Universality of the Local Marginal Polytope

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Overview

1. Introduction to min-sum problem, its usage in computer vision.
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2. Linear programming (LP) relaxation of the problem.
1 Introduction to min-sum problem, its usage in computer vision.
2 Linear programming (LP) relaxation of the problem.
3 How hard is to solve the LP relaxation, what are fundamental limitations.
Min-sum problem
(a.k.a. MAP inference in graphical models or discrete energy minimization problem)

Pairwise min-sum problem with graph \((V, E)\) and label set \(K\):

\[
\min_{k \in K^V} \left[ \sum_{u \in V} f_u(k_u) + \sum_{\{u, v\} \in E} f_{uv}(k_u, k_v) \right].
\]

All weights \(f_u(k), f_{uv}(k, \ell) \in \mathbb{R} \cup \{\infty\}\) form a vector \(f\). Problem instance is defined by \((V, E, K, f)\).
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![Graph representation](image-url)
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Min-sum problem in computer vision

Segmentation

Stereo (correspondences)

Multiview reconstruction, surface fitting, shape matching, deconvolution, texture restoration, super resolution, ...
Complexity of min-sum problem

In general, $NP$-hard.

Certain classes of instances are tractable.

- min-sum problems on trees (restricting structure of graph)
- submodular min-sum problems (restricting weight functions $f$)
- ...
Linear programming relaxation of min-sum problem

LP relaxation = linear optimization over local marginal polytope:

\[
\langle f, \mu \rangle \rightarrow \min \sum_{k \in K} \mu_u(k) = 1, \quad u \in V
\]

\[
\sum_{\ell \in K} \mu_{uv}(k, \ell) = \mu_u(k), \quad \{u, v\} \in E, \quad k \in K
\]

\[
\mu \geq 0
\]

where in scalar product \( \langle f, \mu \rangle \) we define \( \infty \cdot 0 = 0 \).

Components \( \mu_u(k) \) and \( \mu_{uv}(k, \ell) \) of \( \mu \) are pseudomarginals.
Solving the LP relaxation

2 labels

- the optimal solution is half-integral (pseudomarginals in \( \{0, \frac{1}{2}, 1\} \))
- efficiently solvable by max-flow/min-cut algorithms \([\text{Boros & Hammer 1991}]\)
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▶ recently, other algorithms with linear space (using subgradients [Komodakis et al. 2010], bundle methods [Kappes et al. 2012], steepest descent methods [Schwing et al. 2012, 2014], etc). But these are considerably slower than message-passing.
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Is there a chance of inventing something better?
Reductions inside the class P

\[ X \leq_P Y \] (problem \(X\) is polynomial time reducible to problem \(Y\))

Assuming, \(X\) is a well known problem, what does it say about \(Y\)?
Why it can be difficult to design a special efficient algorithm for \(Y\)?
Reductions inside the class P

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Assuming, \( X \) is a well known problem, what does it say about \( Y \)? Why it can be difficult to design a special efficient algorithm for \( Y \)?

1. [stronger argument] Proposing a very fast algorithm for \( Y \) might result in a new, faster algorithm for \( X \).
Reductions inside the class $P$

$X \leq_P Y$ (problem $X$ is polynomial time reducible to problem $Y$)

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Why it can be difficult to design a special efficient algorithm for $Y$?

1. [stronger argument] Proposing a very fast algorithm for $Y$ might result in a new, faster algorithm for $X$.

2. [weaker argument] Proposing an algorithm for $Y$ might bring a new principle for solving $X$. 
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2. [weaker argument] Proposing an algorithm for \( Y \) might bring a new principle for solving \( X \).

In our case, \( X \) is general LP, \( Y \) is the LP relaxation of min-sum problem.
Simplex algorithm [Dantzig 1947]

- quadratic space, polynomial time not guaranteed
Linear programming - history

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Ellipsoid algorithm [Khachiyan 1979]
▶ first polynomial time algorithm for LP
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Karmarkar’s algorithm [Karmarkar 1984]
  ▶ interior point method
  ▶ fastest known algorithm for LP
    $\mathcal{O}(n^{3.5} L^2 \log L \log \log L)$
Main result

Theorem (Průša-Werner-CVPR2013)

Any linear program can be reduced in linear time to the LP relaxation of a pairwise min-sum problem with 3 labels.

Consequences:

▶ Finding an efficient algorithm to solve LP relaxation of min-sum problem might be as hard as improving the complexity of the best known algorithm for LP.

▶ LP relaxation of min-sum problem with 3+ labels is inherently more complex than for 2 labels.
Main result

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Elementary min-sum problems

The reduction is done by combining elementary min-sum problems.

- They perform simple operations on unary pseudomarginals.
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- Depicting a pair \( \{ u, v \} \in E \) with \( |K| = 3 \) labels:

  \[
  p + q + r = a \\
  a + b + c = 1
  \]
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- They perform simple operations on unary pseudomarginals.

- Depicting a pair \( \{u, v\} \in E \) with \( |K| = 3 \) labels:

\[
\begin{align*}
&\quad p + q + r = a \\
&\quad a + b + c = 1
\end{align*}
\]

- Visible edges have weights \( f_{uv}(k, \ell) = 0 \).
  Invisible edge have weights \( f_{uv}(k, \ell) = \infty \), implying \( \mu_{uv}(k, \ell) = 0 \).
Elementary min-sum problem

Precisely:
Given any feasible unary pseudomarginals \( a, b, c, d, e, f \), feasible pairwise pseudomarginals exist if and only if \( a = d \).
Elementary min-sum problem **Addition**

Enforces $c = a + b$. 
Enforces $a = b$. 
Elementary min-sum problem **EQUALITY**

Enforces $a = b$.

shorthand
Constructs unary pseudomarginals with values $2^i a$ for $i = 0, \ldots, d$, where $d$ is the depth of the problem.
Constructs unary pseudomarginals with values $2^{-i}$ for $i = 0, \ldots, d$. 
Example of combining elementary min-sum problems

Constructs a unary pseudomarginal with value $\frac{5}{8} = 5 \cdot 2^{-d}$. Similarly, we can construct any multiple of $2^{-d}$ (not greater than 1).
The input of the reduction is the LP

$$\min \{ \langle c, x \rangle \mid Ax = b, \ x \geq 0 \}$$

where $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^m$, $c \in \mathbb{Z}^n$, $m \leq n$. 
The input LP

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Before reduction, the system $$Ax = b$$ is rewritten as

$$A^+x = A^-x + b$$

where all entries of $$A^+, A^-, b$$ are non-negative and $$A = A^+ - A^-.$$
Lemma

Let \( \mathbf{x} \) be a vertex of the polyhedron \( \{ \mathbf{x} \mid A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq 0 \} \). Then each component \( x_j \) of \( \mathbf{x} \) satisfies either \( x_j = 0 \) or \( M^{-1} \leq x_j \leq M \), where

\[
M = m^{m/2}(B_1 \times \cdots \times B_{n+1})
\]

\[
B_j = \max\{1, |a_{1j}|, \ldots, |a_{mj}|\}, \quad j = 1, \ldots, n
\]

\[
B_{n+1} = \max\{1, |b_1|, \ldots |b_m|\}.
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Lemma

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Lemma

Let the polyhedron \( \{ \mathbf{x} \mid A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0} \} \) be bounded. Then for any $\mathbf{x}$ from the polyhedron, each component of $A^+\mathbf{x}$ and $A^-\mathbf{x} + \mathbf{b}$ is not greater than $N = M(B_1 + \cdots + B_{n+1})$. 
Initializing the reduction

The reduction algorithm:

- Its input is \((A, b, c)\), assuming w.l.o.g. that the polyhedron \(\{x \mid Ax = b, \ x \geq 0\}\) is bounded.

- Its output will be a min-sum problem \((V, E, K, f)\) with \(V = \{1, \ldots, |V|\}\) and \(K = \{1, 2, 3\}\).
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The algorithm is initialized as follows:

1. For each variable \(x_j\) in the input LP, introduce a new object \(j\) into \(V\) and set \(f_j(1) = c_j\).
   (Pseudomarginal \(\mu_j(1)\) will represent variable \(x_j\).)
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\(d_j = \lfloor \log_2 B_j \rfloor\) based on label 1.
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3. Build \textsc{NegPowers} with the depth \(d = \lceil \log_2 N \rceil\).
Encoding the equality constraints

Each equation

\[ a_{i1}^+ x_1 + \cdots + a_{in}^+ x_n = a_{i1}^- x_1 + \cdots + a_{in}^- x_n + b_i \]

of the system \( A^+ x = A^- x + b \) is encoded as follows:
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1. Construct pseudomarginals with values \( a_{ij}^+ x_j \) and \( a_{ij}^- x_j \) by summing selected values from the \( \text{Powers} \).

2. Construct a pseudomarginal with value \( 2 - d b_i \) by summing selected values from the \( \text{NegPowers} \). (The number \( 2 - d \) plays the role of the unit.)

3. Sum the terms on each side of the equation by repetitively applying \( \text{Addition} \) and \( \text{Copy} \).

4. Enforce equality of the two sides of the equation by \( \text{Copy} \).

Finally, set \( f_i(k) = 0 \) for all \( i \) or \( k \) > 1.
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Finally, set \( f_i(k) = 0 \) for all \( i > n \) or \( k > 1 \).
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Let $L$ be the number of bits of the binary representation of $(A, b, c)$. Want to prove that the reduction time is $O(L)$.
Complexity of the reduction

Let $L$ be the number of bits of the binary representation of $(A, b, c)$. Want to prove that the reduction time is $O(L)$.

This is easy:

▶ Let the output of the reduction be $(V, E, K, f)$.

▶ Clearly, the reduction time is $O(|E|)$.

▶ Clearly, $|E| = O(|V|)$.

▶ Thus we need to prove $|V| = O(L)$.

▶ For that, it suffices to prove that the numbers $d_j = \lceil \log_2 B_j \rceil$ and $d = \lceil \log_2 N \rceil$ are $O(L)$.
Corollary

Every polytope is (up to scale) a coordinate-erasing projection of a face of a local marginal polytope with 3 labels, whose description can be computed from the description of the original polytope in linear time.
**Corollary**

Every polytope is (up to scale) a coordinate-erasing projection of a face of a local marginal polytope with 3 labels, whose description can be computed from the description of the original polytope in linear time.

If only finite weights are allowed \((f_u(k), f_{uv}(k, \ell) \in \mathbb{R})\) then:

**Theorem**

Any linear program reduces in time and space \(O(L^2)\) to a linear optimization over a local marginal polytope with 3 labels.
Planar graphs

Vision applications usually induce sparse, planar graphs (like grids).

Is it possible to reduce every LP to a min-sum problem with the underlying planar graph?

Theorem (Průša-Werner-PAMI2014)

Every LP reduces to a linear optimization (with infinite costs) over a local marginal polytope with 3 labels over a planar graph. The size of the output and the reduction time are $O(mL)$.
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Planar graphs – eliminating one edge crossing
Reduction to a grid

Theorem (Tamassia 1989)

Any planar graph $G = (V, E)$ with maximal node degree 4 can be embedded in linear time into a grid with the area $\mathcal{O}(|V|^2)$.

All degrees of nodes in a planar min-sum problem can be reduced to 3 (for a chosen node, create its copies and distribute incident edges among them).

