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A Fast Algorithm for Confidently Stable Matching

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Abstract

An efficient $O(d^2 n)$ algorithm for confidently stable matching for general inhibition zones and an $O(dn \log n)$ algorithm for X-zones are proposed and proved, where n is matching table size and d is inhibition zone depth.

1 Introduction

This report deals with the problem of finding Confidently Stable Matching. Confidently stable matching is a special case of $\sigma\delta$-stable matching discussed in [1].[1] Special properties of confidently stable matching allow for the existence of an efficient algorithm whose implementation runs significantly faster than the general $\sigma\delta$-stable matching algorithm, although it is of the same asymptotic complexity.

We do not review any of the basic concepts or results needed to understand this report and we do not motivate our effort to find confidently stable matching here. We assume the reader knows our previous work [1].

Some of the concepts in this paper may be formulated in a slightly different way or have more precise meaning. If there is any difference from [1], the version which is reported here is the valid one.

2 Confidently Stable Matching

Definition 1 (Matching Problem). *Let P be a matching table, let $c\colon P \to \mathbb{R}$ and $\Delta\colon P \to \mathbb{R}^{+0}$ be bounded functions such that $\big[c(p) - \Delta(p),\, c(p)\big]$ is the confidence interval associated with signature similarity for pair $p \in P$, and let $Z(p)$ be symmetric inhibition zone for each $p \in P$. Then (P, c, Δ, Z) will be called a matching problem.*

Definition 2 (Confident Stability). *Let (P, c, Δ, Z) be a matching problem. A subset $S \subseteq P$ is confidently stable if for each $p \in S$ and every $q \in Z(p)$ the following holds:*

$$\text{If } c(q) \geq c(p) - \Delta(p) \text{ then there must be } r \in S \cap Z(q) \text{ such that } c(r) > c(q) + \Delta(r). \quad (1)$$

[1] We refer to version 1.6 of [1] or newer.

Observation 1. *If $X \subseteq Z$ then S is a matching. If $F \subseteq Z$ then S is monotonic.*

Some discussion related to this observation is given in [1, Sec. 1].

Definition 3 (Completeness). *A subset $M \subseteq P$ is complete iff for each pair $p \notin M$ it holds that the intersection $Z(p) \cap M$ is non-empty.*

Lemma 1. *Let (P, c, Δ, Z) be a matching problem. If $\Delta(p) \equiv 0$ for all pairs p in confidently stable subset $S \subseteq P$ and no two discordant pairs are of equal cost $c(\cdot)$ then S is complete.*

In general, the converse is not true (S can be complete even if $\Delta \not\equiv 0$). The proof of this lemma is given in [1, Lemma 1].

3 Existence and Uniqueness

We prove the following theorem:

Theorem 1. *For any given matching problem (P, c, Δ, Z) there is a unique (possibly empty) confidently stable subset $S \subseteq P$ of maximum cardinality.*

The theorem is proven by defining quasi-stable subset $M \subseteq P$ which is not unique but always exists and is always complete and by showing that any confidently stable subset S must be a subset of some quasi-stable subset M. Then the observation is made that any two complete quasi-stable subsets M differ in equal-cost discordant pairs only. But such pairs cannot be in S, which means S is a subset of the intersection of all quasi-stable subsets of P. It is then shown that if there are two confidently stable subsets in P they must be nested. By induction one gets there is a unique confidently stable matching of maximum cardinality, which completes the proof. The details of the proof follow.

We first define quasi-stable matching. It ignores Δ and weakens the second inequality:

Definition 4 (Quasi-Stable Subset). *Let (P, c, Δ, Z) be a matching problem. Subset $M \subseteq P$ is quasi-stable if for each $p \in M$ and every $q \in Z(p)$ the following holds:*

$$\text{If } c(q) \geq c(p) \text{ then there must be } r \in S \cap Z(q) \text{ such that } c(r) \geq c(q). \tag{2}$$

We will be interested in the largest (maximum-cardinality) quasi-stable subset M on (P, c, Δ, Z); there may be more than one. Some subsets of such M are also stable but they are not complete.

Lemma 2. *There is always at least one complete non-empty quasi-stable subset for any non-empty matching problem (P, c, Δ, Z).*

Proof. The proof is algorithmic and is given in [1, Theorem 1]. □

Lemma 3 (Weak Existence). *Let (P, c, Δ, Z) be a matching problem. Then any confidently stable subset $S \subseteq P$ is a (possibly empty) subset of some complete quasi-stable subset $M \subseteq P$.*

Proof. Let M be some complete quasi-stable subset on (P, c, Δ, Z) and let S be confidently stable subset on (P, c, Δ, Z). We prove the lemma by contradiction. Let $s_0 \in S$ and $s_0 \notin M$. From the assumption that $s_0 \notin M$ and from completeness of M it follows that there is a pair $t_0 \in Z(s_0)$ such that $t_0 \in M$. There are three mutually exclusive cases:

1. $c(t_0) > c(s_0)$. The t_0 must satisfy (1) so there must be $s_1 \in Z(t_0) \cap S$ such that $c(s_1) > c(t_0)$. Since $t_0 \in M$ it holds that $s_1 \notin M$ since $s_1 \in Z(t_0)$. In order for t_0 to be in M there must exist $t_1 \in Z(s_1) \cap M$ such that $c(t_1) \geq c(s_1)$. Since $\Delta \geq 0$ the argument continues along a non-decreasing path until
   (a) There is no such t_i. Then t_{i-1} is not stable which is a contradiction.
   (b) There is no such s_{i+1}. Then s_i is not confidently stable which is a contradiction.
2. $c(t_0) < c(s_0)$. Then, since $t_0 \in M$, there must be $t_1 \in Z(s_0) \cap M$ such that $c(t_1) \geq c(s_0)$. Since s_0 is confidently stable there must be $s_1 \in Z(t_1) \cap S$ such that $c(s_1) > c(t_1) + \Delta(s_1) \geq c(t_1)$. The argument continues the same way as in the previous case and also leads to contradiction.
3. $c(t_0) = c(s_0)$. Then, since s_0 is confidently stable there must be $s'_0 \in Z(t_0) \cap S$ such that $c(s'_0) > c(s_0)$ and we thus have Case 2. □

Lemma 4. *Let M_1, M_2 be two different complete quasi-stable matchings on (P, c, Δ, Z). Let $p_1 \in M_1$. Then for every $p_2 \in Z(p_1) \cap M_2$ it holds that $c(p_2) = c(p_1)$.*

Proof. From completeness of M_1 it follows that any $p_2 \notin M_1$ must have a discordant pair in M_1. The statement then requires two conditions simultaneously hold for each $p_1 \in M_1$, $p_2 \in M_2$, $p_1 \neq p_2$:

$$c(p_2) < c(p_1) \text{ or } c(p_2) \geq c(p_1) \text{ and } \exists q_1 \in M_1 \text{ such that } c(q_1) \geq c(p_2), \tag{3}$$
$$c(p_2) > c(p_1) \text{ or } c(p_2) \leq c(p_1) \text{ and } \exists q_2 \in M_2 \text{ such that } c(q_2) \leq c(p_1). \tag{4}$$

It follows that $c(p_1) = c(p_2)$. □

Lemma 5 (Uniqueness). *Let S_1 and S_2 be two confidently stable matchings for a given matching problem (P, c, Δ, Z). Let $|S_1| \leq |S_2|$. Then $S_1 \subseteq S_2$. Moreover, if $|S_1| = |S_2|$ then $S_1 = S_2$.*

Proof. By Lemma 3 every confidently stable subset S must be a subset of some quasi-stable matching on (P, c, Δ, Z). Suppose $S_1 \subseteq M_1$ and $S_2 \subseteq M_2$, where M_1, M_2 are two complete quasi-stable matchings on (P, c, Δ, Z). Let $|S_1| \leq |S_2|$. First, by Lemma 4 and by completeness, any two quasi-stable subsets of P are either nested or they differ in equal-cost pairs only. Second, it follows that all pairs $q_2 \in Z(p_1) \cap S_2$ have $c(q_2) = c(p_1)$, therefore p_2 is not confidently stable in S_2 for any choice of Δ. A similar observation holds for p_2. It follows that the only possibility left is that $S_1 \subseteq S_2$ $(\subseteq M_1 \cap M_2)$. □

The corollary of Lemmas 3, 4, and 5 is that confidently stable matching is a subset of the intersection of all quasi-stable subsets of P. In addition, by Lemma 5, all such subsets are mutually nested. This means there is a unique confidently stable subset of the largest cardinality. This completes the proof of Theorem 1.

4 Algorithm

Algorithm 1 (Confidently Stable Matching).

Input: *Confidently stable matching problem* (P, c, Δ, Z).

Output: *Confidently stable subset* S.

Working data structures:

U *the set of potentially converting pairs (the union* $\bigcup_{p \in M} Z(p)$*),*
R *the set of potentially converted pairs (the set* $R \subseteq P$ *such that* $S \cap R = \emptyset$*),*
M *the quasi-stable subset,*
$\tilde{c}(p)$ *current similarity value for each* $p \in P$.

Procedure:

1. *Initialize* $U := \emptyset$, $M := \emptyset$, $S := \emptyset$, *for all* $p \in P$ *let* $\tilde{c}(p) := c(p)$.
2. *If* P *is empty, terminate.*
3. *Remove the subset* $L \subseteq P$ *of pairs that have the largest cost* $\tilde{c}(\cdot)$ *in* P.
4. *For all* $r \in L$ *do:*
   (a) *if* $r \in U$ *then*
      i. $R := R \cup Z(r)$
   (b) *else if* $r \notin M$ *then*
      i. $\tilde{c}(r) := c(r) - \Delta(r)$,
      ii. $P := P \cup \{r\}$,
      iii. $M := M \cup \{r\}$,
      iv. $U := U \cup Z(r)$.
5. *For all* $r \in (L \cap M) \setminus R$ *do:*
   (a) $S := S \cup \{r\}$,
   (b) $P := P \setminus (Z(r) \cup \{r\})$.
6. *Go to Step 2.*

Comment It is essential Steps 4 and 5 to be kept separate, otherwise S would not be confidently stable because of equal-cost pairs. All pairs from $L \cap U$ must be processed before other pairs are considered.

5 Algorithm Correctness

Theorem 2 (Alg. 1 Correctness). *For any matching problem* (P, c, Δ, Z) *Alg. 1 terminates and finds the confidently stable matching of maximum cardinality.*

In the proof one first notices the difference between confidently stable matching S and quasi-stable matching M is that S contains no pairs in inhibition zones of *converting pairs* $C \subseteq P \setminus M$. One then shows there is a strict partial ordering on C for any Δ that allows us to uniquely identify C in P. One shows that the set M in the algorithm is quasi-stable matching on (P, c, Δ, Z). The statement of the theorem follows as a corollary. Details of the proof follow.

First, we will be interested in converting complete quasi-stable matching M to confidently stable matching S by removing some pairs from M. Pair q participates in the conversion by removing a pair $s \in M \cap Z(q)$ if $c(q) \geq c(s) - \Delta(s)$ and there is no pair $p \in S \cap Z(q)$ such that $c(p) > c(q) + \Delta(p)$. Note it follows that $q \notin M$ and $p \neq s$.

Let M be a quasi-stable matching on (P, c, Δ, Z). Let $q \in P$. We introduce two sets $T(q)$ and $D(q)$ as follows:

$$T(q) = \{t \mid t \in Z(q),\ c(t) - \Delta(t) > c(q)\}\,, \tag{5}$$
$$D(q) = \{d \mid d \in Z(q),\ c(d) - \Delta(d) \leq c(q)\}\,. \tag{6}$$

Note that if $q \in M$ then $T(q) \cap M = D(q) \cap M = \emptyset$. We will now state the converting pair condition precisely:

Observation 2. *Let (P, c, Δ, Z) be a matching problem. Let S be maximum-cardinality confidently stable subset of P. Let $C \subseteq P$ be the set of pairs that convert maximum-cardinality quasi-stable matching M to S. Then, for each $q \in C$ it holds that $T(q) \cap S = \emptyset$ and $D(q) \nsubseteq S$.*

Lemma 6. *Let (P, c, Δ, Z) be a matching problem and let C be the set of converting pairs. Then there is a strict partial ordering on C for any choice of Δ.*

Proof. Let $r \in P \setminus M$ be such that $T(r) \cap M = \emptyset$, where M is maximum cardinality quasi-stable matching. Then $r \in C$. It is easy to see that if there is no such r then $S = M$. Suppose there is a pair $r \in C$ such that $T(r) \cap M \neq \emptyset$. Then $r \in C$ iff every $t_i \in T(r) \cap M$ is in inhibition zone $Z(r'_i)$ of some pair $r'_i \in C$, $r'_i \neq r$. Then, if $r'_i \in C$, it must holds that $c(r'_i) > c(r)$, since, by (5), $c(t_i) - \Delta(t_i) > c(r)$ and, by (6), $c(t_i) - \Delta(t_i) \leq c(r'_i)$. □

We say r is a descendant of all pairs r'_i. The descendance relation is a strict partial ordering. Pairs $r \in C$ such that $T(r) \cap M = \emptyset$ will be called *root converting pairs.* Therefore, every $r \in C$ must either be a root converting pair or a descendant of at least one root converting pair. Each $r \in C$ may remove some pairs from M but it removes no other pairs than those violating condition (1). It follows the result of successive pair removal leaves us with the *largest* confidently stable subset S.

Alg. 1 proceeds by first finding the root converting pairs r_i and removing $\bigcup_i D(r_i)$ from M. Then it identifies the descendants of the root pairs and repeats the process. The strict partial ordering implies there is no freedom left in determining the descendants. The algorithm thus finds a *unique* S.

To identify the converting pairs correctly the following must hold:

Lemma 7. *The set M in Alg. 1 is always a quasi-stable matching.*

The statement follows from the nesting property of quasi-stable matching proven in [1].

To summarize, the algorithm constructs a quasi-stable subset M in the order of decreasing cost c and, simultaneously, it identifies converting pairs in the order of their decreasing cost c. At the time a pair r is about to be identified as converting in Step 4(a)i, the set $Z(r) \cap M$ is known up to some pairs p such that $c(p) - \Delta(p) = c(r)$. Such pairs p play no role in identifying r as converting, cf. (5). Then, if $r \in U$ in Step 4, it must satisfy the following condition:

$$\forall t \in Z(r)\colon \begin{cases} c(t) - \Delta(t) \leq c(r) & \text{if } t \in S, \\ c(t) \leq c(r) & \text{otherwise.} \end{cases} \tag{7}$$

This means $T(r) \cap M = \emptyset$ and r is thus a converting pair.

The last thing to show is that the Step 5 is correct: Suppose $r \notin R$. Then there is no converting pair in $Z(r)$ since for all $p \in Z(r)$ it holds that either $p \notin U$ or $c(p) < c(r) - \Delta(r)$. Thus, (1) $r \in S$ and (2) $Z(r) \cup \{r\}$ can be removed from P.

6 Time Complexity

We say an inhibition zone $Z(p)$ is of depth d if $p = (i, j)$ and that for each $q \in Z(p)$, $q = (k, l)$ it holds that $D(p, q) = max(|k - i|, |l - j|) \leq d$, where $D(p, q)$ is the relative disparity of p and q.

Let d be inhibition zone depth, then $d = O(|M|)$, where $|M|$ is the number of pairs in complete quasi-stable matching M.

Let all tests of the type $r \in Q$, where $Q = \{R, U, M\}$ take $O(d)$ time and let the update of the type $Q := Q \cup Z(r)$ also takes $O(d)$ time. Then

- The pre-sorting of P takes $O(|P| \log |P|)$ time.
- Step 4(a)i is executed $O(|P|)$ times and each execution takes $O(d)$ time,
- Steps 4(b)i– 4(b)iii is executed at most $O(|M|)$ times, each execution takes $O(d)$ time,
- Steps 5a and 5b are executed $O(|M|)$ times, each execution takes $O(d)$ time.

Then the overall time complexity is $O(d \cdot |P|)$, where $|P|$ is the initial number of pairs in P. If $|P| \sim n \times n$ is a fully populated matching table, it is $O(d^2 n)$.

It is easy to make the tests $r \in Q$ in $O(\log n)$ time or better. If we do this, the bottleneck of the algorithm becomes Step 4(a)i. By reducing the complexity of this step to $O(\log n)$ we would get an $O(|P| \log |P|)$ algorithm.

7 Implementation Remarks

For the analysis given in this section we assume the matching table $P \sim m \times n$ is populated by matching candidates within disparity range d, where $d \leq m \leq n$, without any loss of generality.

We will discuss two cases: matchings with X inhibition zone (X-stable matching) and FX inhibition zone (FX-stable matching). They are the two most useful cases in computer vision: representing solutions to ordinary matching and matching under ordering constraint, respectively.

7.1 X-Stable Matching

In X-stable matching problem it is possible to make all tests $r \in Q$ and all updates of the type $Q := Q \cup Z(p)$ or $Q := Q \cup Z(p) \cup \{p\}$, where $Q = \{R, U, M\}$, to take $O(1)$ time. This is sufficient to make the X-stable matching algorithm of $O(d\,n \log(n))$ complexity.

To represent a union of X-zones one needs two logical arrays $\texttt{A}[1 : m]$ and $\texttt{B}[1 : n]$ to hold a flag and one integer array $\texttt{C[1 : m]}$ to represent pairs $p = (k, \texttt{C}[k])$ left out when deleting $X(p)$. The array C is initially set to zeros. The A represents columns and the B represents rows of matching table that are in Q. The flags are initially set to zero and are set to unity in one of the update operations $Q{:=}Q{\cup}Z(p)$ or $Q{:=}Q{\cup}Z(p){\cup}\{p\}$. For a pair $p = (i, j)$, the update operation $Q{:=}Q{\cup}Z(p)$ requires three steps $\texttt{A}[i]{:=}1;\ \texttt{B}[j]{:=}1;\ \texttt{C}[i]{:=}j$, and the update operation $Q := Q \cup Z(p) \cup \{p\}$ requires only two steps $\texttt{A}[i] := 1;\ \texttt{B}[j] := 1$. The test of the type $p \in Q$ is done as follows: $\texttt{C}[i] \neq j \wedge (\texttt{A}[i] \vee \texttt{B}[j])$ where $\vee$ and $\wedge$ are logical or and logical and operation, respectively.

The expected time complexity of the algorithm can be significantly improved when heaps are used to represent the sorted matching table. Each column of the table is represented as a max-rooted heap $\texttt{ColHeap}[1 : n]$ and the whole table is represented as a max-rooted heap H, whose elements are the column heaps. The keys of the column heaps are their root values. Each of the column heaps $\texttt{ColHeap}[1 : n]$ is initialized in $O(d)$ time and the heap H is initialized in $O(n)$ time, since we know the number of heap elements beforehand [2]. We thus need $O(n\,d)$ time for initialization.

Extracting the largest value from P is equivalent to extracting the root in $\texttt{ColHeap}[j]$ of the root heap in H. If the column of P becomes a subset of the union $\bigcup_{p \in S} X(p)$, we remove the root element in H, which is done in $O(\log n)$ time; otherwise we just decrease its root key to a new value, which is also done in $O(\log n)$ time.

If the solution to the confidently X-stable matching problem is Δ-dominant,[2] the speedup is the most significant. In this case the algorithm runs in just $O(n\,d)$ time (strongly dominated by the initialization). Thus the algorithm becomes progressively slower as the uncertainty in data increases, but its run time never exceeds the upper bound of $O(d\,n \log n)$.

[2] See [1] for the definition of dominance which easily generalizes to Δ-dominance.

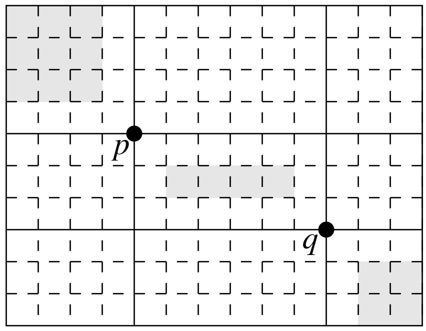

Figure 1: The complement in P of the intersection $FX(p) \cap FX(q)$ is blockwise ordered.

7.2 FX-Stable Matching

In FX-stable matching we make use of the following observation:

Observation 3. *The intersection of complements to FX zones in P*

$$\bigcap_{p \in T} (P \setminus FX(p))$$

is blockwise ordered for any subset $T \subseteq P$ (see Fig. 1).

This allows us to do two spedups:

1. do the update $Q := Q \cup FX(p)$ in $O(d)$ time,
2. recognize if j-th column of P is a subset of Q in $O(1)$ time,
3. recognize if $FX(p) \subseteq Q$ in $O(1)$ time.

The second property allows us to do the test $p \in Q$ in $O(1)$ time. The overall complexity of FX-stable matching algorithm is thus $O(d^2 n)$. The third property helps reduce the average time complexity of the $Q := Q \cup FX(p)$ update. Further average time complexity reduction is possible using heaps as in X-stable matching.

8 Ambiguity Detection

Confidently stable matching automatically identifies zones in matching table that are ambiguous. Let S be confidently stable matching found by Alg. 1. Then the ambiguous subset of P is the union of pairs $p \notin S$ such that $Z(p) \cap S = \emptyset$. If $Z = FX$ the ambiguous subset is blockwise ordered.

Due to this property of confidently stable matching we can classify image features (subsets) that are matched into three classes:

1. unambiguously matched,
2. unambiguously half-occluded,

3. ambiguous.

The half-occlusion can be due to some object obscuring the feature in one of the images or due to non-overlapping fields of view of the cameras.

Confidently stable matching thus uses all unambiguous image information to establish the reliable set of matches and identifies the subset of matching table where strong regularization approach has to be used to diambiguate the matching.

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