The Fastest Learning in the West:

Practical Tracking with Correlation Filters

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Visual tracking

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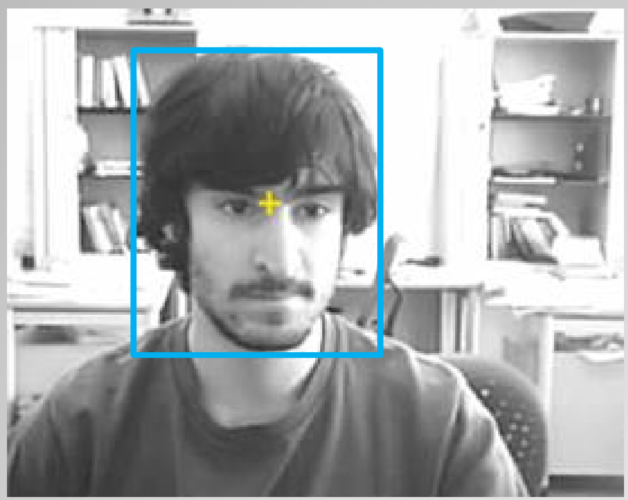

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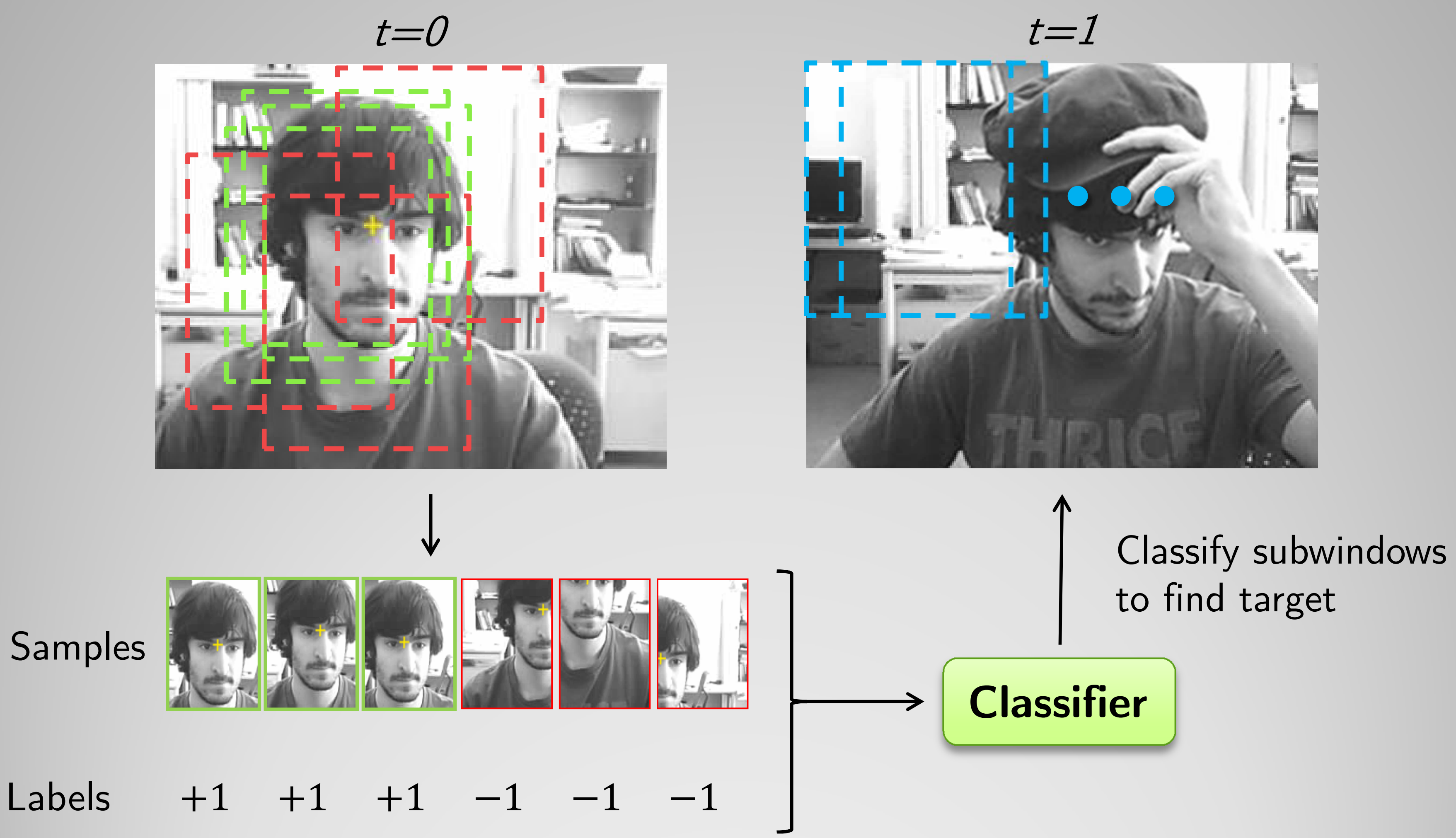
- We are given the **initial** bounding box (BB) of a target.
- **Estimate** its BB in later frames of a video (“track the target”).
- Important component of many Computer Vision pipelines (simpler/faster than detection; ensures temporal consistency).
- Successfully tracked frames yield **more** information on target appearance.

Visual tracking – discriminative



The Convolution Connection

- Linear classifier with weights $\mathbf{w}$:

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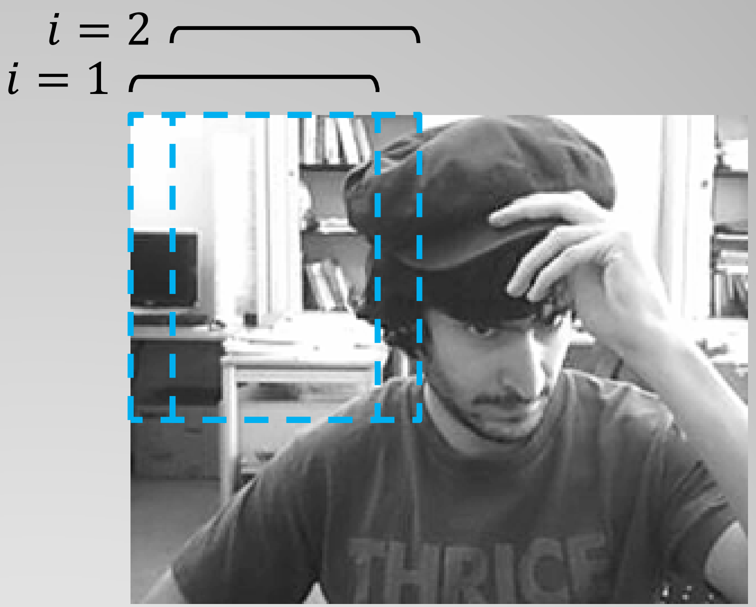

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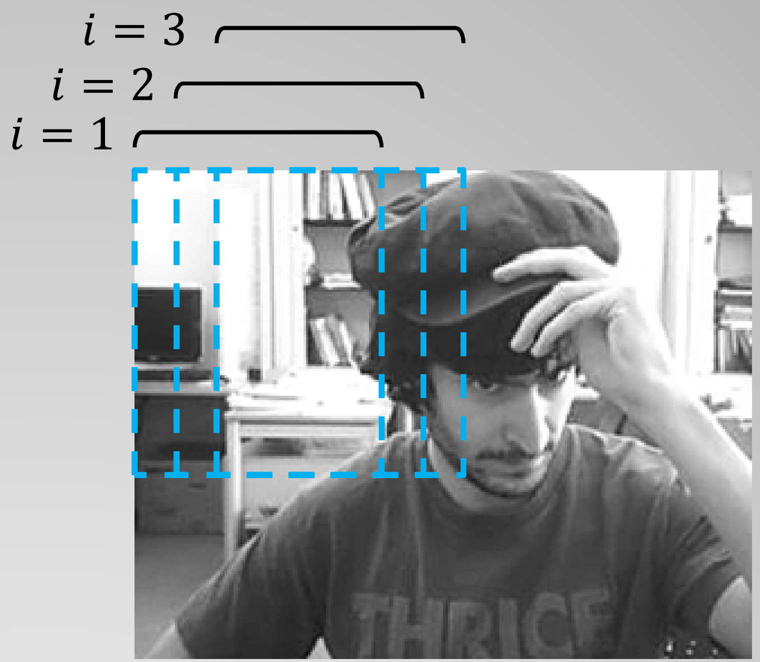

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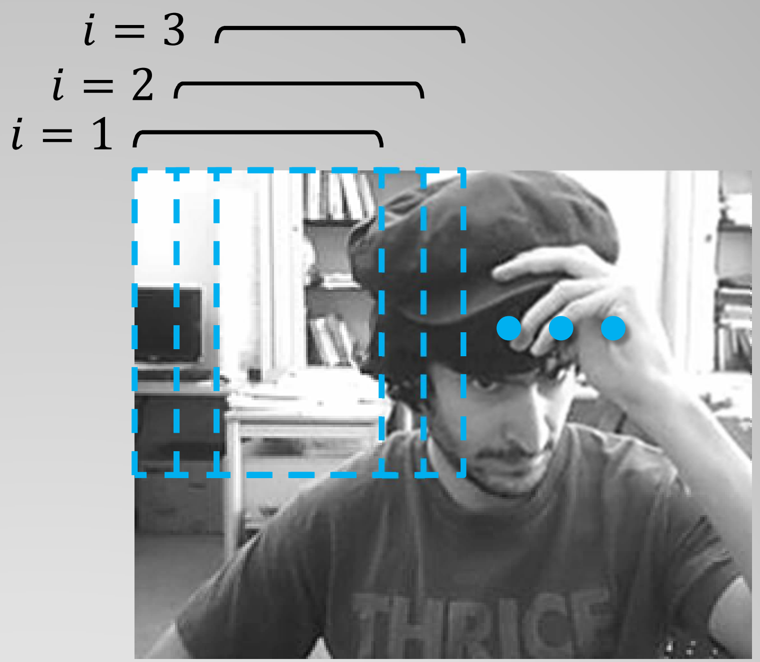

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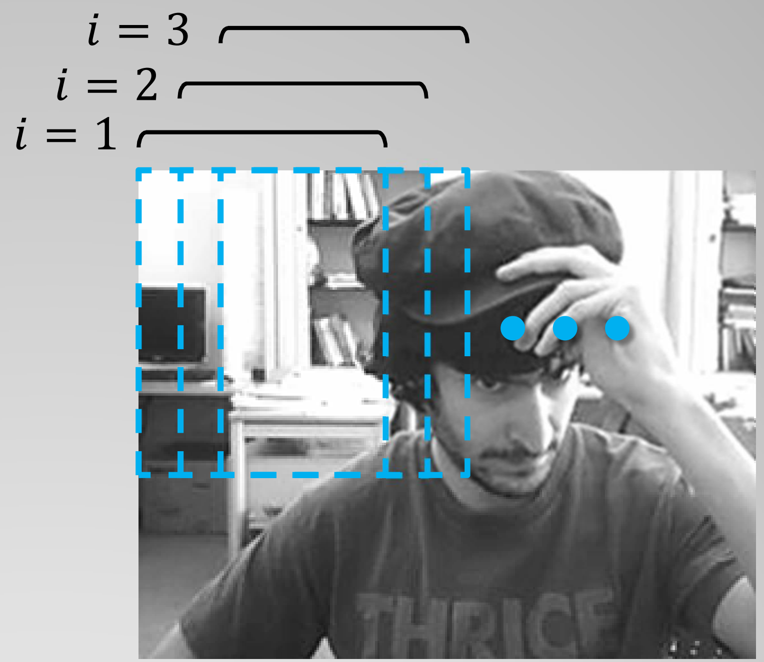

- Linear classifier with weights $\mathbf{w}$:

$$y = \mathbf{w}^T \mathbf{x}$$

- Evaluate it at subwindows $\mathbf{x}_i$:

$$y_i = \mathbf{w}^T \mathbf{x}_i$$

- Concatenate y_i into a vector $\mathbf{y}$.
- Equivalent to **cross-correlation** (or *correlation* for short)

$$\mathbf{y} = \mathbf{x} \circledast \mathbf{w}$$

- Note: Convolution is related; it is the same as cross-correlation, but with the flipped image of $\mathbf{w}$ (A → ∀).

The Convolution Theorem

- Cross-correlation is **equivalent** to an **element-wise product** in Fourier domain:

$$\mathbf{y} = \mathbf{x} \circledast \mathbf{w} \quad \Longleftrightarrow \quad \hat{\mathbf{y}} = \hat{\mathbf{x}}^* \times \widehat{\mathbf{w}}$$

where
- $\hat{\mathbf{y}} = \mathcal{F}(\mathbf{y})$ is the Discrete Fourier Transform (DFT) of $\mathbf{y}$. (likewise for $\hat{\mathbf{x}}$ and $\widehat{\mathbf{w}}$).
- $\times$ is element-wise product.
- $.^*$ is complex-conjugate (i.e. negate imaginary part).

- Note that cross-correlation, and the DFT, are **cyclic** (the window wraps at the image edges). Not an issue in practice.

The Convolution Theorem

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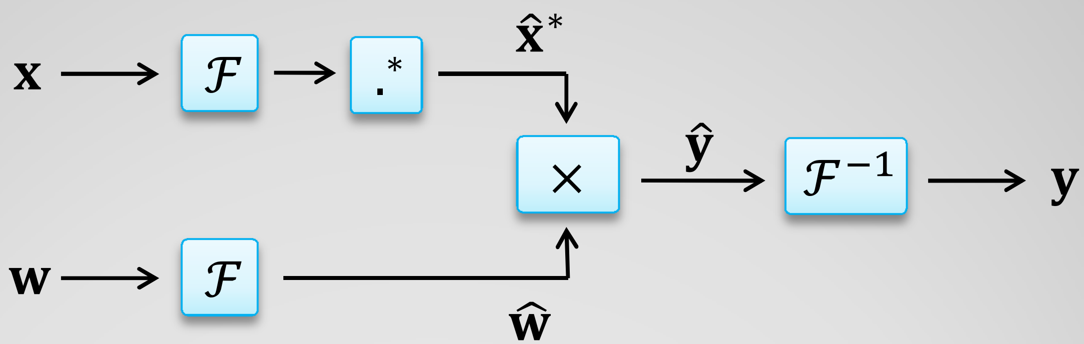
- In practice:



- Can be **orders of magnitude faster**:
  - For $n \times n$ images, cross-correlation is $\mathcal{O}(n^4)$.
  - Fast Fourier Transform (and its inverse) are $\mathcal{O}(n^2 \log n)$.

The Convolution Theorem

- **The evaluation of any linear classifier can be accelerated** → with the Convolution Theorem.

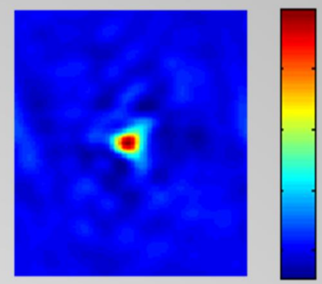
  (Not just for tracking.)

- What about **training**?

- It turns out that **Signal Processing** studied this problem for decades, almost separately from mainstream Computer Vision!

Objective

$\circledast$ $\mathbf{w}$ $=$

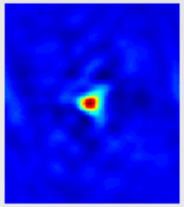


Intuition of training objective:

- Cross-correlation of classifier $\mathbf{w}$ and a training image $\mathbf{x}$ should have:
  - A **high peak** near the true location of the target.
  - **Low values** elsewhere (to minimize false positives).

Historical Perspective

- **Synthetic Discriminant Functions** (1970's – 1980's)
  - Different criteria to optimize separation between a positive and a negative class.
  - Very similar to Linear Discriminant Analysis.
  - Less interesting, compared to today's generic classifiers (SVM, Logistic Regression, Boosting...).
  - **No guarantee** that correlation output will yield a sharp peak, i.e.:

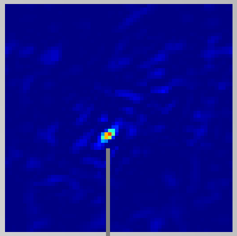
$\mathbf{x} \circledast \mathbf{w} \neq$ 

Historical Perspective

Now the good stuff (1980's – 1990's)

- **Minimum Average Correlation Energy** (MACE) filters
  - Bring the average correlation output towards 0:

$$\min_{\mathbf{w}} \sum_i \|\mathbf{x}_i \circledast \mathbf{w}\|^2$$



  - While keeping the peak value fixed for all samples:

$$\text{subject to: } \mathbf{w}^T \mathbf{x}_i = y_i$$

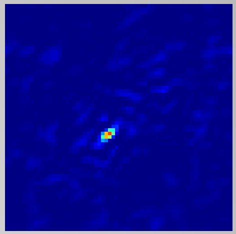
  - The goal is to produce a **sharp peak** at the target location.

Historical Perspective

Now the good stuff (1980's – 1990's)

- **Minimum Average Correlation Energy** (MACE) filters
  - The solution is:

$\mathbf{x}_i \circledast \mathbf{w}$



$$\hat{\mathbf{w}} = \hat{D}^{-1}\hat{X}\left(\hat{X}\hat{D}^{-1}\hat{X}^T\right)^{-1}\mathbf{y}$$

where
- $\mathbf{y}$ is vector of class labels (y_i).
- $\hat{X}$ is data matrix in Fourier domain (its columns are $\hat{\mathbf{x}}_i$).
- $\hat{D}$ is a *diagonal* matrix, with elements $\sum_i \hat{\mathbf{x}}_i{}^2$.

  - Easy to compute; only one expensive matrix inversion.
  - **Sharp peak = good localization**! Are we done?

Historical Perspective

The MACE filter suffers from 2 issues:

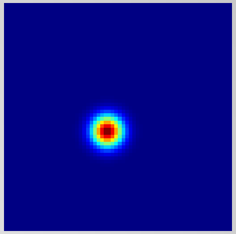
- **Hard constraints easily lead to overfitting**.
  - UMACE (“Unconstrained MACE”) attempts to solve this by instead maximizing the average classifier output on positive samples.
  - Unfortunately, it still suffers from the second problem...

Historical Perspective

The MACE filter suffers from 2 issues:

- Enforcing a **sharp peak** is also a too strong condition; overfits.
  - This led to the development of **Gaussian-MACE** / **MSE-MACE**, which encourages the peak to follow a nice Gaussian shape:

$$\min_{\mathbf{w}} \sum_i \|\mathbf{x}_i \circledast \mathbf{w} - \mathbf{g}\|^2, \qquad \mathbf{g} =$$



$$\text{subject to: } \mathbf{w}^T \mathbf{x}_i = y_i$$

  - In the original papers, the minimization was *still* subject to the MACE hard constraints: $\mathbf{w}^T \mathbf{x}_i = y_i$.
    (They later turned out to be unnecessary!)

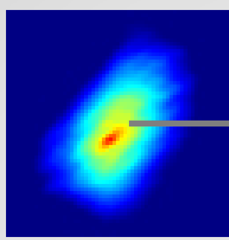
Sharp vs. Gaussian peaks

Training image: $\mathbf{x} =$ 

Naïve filter
$(\mathbf{w} = \mathbf{x})$

Classifier
$(\mathbf{w})$

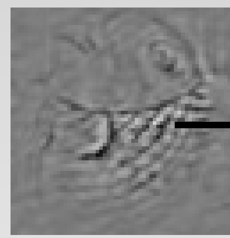
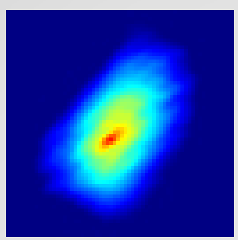
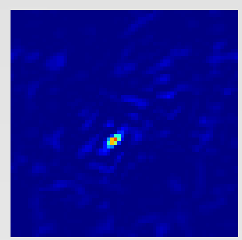
Output
$(\mathbf{w} * \mathbf{x})$



- Very broad peak is hard to localize (especially with clutter).
- State-of-the-art classifiers (e.g. SVM) show **same** behavior!

Sharp vs. Gaussian peaks

Training image: $\mathbf{x} =$ 

	Naïve filter ($\mathbf{w} = \mathbf{x}$)	Sharp peak (UMACE)
Classifier ($\mathbf{w}$)		
Output ($\mathbf{w} * \mathbf{x}$)		

- A very sharp peak is obtained by emphasizing **small image details** (like the fish's scales here).

- Unfortunately, this classifier **generalizes poorly**:
  If the details are not exactly the same, the output is 0.

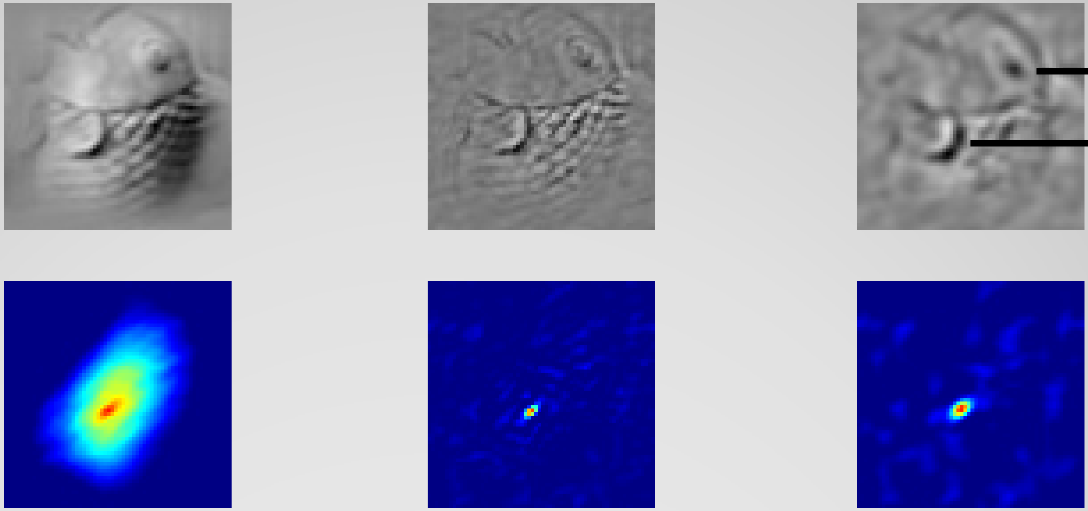
Sharp vs. Gaussian peaks

Training image: $\mathbf{x} =$ 

Naïve filter ($\mathbf{w} = \mathbf{x}$) | Sharp peak (UMACE) | Gaussian peak (GMACE)

Classifier ($\mathbf{w}$)

Output ($\mathbf{w} * \mathbf{x}$)



- A Gaussian peak is a good compromise.
- Tiny details are ignored.
- Instead, the classifier focuses on **larger, more robust structures**.

Min. Output Sum of Sq. Errors (MOSSE)

In their CVPR 2010 paper, David Bolme and colleagues brought these techniques back to the spotlight.

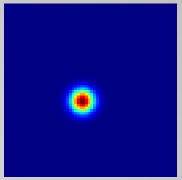
- They presented a tracker that:
  - Processed videos at over **600 frames-per-second** (!)
  - Was **very simple** to implement
    - No features.
    - Only FFT and element-wise operations on raw pixels.
  - Despite this fact, it performed **similarly** to the **most sophisticated** trackers of the time.

Min. Output Sum of Sq. Errors (MOSSE)

How did they do it?

- They focused on just the "Gaussian peak" objective (no constraints):

$$\min_{\mathbf{w}} \sum_i \|\mathbf{x}_i \circledast \mathbf{w} - \mathbf{g}\|^2, \qquad \mathbf{g} =$$



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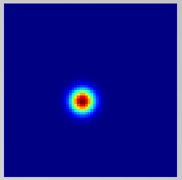

- Found the following solution using the Convolution Theorem:

$$\hat{\mathbf{w}} = \frac{\sum_i \hat{\mathbf{g}}^* \times \hat{\mathbf{x}}_i}{\sum_i \hat{\mathbf{x}}_i^* \times \hat{\mathbf{x}}_i + \lambda}$$

(where a small constant $\lambda = 10^{-4}$ is added to prevent divisions by 0)

No expensive matrix operations! $\Rightarrow$ Only FFT and element-wise.

Circulant matrices

Why does the MOSSE filter work so well in practice?

→ We need tools to connect **correlation filters** with **machine learning**.

- Consider the problem for one sample $\mathbf{x}$: $\min_{\mathbf{w}} \|\mathbf{x} \circledast \mathbf{w} - \mathbf{g}\|^2$

Circulant matrices

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- Consider the problem for one sample $\mathbf{x}$:

$$\min_{\mathbf{w}} \|\mathbf{x} \circledast \mathbf{w} - \mathbf{g}\|^2$$

- We can **replace** the correlation with a **special matrix** $C(\mathbf{x})$:

$$\min_{\mathbf{w}} \|C(\mathbf{x})\mathbf{w} - \mathbf{g}\|^2$$

- $C(\mathbf{x})$ is a **circulant matrix**:

$$C(\mathbf{u}) = \begin{bmatrix} u_0 & u_1 & u_2 & \cdots & u_{n-1} \\ u_{n-1} & u_0 & u_1 & \cdots & u_{n-2} \\ u_{n-2} & u_{n-1} & u_0 & \cdots & u_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ u_1 & u_2 & u_3 & \cdots & u_0 \end{bmatrix}$$

Circulant matrices

- We can see $X = C(\mathbf{x})$ as a **dataset** with **cyclically shifted** versions of $\mathbf{x}$:

$$X = \begin{bmatrix} (P^0\mathbf{x})^T \\ (P^1\mathbf{x})^T \\ \vdots \\ (P^{n-1}\mathbf{x})^T \end{bmatrix}$$

- P is a permutation matrix that shifts the pixels down 1 element.
- Arbitrary shift i obtained with power $P^i\mathbf{x}$.
- Cyclic: $P^n\mathbf{x} = P^0\mathbf{x} = \mathbf{x}$.

$P^0\mathbf{x}$ | $P^1\mathbf{x}$ | $P^2\mathbf{x}$ | $\cdots$ | $P^{n-1}\mathbf{x}$

Circulant matrices

- Circulant matrices have many nice properties.

$$X = \begin{bmatrix} (P^0\mathbf{x})^T \\ (P^1\mathbf{x})^T \\ \vdots \\ (P^{n-1}\mathbf{x})^T \end{bmatrix} \qquad \mathcal{F}(X) = \begin{bmatrix} \hat{\mathbf{x}}_1 & 0 & \cdots & 0 \\ 0 & \hat{\mathbf{x}}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \hat{\mathbf{x}}_n \end{bmatrix}$$

Data matrix is **circulant** ⇒ Becomes **diagonal** in Fourier domain

- Similar role to the Convolution Theorem.
- **Most of the "data" is 0 and can be ignored!** ⇒ Massive speed-up

Circulant matrices

Back to our question:
Why does the MOSSE filter work so well in practice?

- Consider a simple Ridge Regression (RR) problem:

$$\min_{\mathbf{w}} \|X\mathbf{w} - \mathbf{y}\|^2 + \lambda\|\mathbf{w}\|^2$$

RR = Least-squares with regularization (avoids overfitting!)

- Closed-form solution: $\mathbf{w} = (X^T X + \lambda I)^{-1} X^T \mathbf{y}$

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- Now replace X = $C(\mathbf{x})$ (circulant data), and $\mathbf{y} = \mathbf{g}$ (Gaussian targets).
- **Diagonalizing** the involved circulant matrices with the DFT yields:

$$\hat{\mathbf{w}} = \frac{\hat{\mathbf{x}}^* \times \hat{\mathbf{y}}}{\hat{\mathbf{x}}^* \times \hat{\mathbf{x}} + \lambda}$$

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$$\hat{\mathbf{w}} = \frac{\hat{\mathbf{x}}^* \times \hat{\mathbf{y}}}{\hat{\mathbf{x}}^* \times \hat{\mathbf{x}} + \lambda} \Rightarrow$$

- Which is exactly the MOSSE solution!
- So MOSSE is equivalent to a **good learning algorithm** (RR) with **lots of data** (circulant/shifted samples).

Kernelized Correlation Filters

- Circulant matrices are a **very general tool**, replacing standard operations with fast Fourier operations.

- For example, we can apply the same idea to **Kernel Ridge Regression**:

$$\boldsymbol{\alpha} = (K + \lambda I)^{-1}\mathbf{y} \qquad (K \text{ kernel matrix})$$

- For many kernels, circulant data $\Rightarrow$ circulant K:

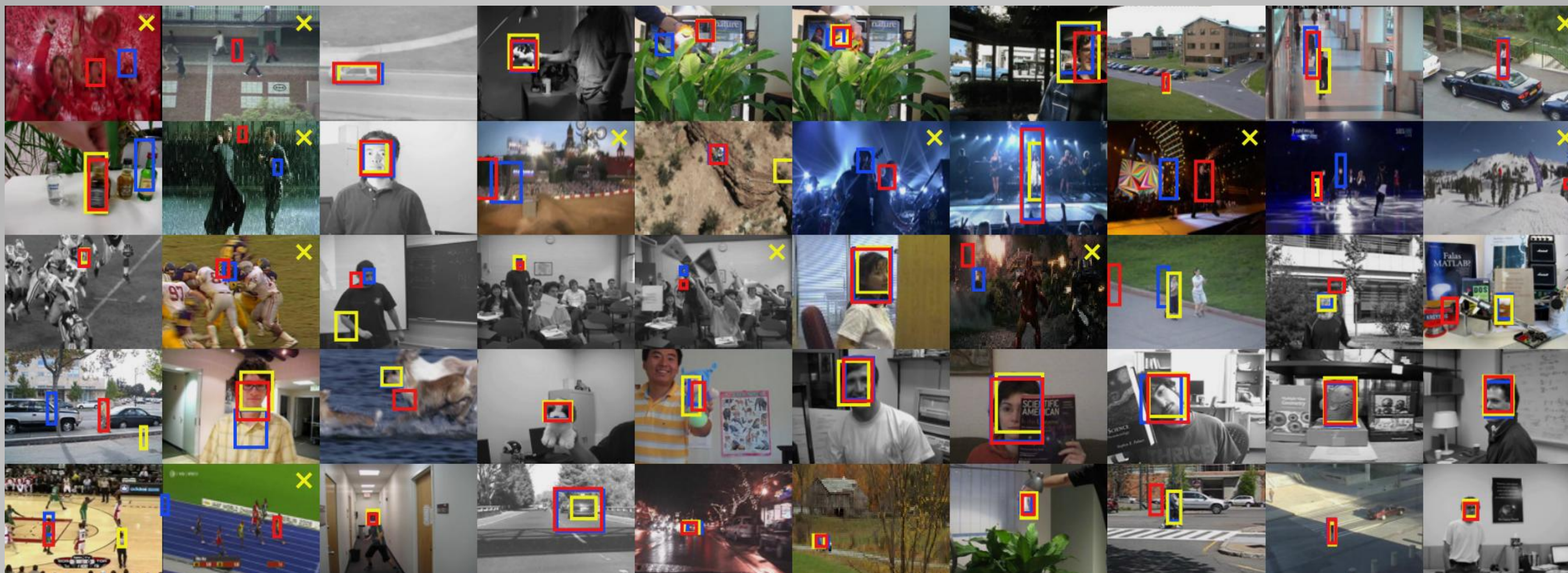
$$K = C(\mathbf{k}),$$ where $\mathbf{k}$ is the first row of K (small, and easy to compute)

- Diagonalizing with the DFT yields:

$$\hat{\boldsymbol{\alpha}} = \frac{\hat{\mathbf{y}}}{\hat{\mathbf{k}} + \lambda} \quad \Rightarrow$$

Fast solution in $\mathcal{O}(n \log n)$. Typical kernel algorithms are $\mathcal{O}(n^2)$ or higher!

Kernelized Correlation Filters

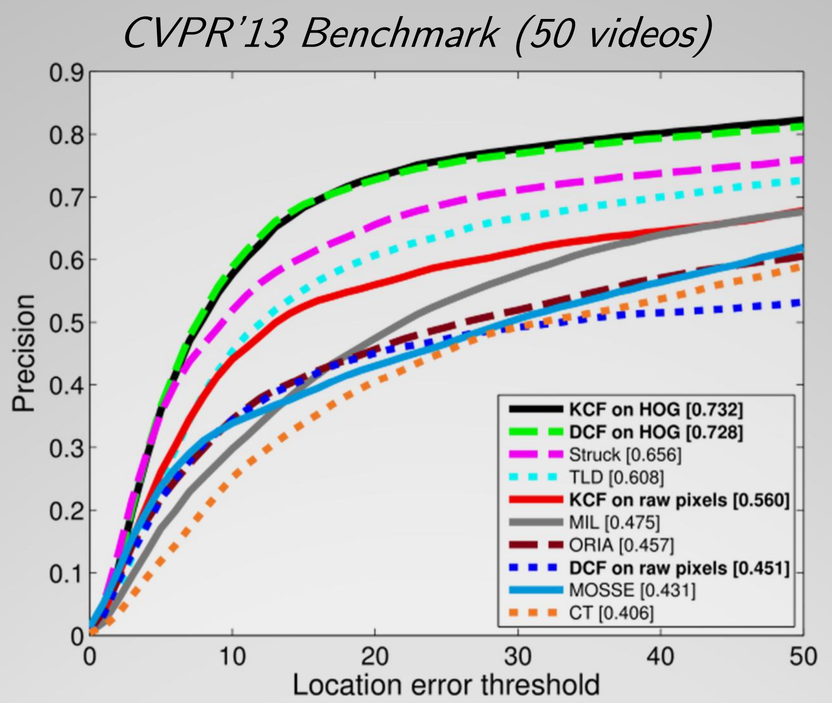


Kernelized Correlation Filter (KCF) TLD Struck

KCF Tracker

- Open-source (ported to Matlab/Python/Java/C)
- ~ **300 FPS**
- Base for **top 3 trackers in VOT 2014**.
- Train + detect: **13 lines of MATLAB code**.

Kernelized Correlation Filters



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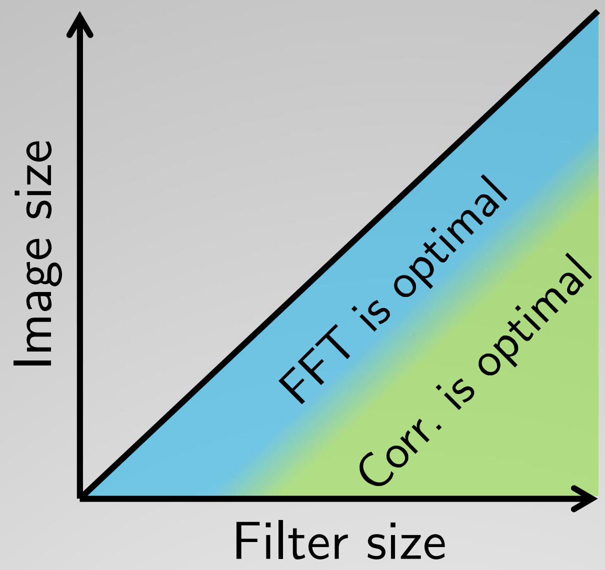
Practical considerations



- As a rule of thumb, **similarly sized** cross-correlation arguments (e.g. image and filter) take the **best** advantage of the FFT.

- Consider a $n \times n$ image and a $f \times f$ filter.
  - FFT complexity is $\mathcal{O}(n^2 \log n)$ (*independent* of f, big or small!).
  - Cross-correlation complexity is $\mathcal{O}(n^2 f^2)$ (better when $f \ll n$).

Practical considerations

- When performing FFTs, the "classic advice" is to set the image size to a power-of-two if possible:

$$\text{size}(\mathbf{x}) = 2^r \times 2^s, \quad \text{with integer } r, s.$$

- While this theoretically achieves the best speed, modern FFT libraries (such as FFTW) are **optimized for arbitrary sizes**.

- Rounding the size up to the next power-of-two has 2 drawbacks:

  - A mismatched size can **degrade** recognition performance (e.g. by including unnecessary background regions in a filter).

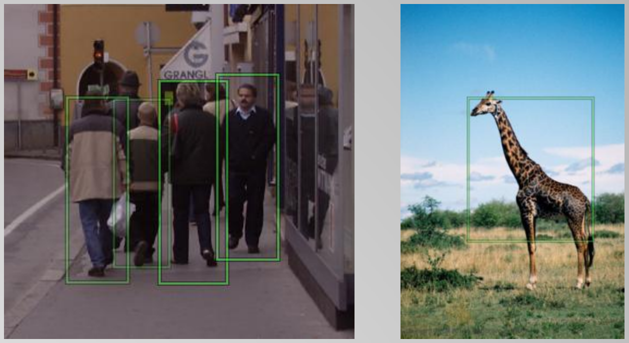
  - If the next power-of-two is significantly larger, we can end up with actually **slower** FFTs!

Other topics

Topics not covered here:

- Considering multiple samples and features simultaneously.
- Circulant trick for other algorithms (Support Vector Regression, etc).

ICCV'13 (example detections)



- Generalizing shifts to other transformations (rotations, etc).
- Fast training of classifier ensemble (pose estimator).

NIPS'14 (example pose estimates)