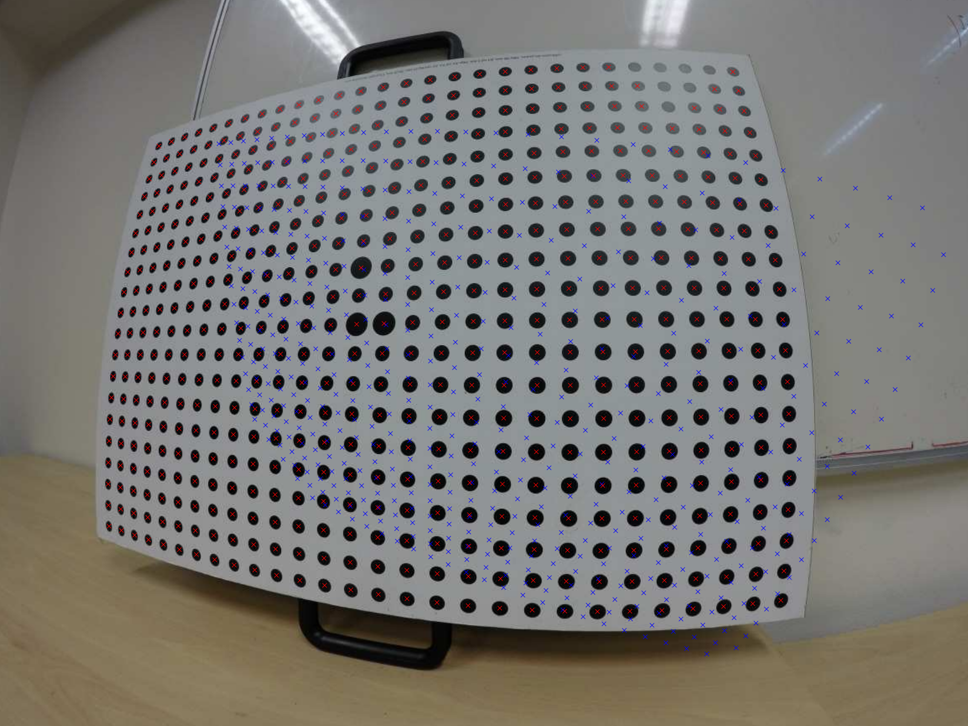
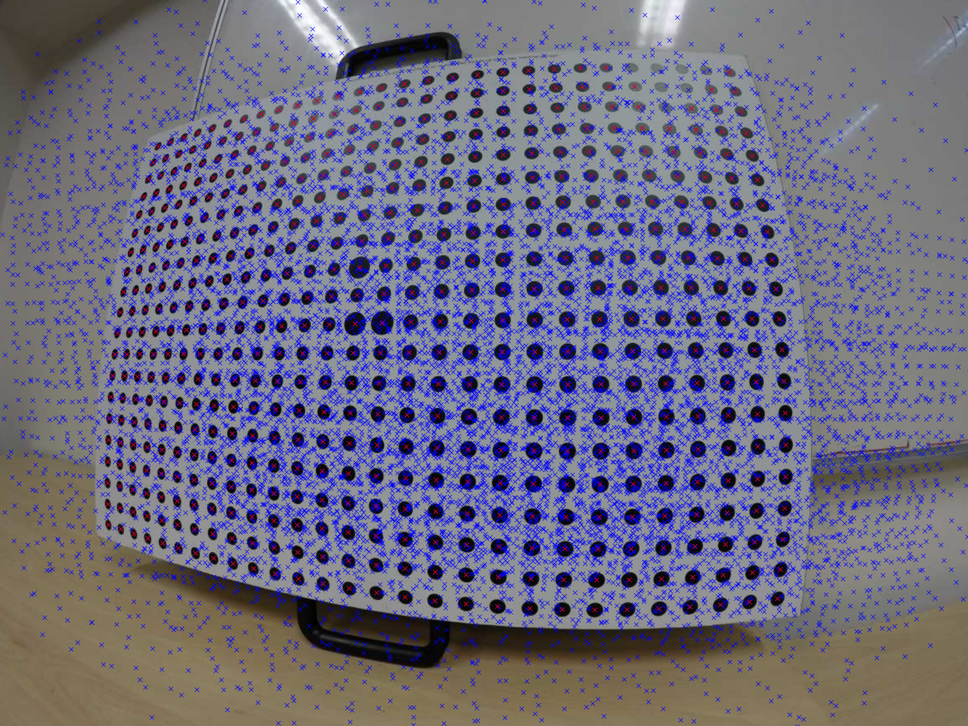
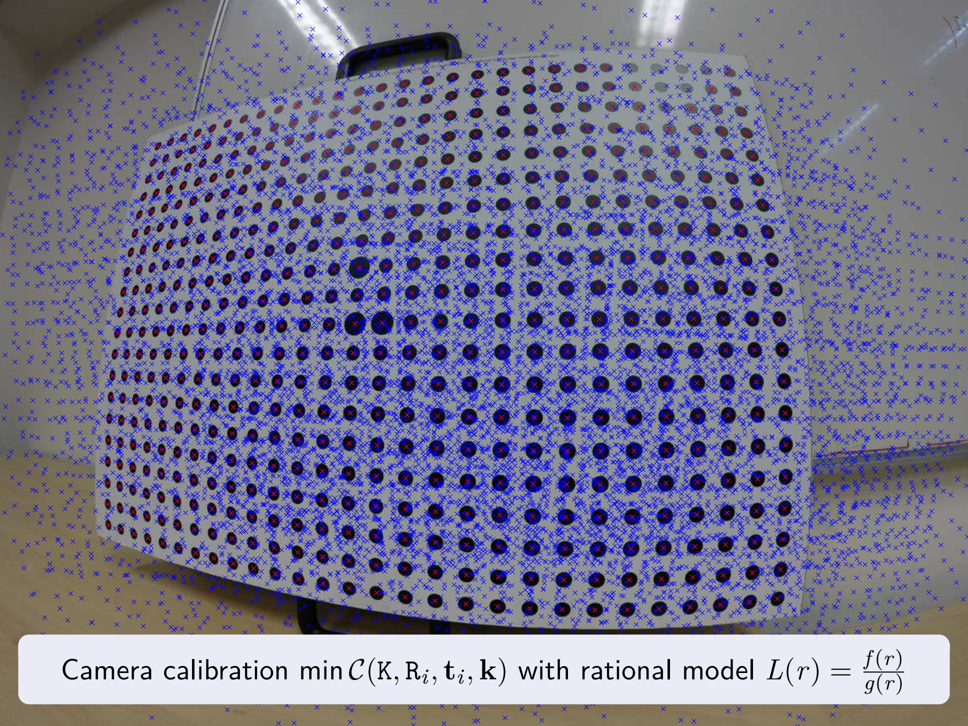
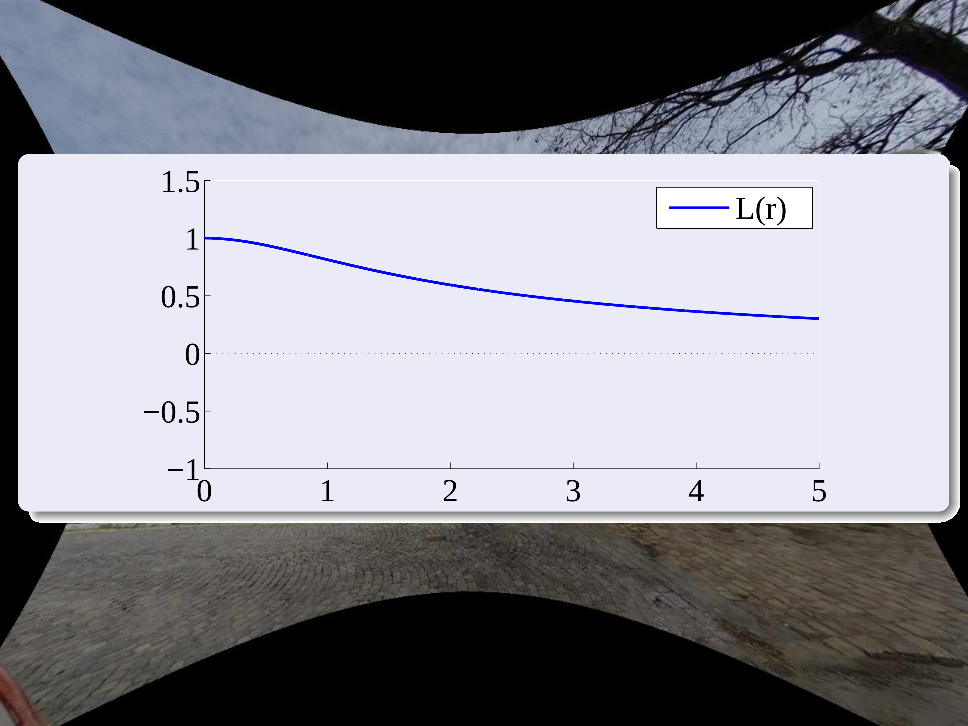








Camera calibration $\min \mathcal{C}(\mathtt{K}, \mathtt{R}_i, \mathbf{t}_i, \mathbf{k})$ with rational model $L(r) = \frac{f(r)}{g(r)}$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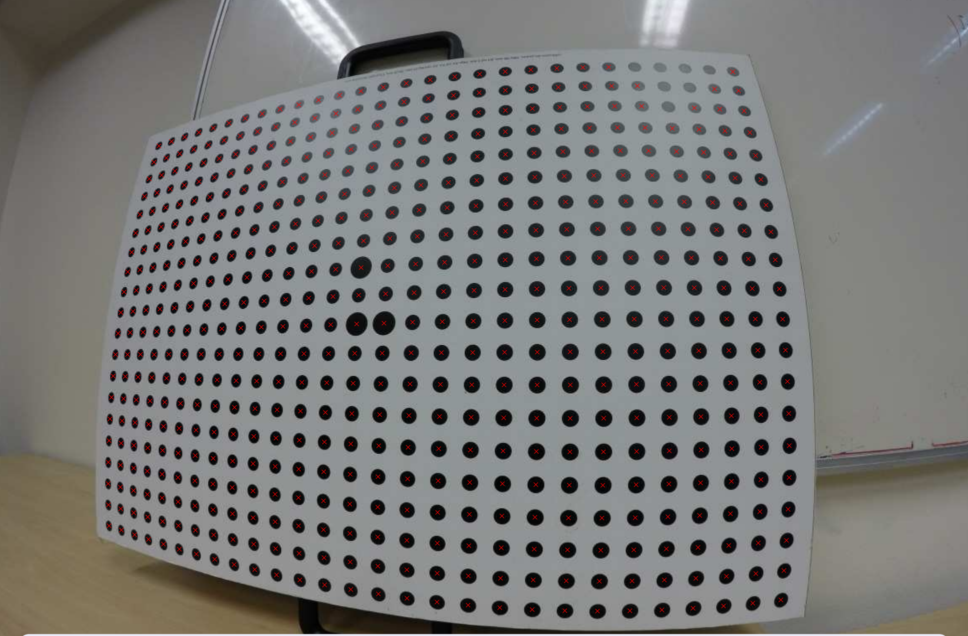
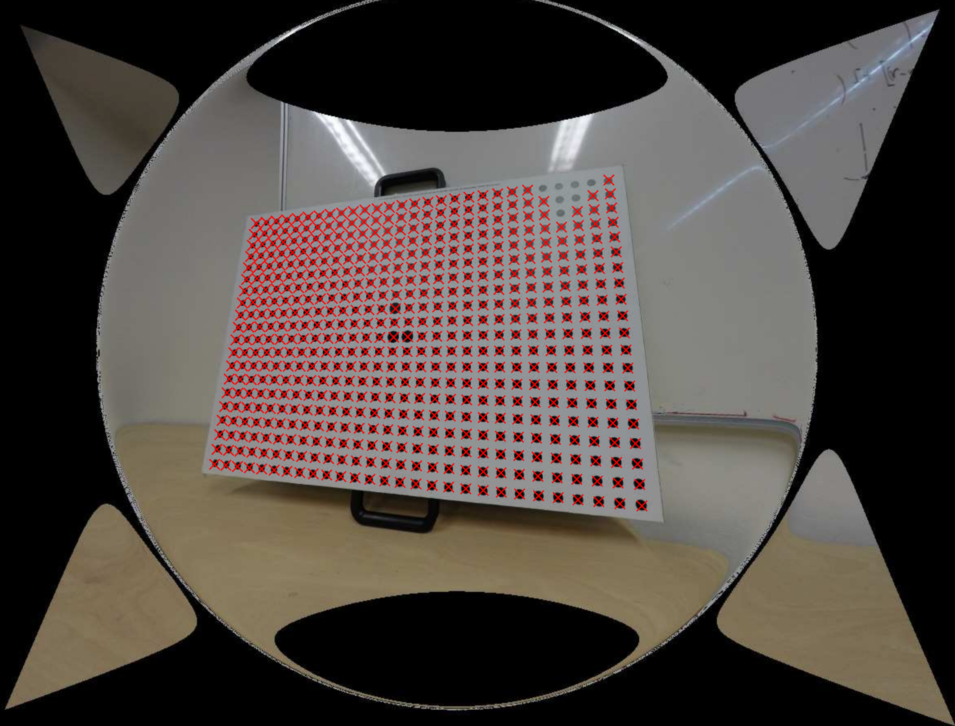
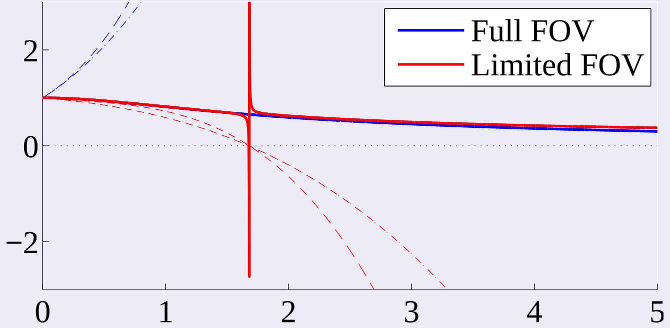


Homography estimation $\min \mathcal{H}(\mathtt{H}, \mathbf{c}, \mathbf{k})$ with rational model $L(r) = \frac{f(r)}{g(r)}$

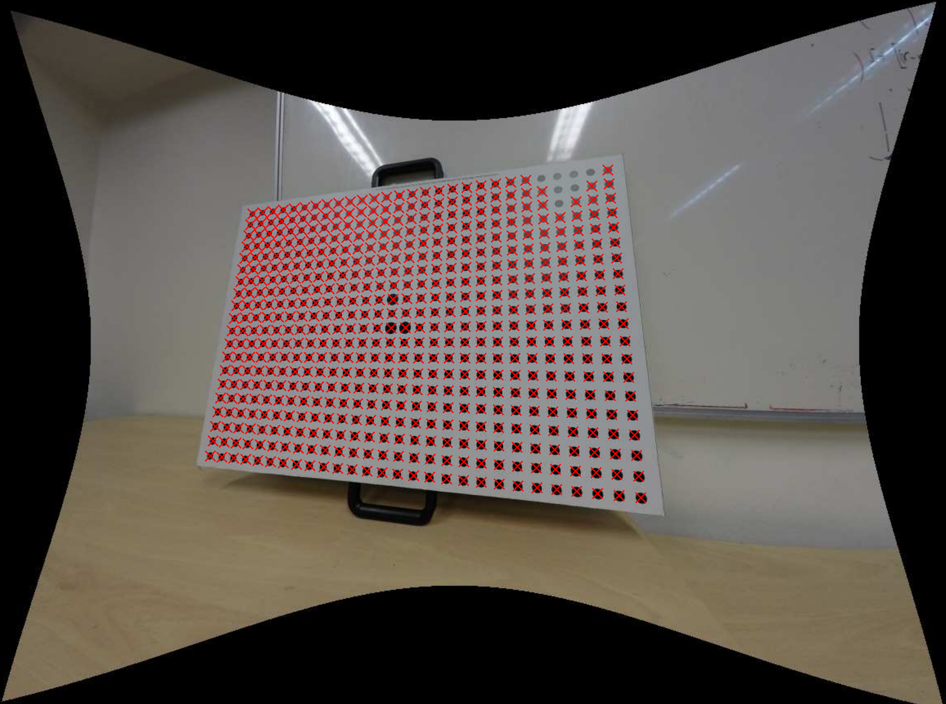


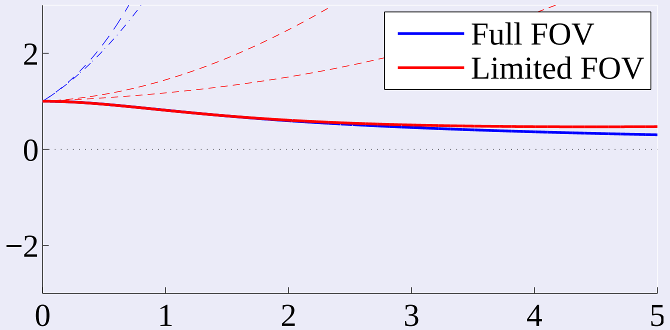


Zero-crossing problem

$$L_f(r) = \frac{1+1.59\,r-0.13\,r^2-0.00872\,r^3}{1+1.6\,r+1.13\,r^2+0.681\,r^3} = \frac{-0.00872}{0.681}\,\frac{(r-131)(r+0.698\pm0.622i)}{(r+1.15)(r+0.514\pm1.07i)}$$

$$L_l(r) = \frac{1-0.218\,r-0.145\,r^2-0.048\,r^3}{1-0.227\,r+0.191\,r^2-0.244\,r^3} = \frac{-0.0478}{-0.244}\,\frac{(r-1.682)(r+2.35\pm2.63i)}{(r-1.678)(r+0.449\pm1.5i)}$$

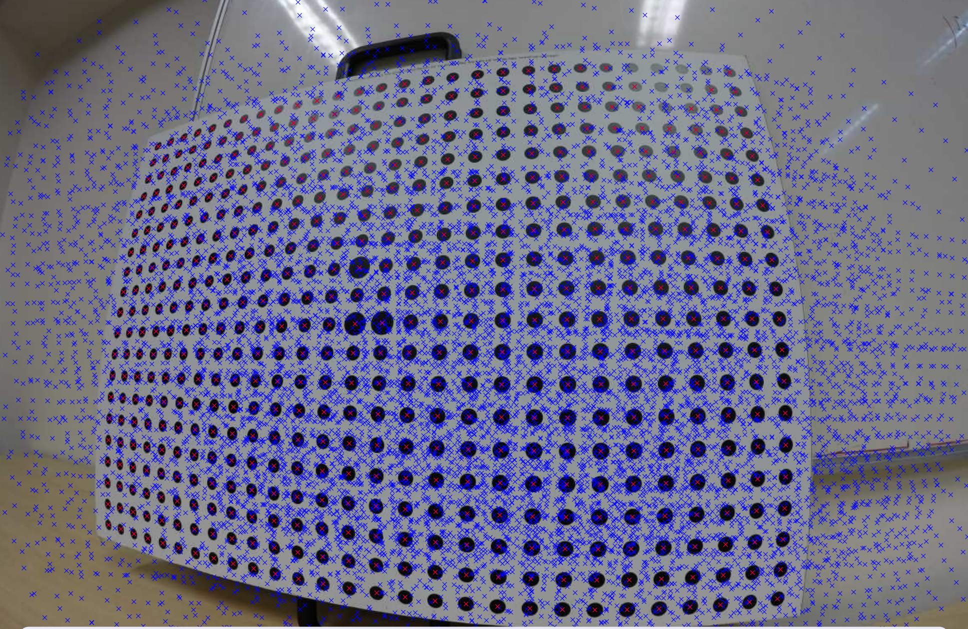




Zero-crossing problem – correction

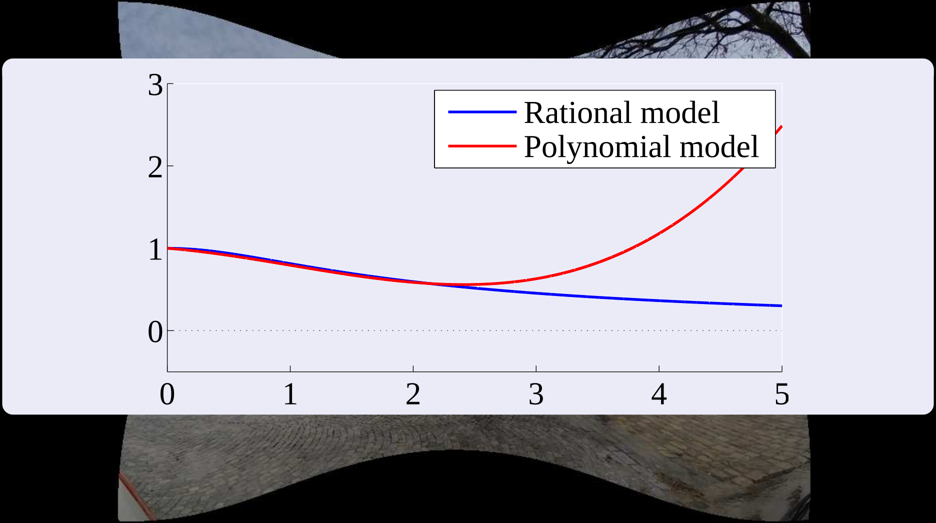
$$L_f(r) = \frac{1{+}1.59\,r{-}0.13\,r^2{-}0.00872\,r^3}{1{+}1.6\,r{+}1.13\,r^2{+}0.681\,r^3} = \frac{-0.00872}{0.681}\,\frac{(r{-}131)(r{+}0.698{\pm}0.622i)}{(r{+}1.15)(r{+}0.514{\pm}1.07i)}$$

$$L_l(r) = \frac{1{+}0.111\,r{+}0.0546\,r^2{-}0.00805\,r^3}{1{+}0.118\,r{+}0.342\,r^2{-}0.0144\,r^3} = \frac{-0.00805}{-0.0144}\,\frac{(r{+}7.25)(r{-}0.231{\pm}0.4.13i)}{(r{-}24.2)(r{+}0.228{\pm}1.68i)}$$



Camera calibration $\min \mathcal{C}(\mathtt{K}, \mathtt{R}_i, \mathbf{t}_i, \mathbf{k})$ with polynomial model

$$L(r) = f(r) = 1 + rk_1 + r^2k_2 + r^3k_3$$


Rational model
Polynomial model
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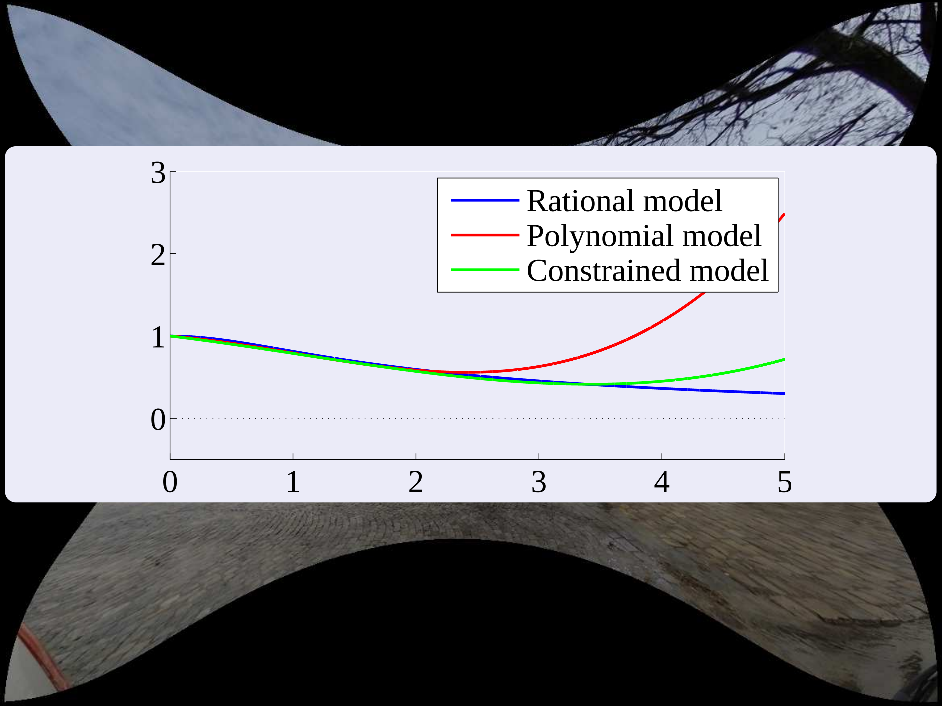
Rational model
Polynomial model
Constrained model
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Stabilizing radial distortion function L

Motivation

+ polynomial and rational functions are easily manipulated and yet provide sufficient fitting power for wide range or distortions.

– Several extrapolation issues arise, mainly for wide angle cameras.

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Minimization constrained by polynomials non-negative on an interval

$$\min f(\mathbf{x}) \quad \Longrightarrow \quad \begin{array}{c} \min\ f(\mathbf{x}) \\ \text{subject to } p_i(y_i, \mathbf{x}) \geq 0 \text{ for } y_i \in [a_i, b_i],\ i = 1, \ldots, n. \end{array}$$

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Stabilized radial distortion calibration example

$$\min \mathcal{C}(\mathtt{K}, \mathtt{R}_i, \mathbf{t}_i, \mathbf{k}) \quad \Longrightarrow \quad \begin{array}{r} \min\ \mathcal{C}(\mathtt{K}, \mathtt{R}_i, \mathbf{t}_i, \mathbf{k}) \\ \text{subject to } f(r) - 1 \geq 0 \text{ for } r \in [0, r_{\max}], \\ g(r) - 1 \geq 0 \text{ for } r \in [0, r_{\max}]. \end{array}$$

Representation of polynomials

Univariate polynomials

An univariate polynomial $p(x) \in \mathbb{R}_n[x]$ of degree $n \in \mathbb{N}$ is a real function

$$p(x) = p_n x^n + p_{n-1} x^{n-1} + \cdots + p_1 x + p_0 = \mathbf{p}^\top \boldsymbol{\psi}_n(x),$$

where $\mathbf{p} = (p_0, p_1, \ldots, p_n)^\top \in \mathbb{R}^{n+1}$ and $\boldsymbol{\psi}_n(x)$ is the canonical basis

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Gram matrix associated with a polynomial

Let $q(x) \in \mathbb{R}_{2n}[x]$. A symmetric matrix $\mathtt{Q} \in \mathbb{R}^{n' \times n'}$, where $n' = n + 1$, is called *Gram matrix* associated with $q(x)$ and the basis $\boldsymbol{\psi}_n(x)$ if

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Non-negative polynomials

A polynomial $p(x) \in \mathbb{R}_n[x]$ is $p(x) \geq 0$ for $\forall x$, iff there exists a *Gram matrix* $\mathtt{Q}$ assoc. with $p(x)$ such that $\mathtt{Q} \succeq 0$, *i.e.*, $\mathtt{Q}$ is positive semidefinite.

Markov-Lukacs Theorem

Computationaly efficient characterization of polynomials non-negative on an interval

Let $\alpha < \beta$, $p(x) \in \mathbb{R}[x]$ and $\deg p(x) = 2n$. Then $p(x) \geq 0$ for all $x \in [\alpha, \beta]$ if and only if

$$p(x) = s(x) + (x - \alpha)(\beta - x)t(x),$$

where $s(x) = \boldsymbol{\psi}_n^\top(x)\,\mathtt{S}\,\boldsymbol{\psi}_n(x)$, $t(x) = \boldsymbol{\psi}_{n-1}^\top(x)\,\mathtt{T}\,\boldsymbol{\psi}_{n-1}(x)$, such that $\mathtt{S}, \mathtt{T} \succeq 0$ ($\mathtt{S}, \mathtt{T}$, are positive semidefinite Gram matrices of $s(x)$ and $t(x)$).
If $\deg p(x) = 2n + 1$, then $p(x) \geq 0$ for all $x \in [\alpha, \beta]$ if and only if

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NB: Even though M-L theorem is an equivalence, we will only use it as an implication: as long as we will have matrices $\mathtt{S}, \mathtt{T} \succeq 0$, M-L theorem guarantees that $p(x)$ constructed using these matrices will be nonnegative on a given interval.

Markov-Lukacs Theorem: Example I

Non-negativity on a interval constraints to positive semidefinite constraints

Constraints on the rational model $L(r) = f(r)/g(r)$

$$f(r) \geq 1 \text{ and } g(r) \geq 1 \text{ for } r \in [0, \bar{r}]$$

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Constraint $f(r) \geq 1$

- According to M-L theorem, $f(r) - 1 \geq 0$ for $r \in [0, \bar{r}]$ iff

$$f(r) - 1 = k_1 r + k_2 r^2 + k_3 r^3 = r \boldsymbol{\psi}_1(r)^\top \mathtt{S}_1 \boldsymbol{\psi}_1(r) + (\bar{r} - r) \boldsymbol{\psi}_1(r)^\top \mathtt{T}_1 \boldsymbol{\psi}_1(r),$$

$$\text{where } \mathtt{S}_1 = \begin{pmatrix} s_{11} & s_{12} \\ s_{12} & s_{13} \end{pmatrix} \succeq 0, \ \mathtt{T}_1 = \begin{pmatrix} t_{11} & t_{12} \\ t_{12} & t_{13} \end{pmatrix} \succeq 0.$$

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- After expanding and comparing coefficients

$$\begin{aligned} k_1 &= s_{11} - t_{11} + 2\bar{r} t_{12}, \\ k_2 &= 2s_{12} - 2t_{12} + \bar{r} t_{13}, \\ k_3 &= s_{13} - t_{13}, \\ 0 &= \bar{r} t_{11} (\Rightarrow t_{11} = 0). \end{aligned}$$

Markov-Lukacs Theorem: Example I

Non-negativity on a interval constraints to positive semidefinite constraints

Constraint $g(r) \geq 1$

- According to M-L theorem, $g(r) - 1 \geq 0$ for $r \in [0, \bar{r}]$ iff

$$g(r) - 1 = k_4 r + k_5\, r^2 + k_6\, r^3 = r \boldsymbol{\psi}_1(r)^\top \mathtt{S}_2 \boldsymbol{\psi}_1(r) + (\bar{r} - r) \boldsymbol{\psi}_1(r)^\top \mathtt{T}_2 \boldsymbol{\psi}_1(r),$$

$$\text{where } \mathtt{S}_2 = \begin{pmatrix} s_{21} & s_{22} \\ s_{22} & s_{23} \end{pmatrix} \succeq 0,\ \mathtt{T}_2 = \begin{pmatrix} t_{21} & t_{22} \\ t_{22} & t_{23} \end{pmatrix} \succeq 0.$$

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Constraint $g(r) \geq 1$

- According to M-L theorem, $g(r) - 1 \geq 0$ for $r \in [0, \bar{r}]$ iff

$$g(r)-1 = k_4 r + k_5\, r^2 + k_6\, r^3 = r\boldsymbol{\psi}_1(r)^\top \mathtt{S}_2 \boldsymbol{\psi}_1(r) + (\bar{r}-r)\boldsymbol{\psi}_1(r)^\top \mathtt{T}_2 \boldsymbol{\psi}_1(r),$$

$$\text{where } \mathtt{S}_2 = \begin{pmatrix} s_{21} & s_{22} \\ s_{22} & s_{23} \end{pmatrix} \succeq 0,\ \mathtt{T}_2 = \begin{pmatrix} t_{21} & t_{22} \\ t_{22} & t_{23} \end{pmatrix} \succeq 0.$$

- After expanding and comparing coefficients

$$\begin{aligned} k_4 &= s_{21} - t_{21} + 2\bar{r}t_{22}, \\ k_5 &= 2s_{22} - 2t_{22} + \bar{r}t_{23}, \\ k_6 &= s_{23} - t_{23}, \\ 0 &= \bar{r}t_{21} (\Rightarrow t_{21} = 0). \end{aligned}$$

Markov-Lukacs Theorem: Example I

Non-negativity on a interval constraints to positive semidefinite constraints

Cost function

$$\begin{array}{ccc} \min\ \mathcal{C}(\mathtt{K}, \mathtt{R}_i, \mathbf{t}_i, \mathbf{k}) & \text{substitution} & \min\ \mathcal{C}(\mathtt{K}, \mathtt{R}_i, \mathbf{t}_i, \mathbf{k}(\mathbf{s}, \mathbf{t})) \\ \mathcal{H}(\mathtt{H}, \mathbf{c}, \mathbf{k}) & \Longrightarrow & \mathcal{H}(\mathtt{H}, \mathbf{c}, \mathbf{k}(\mathbf{s}, \mathbf{t})) \end{array}$$

$$\text{where } \mathbf{k}(\mathbf{s}, \mathbf{t}) = (s_{11} - t_{11} + 2\bar{r}t_{12}, 2s_{12} - 2t_{12} + \bar{r}t_{13}, s_{13} - t_{13},\\ s_{21} - t_{21} + 2\bar{r}t_{22}, 2s_{22} - 2t_{22} + \bar{r}t_{23}, s_{23} - t_{23}).$$

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$$\text{where } \mathbf{k}(\mathbf{s},\mathbf{t}) = (s_{11}-t_{11}+2\bar{r}t_{12}, 2s_{12}-2t_{12}+\bar{r}t_{13}, s_{13}-t_{13}, \\ s_{21}-t_{21}+2\bar{r}t_{22}, 2s_{22}-2t_{22}+\bar{r}t_{23}, s_{23}-t_{23}).$$

Constraints

$$f(r) \geq 1 \text{ and } g(r) \geq 1 \text{ for } r \in [0,\bar{r}] \quad \Longrightarrow$$

$$\mathtt{S}_1 = \begin{pmatrix} s_{11} & s_{12} \\ s_{12} & s_{13} \end{pmatrix} \succeq 0,\ \mathtt{T}_1 = \begin{pmatrix} 0 & t_{12} \\ t_{12} & t_{13} \end{pmatrix} \succeq 0,$$

$$\mathtt{S}_2 = \begin{pmatrix} s_{21} & s_{22} \\ s_{22} & s_{23} \end{pmatrix} \succeq 0,\ \mathtt{T}_2 = \begin{pmatrix} 0 & t_{22} \\ t_{22} & t_{23} \end{pmatrix} \succeq 0.$$

Markov-Lukacs Theorem: Example II

Non-negativity on a interval constraints to positive semidefinite constraints

Constraints on the polynomial model $L(r) = f(r)$

$$f(r) \geq 0 \text{ and } f(r) \leq 1 \text{ for } r \in [0, \bar{r}]$$

Markov-Lukacs Theorem: Example II

Non-negativity on a interval constraints to positive semidefinite constraints

Constraints on the polynomial model $L(r) = f(r)$

$$f(r) \geq 0 \text{ and } f(r) \leq 1 \text{ for } r \in [0, \bar{r}]$$

Constraint $f(r) \geq 0$

- According to M-L theorem, $f(r) \geq 0$ for $r \in [0, \bar{r}]$ iff

$$f(r) = 1 + k_1 r + k_2\, r^2 + k_3\, r^3 = r\boldsymbol{\psi}_1(r)^\top \mathtt{S}_1 \boldsymbol{\psi}_1(r) + (\bar{r} - r)\boldsymbol{\psi}_1(r)^\top \mathtt{T}_1 \boldsymbol{\psi}_1(r),$$

$$\text{where } \mathtt{S}_1 = \begin{pmatrix} s_{11} & s_{12} \\ s_{12} & s_{13} \end{pmatrix} \succeq 0,\ \mathtt{T}_1 = \begin{pmatrix} t_{11} & t_{12} \\ t_{12} & t_{13} \end{pmatrix} \succeq 0.$$

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- After expanding and comparing coefficients

$$\begin{aligned} k_1 &= s_{11} - t_{11} + 2\bar{r} t_{12}, \\ k_2 &= 2s_{12} - 2t_{12} + \bar{r} t_{13}, \\ k_3 &= s_{13} - t_{13}, \\ 1 &= \bar{r} t_{11} (\Rightarrow t_{11} = \tfrac{1}{\bar{r}}). \end{aligned}$$

Markov-Lukacs Theorem: Example II

Non-negativity on a interval constraints to positive semidefinite constraints

Constraint $f(r) \leq 1$

- According to M-L theorem, $f(r) \leq 1$ for $r \in [0, \bar{r}]$ iff

$$1-f(r) = -k_1 r - k_2\, r^2 - k_3\, r^3 = r\boldsymbol{\psi}_1(r)^\top \mathtt{S}_2 \boldsymbol{\psi}_1(r) + (\bar{r}-r)\boldsymbol{\psi}_2(r)^\top \mathtt{T}_1 \boldsymbol{\psi}_1(r),$$

$$\text{where } \mathtt{S}_2 = \begin{pmatrix} s_{21} & s_{22} \\ s_{22} & s_{23} \end{pmatrix} \succeq 0,\ \mathtt{T}_2 = \begin{pmatrix} t_{21} & t_{22} \\ t_{22} & t_{23} \end{pmatrix} \succeq 0.$$

Markov-Lukacs Theorem: Example II

Non-negativity on a interval constraints to positive semidefinite constraints

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$$\text{where } \mathtt{S}_2 = \begin{pmatrix} s_{21} & s_{22} \\ s_{22} & s_{23} \end{pmatrix} \succeq 0,\ \mathtt{T}_2 = \begin{pmatrix} t_{21} & t_{22} \\ t_{22} & t_{23} \end{pmatrix} \succeq 0.$$

- After expanding and comparing coefficients

$$\begin{aligned} -k_1 &= s_{21} - t_{21} + 2\bar{r} t_{22}, \\ -k_2 &= 2 s_{22} - 2 t_{22} + \bar{r} t_{23}, \\ -k_3 &= s_{23} - t_{23}, \\ 0 &= \bar{r} t_{21}. \end{aligned}$$

Markov-Lukacs Theorem: Example II

Non-negativity on a interval constraints to positive semidefinite constraints

Constraint $f(r) \leq 1$

- According to M-L theorem, $f(r) \leq 1$ for $r \in [0, \bar{r}]$ iff

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- After expanding and comparing coefficients

$$\left.\begin{array}{rcl} -k_1 & = & s_{21} - t_{21} + 2\bar{r}t_{22}, \\ -k_2 & = & 2s_{22} - 2t_{22} + \bar{r}t_{23}, \\ -k_3 & = & s_{23} - t_{23}, \\ 0 & = & \bar{r}t_{21}. \end{array}\right\} \Rightarrow \left\{\begin{array}{rcl} s_{21} & = & -2\bar{r}t_{22} - 2t_{12}\bar{r} + s_{11} - \frac{1}{\bar{r}}, \\ s_{22} & = & t_{12} - s_{12} + t_{22} - \frac{1}{2}\bar{r}s_{23} - \\ & & \frac{1}{2}\bar{r}t_{13} - \frac{1}{2}\bar{r}(s_{13} - t_{13}), \\ t_{21} & = & 0, \\ t_{23} & = & s_{13} + s_{23} - t_{13}. \end{array}\right.$$

Markov-Lukacs Theorem: Example II

Cost function

$$\begin{array}{ccc} \min\ \mathcal{C}(\mathtt{K}, \mathtt{R}_i, \mathbf{t}_i, \mathbf{k}) & \text{substitution} & \min\ \mathcal{C}(\mathtt{K}, \mathtt{R}_i, \mathbf{t}_i, \mathbf{k}(\mathbf{s}, \mathbf{t})) \\ \mathcal{H}(\mathtt{H}, \mathbf{c}, \mathbf{k}) & \Longrightarrow & \mathcal{H}(\mathtt{H}, \mathbf{c}, \mathbf{k}(\mathbf{s}, \mathbf{t})) \end{array}$$

where $\mathbf{k}(\mathbf{s}, \mathbf{t}) = (s_{11} - t_{11} + 2\bar{r}t_{12}, 2s_{12} - 2t_{12} + \bar{r}t_{13}, s_{13} - t_{13})$

Markov-Lukacs Theorem: Example II

Cost function

$$\begin{array}{ccc} \min\ \mathcal{C}(\mathtt{K}, \mathtt{R}_i, \mathbf{t}_i, \mathbf{k}) & \text{substitution} & \min\ \mathcal{C}(\mathtt{K}, \mathtt{R}_i, \mathbf{t}_i, \mathbf{k}(\mathbf{s}, \mathbf{t})) \\ \mathcal{H}(\mathtt{H}, \mathbf{c}, \mathbf{k}) & \Longrightarrow & \mathcal{H}(\mathtt{H}, \mathbf{c}, \mathbf{k}(\mathbf{s}, \mathbf{t})) \end{array}$$

where $\mathbf{k}(\mathbf{s}, \mathbf{t}) = (s_{11} - t_{11} + 2\bar{r}t_{12}, 2s_{12} - 2t_{12} + \bar{r}t_{13}, s_{13} - t_{13})$

Constraints

$$f(r) \geq 0 \text{ and } f(r) \leq 0 \text{ for } r \in [0, \bar{r}] \quad \Longrightarrow$$

$$\mathtt{S}_1 = \begin{pmatrix} s_{11} & s_{12} \\ s_{12} & s_{13} \end{pmatrix} \succeq 0,\ \mathtt{T}_1 = \begin{pmatrix} \frac{1}{\bar{r}} & t_{12} \\ t_{12} & t_{13} \end{pmatrix} \succeq 0,$$

$$\mathtt{S}_2 = \begin{pmatrix} -2\bar{r}t_{22} - 2t_{12}\bar{r} + s_{11} - \frac{1}{\bar{r}} & t_{12} - s_{12} + t_{22} - \dots \\ t_{12} - s_{12} + t_{22} - \dots & s_{23} \end{pmatrix} \succeq 0,$$

$$\mathtt{T}_2 = \begin{pmatrix} 0 & t_{22} \\ t_{22} & s_{13} + s_{23} - t_{13} \end{pmatrix} \succeq 0.$$

Stable Radial distortion calibration

Minimization constrained by polynomials non-negative on an interval

1. $$\min f(\mathbf{x})$$

$$\Longrightarrow$$

2. $$\begin{aligned} &\min\ f(\mathbf{x}) \\ \text{subject to}\ &p_i(y_i, \mathbf{x}) \geq 0 \\ \text{for}\ &y_i \in [a_i, b_i] \end{aligned}$$

$$\Longleftrightarrow$$

3. $$\begin{aligned} &\min\ f(\mathbf{x}(\mathbf{s}, \mathbf{t})) \\ \text{subject to}\ &\mathtt{S}_i \succeq 0 \\ &\mathtt{T}_i \succeq 0 \end{aligned}$$

Stable Radial distortion calibration

Minimization constrained by polynomials non-negative on an interval

❶ $\min f(\mathbf{x})$ $\Longrightarrow$ ❷ $\begin{array}{rl} \min & f(\mathbf{x}) \\ \text{subject to} & p_i(y_i, \mathbf{x}) \geq 0 \\ \text{for} & y_i \in [a_i, b_i] \end{array}$ $\Longleftrightarrow$ ❸ $\begin{array}{rl} \min & f(\mathbf{x}(\mathbf{s}, \mathbf{t})) \\ \text{subject to} & \mathtt{S}_i \succeq 0 \\ & \mathtt{T}_i \succeq 0 \end{array}$

Solution

- Estimate initial parameters
- Local opt. (L-M) of ❶
- Identify $\mathtt{S}_i, \mathtt{T}_i$ based on L
- Set $\mathtt{S}_i, \mathtt{T}_i = \mathbf{0}$
- Local opt. (SQP, IP) of ❸

Stable Radial distortion calibration

Minimization constrained by polynomials non-negative on an interval

❶ $\min f(\mathbf{x})$ $\Longrightarrow$ ❷ $\min\ f(\mathbf{x})$ subject to $p_i(y_i, \mathbf{x}) \geq 0$ for $y_i \in [a_i, b_i]$ $\Longleftrightarrow$ ❸ $\min\ f(\mathbf{x}(\mathbf{s}, \mathbf{t}))$ subject to $\mathtt{S}_i \succeq 0$, $\mathtt{T}_i \succeq 0$

Solution

- Estimate initial parameters
- Local opt. (L-M) of ❶
- Identify $\mathtt{S}_i, \mathtt{T}_i$ based on L
- Set $\mathtt{S}_i, \mathtt{T}_i = \mathbf{0}$
- Local opt. (SQP, IP) of ❸

Better solution

- Estimate initial parameters
- Local opt. (L-M) of ❶
- Identify $\mathtt{S}_i, \mathtt{T}_i$ based on L
- Initialize $\mathtt{S}_i, \mathtt{T}_i$ using PMI
- Local opt. (SQP, IP) of ❸

Polynomial matrix inequalities (PMI) programming

Polynomial matrix inequalities optimization problem

Let $p_0(\mathbf{x}) \in \mathbb{R}[\mathbf{x}]$, $\mathtt{G}_i \in \mathbb{S}^{n_i}(\mathbb{R}[\mathbf{x}])$, $i = 1, \ldots, m$ and $\mathbf{x} =\in \mathbb{R}^m$. Then the polynomial matrix inequalities optimization problem has the following form:

$$\begin{array}{rl} \min & p(\mathbf{x}) \\ \text{subject to} & \mathtt{G}_i(\mathbf{x}) \succeq 0,\ i = 1, \ldots, m. \end{array}$$

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Calibration problem with rational L to PMI problem

$$\begin{aligned} \text{min} \;\; & \mathcal{C}(\mathtt{K}, \mathtt{R}_i, \mathbf{t}_i, \mathbf{k}(\mathbf{s}, \mathbf{t})) \\ & \mathcal{H}(\mathtt{H}, \mathbf{c}, \mathbf{k}(\mathbf{s}, \mathbf{t})) \\ \text{subject to} \;\; & \mathtt{S}_i(\mathbf{s}, \mathbf{t}) \succeq 0 \\ & \mathtt{T}_i(\mathbf{s}, \mathbf{t}) \succeq 0 \end{aligned} \quad \overset{\text{relaxation}}{\Longrightarrow} \quad \begin{aligned} \text{min} \;\; & \mathcal{C}(\overbrace{\mathtt{K}, \mathtt{R}_i, \mathbf{t}_i}^{\text{fixed}}, \mathbf{k}(\mathbf{s}, \mathbf{t})) \\ & \mathcal{H}(\underbrace{\mathtt{H}, \mathbf{c}}_{\text{fixed}}, \mathbf{k}(\mathbf{s}, \mathbf{t})) \\ \text{subject to} \;\; & \mathtt{S}_i(\mathbf{s}, \mathbf{t}) \succeq 0 \\ & \mathtt{T}_i(\mathbf{s}, \mathbf{t}) \succeq 0 \end{aligned}$$

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– Still, typically a non-convex problem with many local minima
– Still, finding the global minimizer $\mathbf{x}^*$ is an NP-hard problem

Hierarchy of Linear Matrix Inequality Relaxations

J.-B. Lasserre: *Global optimization with polynomials and the problem of moments*, 2001.
D. Henrion, J.-B. Lasserre: *Convergent relaxations of polynomial matrix inequalities and static output feedback*, 2006.

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- Hierarchy of LMI ($\equiv$ SDP) programs $\mathcal{P}_1, \mathcal{P}_2, \ldots$ that produces a monotonically non-decreasing sequence of lower bounds $p(\mathbf{x}_1^*) \leq p(\mathbf{x}_2^*) \leq \ldots$ on the PMI problem, $\lim_{i \to \infty} p(\mathbf{x}_i^*) = p(\mathbf{x}^*)$
- Practically, the series converges to $p(\mathbf{x}^*)$ in finitely many steps, *i.e.*, there exists $j \in \mathbb{N}$, such that $p(\mathbf{x}_j^*) = p(\mathbf{x}^*)$
- Tools of linear algebra can be used to detect this finite convergence and to recover both $p(\mathbf{x}^*)$ and $\mathbf{x}^*$

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LMI relaxation $\mathcal{P}_\delta$ of order δ

$$\begin{array}{ll} \min & p(\mathbf{x}) \\ \text{s.t.} & \mathtt{G}_i(\mathbf{x}) \succeq 0 \end{array} \Rightarrow \begin{array}{ll} \min & \ell_{\mathbf{y}}(p(\mathbf{x})) \\ \text{s.t} & \mathtt{M}_{\delta-\gamma_i}(\mathtt{G}_i, \mathbf{y}) \succeq 0 \\ & \mathtt{M}_\delta(\mathbf{y}) \succeq 0. \end{array}$$

$$\ell_{\mathbf{y}}(p(\mathbf{x})) = \textstyle\sum_{\boldsymbol{\alpha}} p_{\boldsymbol{\alpha}} y_{\boldsymbol{\alpha}}$$
$$\mathtt{M}_\delta(\mathtt{G}, \mathbf{y}) = \ell_{\mathbf{y}}((\boldsymbol{\psi}_\delta(\mathbf{x})\boldsymbol{\psi}_\delta^\top(\mathbf{x})) \otimes \mathtt{G})$$
$$\mathtt{M}_\delta(\mathbf{y}) = \ell_{\mathbf{y}}(\boldsymbol{\psi}_\delta(\mathbf{x})\boldsymbol{\psi}_\delta^\top(\mathbf{x}))$$

Stable Radial Distortion Calibration for Rational $L(r)$

Final Breakdown

- Estimate initial parameters
- Unconstrained nonlinear optimization (`lsqnonlin`, Ceres, ...)

$$\min\ \mathcal{C}(\mathtt{K}, \mathtt{R}_i, \mathbf{t}_i, \mathbf{k}), \mathcal{H}(\mathtt{H}, \mathbf{c}, \mathbf{k})$$

- Identify $\mathtt{S}_i, \mathtt{T}_i$ based on the shape of $L(r)$ (by hand)
- Initialize $\mathtt{S}_i, \mathtt{T}_i$ using PMI prg. (GloptiPoly, YALMIP, GpoSolver)

$$\begin{aligned} \min\ & \mathcal{C}(\overbrace{\mathtt{K}, \mathtt{R}_i, \mathbf{t}_i}^{\text{fixed}}, \mathbf{k}(\mathbf{s}, \mathbf{t})), \mathcal{H}(\overbrace{\mathtt{H}, \mathbf{c}}^{\text{fixed}}, \mathbf{k}(\mathbf{s}, \mathbf{t})) \\ \text{subject to}\ & \mathtt{S}_i(\mathbf{s}, \mathbf{t}) \succeq 0, \mathtt{T}_i(\mathbf{s}, \mathbf{t}) \succeq 0 \end{aligned}$$

- Constrained nonlinear optimization (`fmincon`: SQP, IP, ...)

$$\begin{aligned} \min\ & \mathcal{C}(\mathtt{K}, \mathtt{R}_i, \mathbf{t}_i, \mathbf{k}(\mathbf{s}, \mathbf{t})), \mathcal{H}(\mathtt{H}, \mathbf{c}, \mathbf{k}(\mathbf{s}, \mathbf{t})) \\ \text{subject to}\ & \mathtt{S}_i(\mathbf{s}, \mathbf{t}) \succeq 0, \mathtt{T}_i(\mathbf{s}, \mathbf{t}) \succeq 0 \end{aligned}$$

Stable Radial Distortion Calibration for Rational $L(r)$

Final Breakdown

- Estimate initial parameters
- Unconstrained nonlinear optimization (`lsqnonlin`, Ceres, ...)

$$\min\ \mathcal{C}(\mathtt{K}, \mathtt{R}_i, \mathbf{t}_i, \mathbf{k}), \mathcal{H}(\mathtt{H}, \mathbf{c}, \mathbf{k})$$

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$$\begin{aligned} \min\ & \mathcal{C}(\overbrace{\mathtt{K}, \mathtt{R}_i, \mathbf{t}_i}^{\text{fixed}}, \mathbf{k}(\mathbf{s}, \mathbf{t})), \mathcal{H}(\overbrace{\mathtt{H}, \mathbf{c}}^{\text{fixed}}, \mathbf{k}(\mathbf{s}, \mathbf{t})) \\ \text{subject to}\ & \mathtt{S}_i(\mathbf{s}, \mathbf{t}) \succeq 0, \mathtt{T}_i(\mathbf{s}, \mathbf{t}) \succeq 0 \end{aligned}$$

- Constrained nonlinear optimization (`fmincon`: SQP, IP, ...)

$$\begin{aligned} \min\ & \mathcal{C}(\mathtt{K}, \mathtt{R}_i, \mathbf{t}_i, \mathbf{k}(\mathbf{s}, \mathbf{t})), \mathcal{H}(\mathtt{H}, \mathbf{c}, \mathbf{k}(\mathbf{s}, \mathbf{t})) \\ \text{subject to}\ & \mathtt{S}_i(\mathbf{s}, \mathbf{t}) \succeq 0, \mathtt{T}_i(\mathbf{s}, \mathbf{t}) \succeq 0 \end{aligned}$$

Stable Radial Distortion Calibration for Rational $L(r)$

Final Breakdown

- Estimate initial parameters
- Unconstrained nonlinear optimization (`lsqnonlin`, Ceres, ...)

$$\min\ \mathcal{C}(\mathtt{K}, \mathtt{R}_i, \mathbf{t}_i, \mathbf{k}), \mathcal{H}(\mathtt{H}, \mathbf{c}, \mathbf{k})$$

- Identify $\mathtt{S}_i, \mathtt{T}_i$ based on the shape of $L(r)$ (by hand)
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Stable Radial Distortion Calibration for Rational $L(r)$

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Conclusion

- We were interested in polynomial and rational distortion functions L and the related extrapolation issues (problem mainly for wide angle camera with strong radial distortion)
- We introduced a new prior:

  nonnegativity of certain polynomials on certain intervals

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Thank you for your attention