Principles of Solving Minimal Problems

T o m a s P a j d l a

The Art of Solving Minimal Problems

ICCV 2015 Tutorial

cmp.felk.cvut.cz/iccv-2015-minimal

Czech Technical University Prague
Center for Machine Perception

SOLVING MINIMAL PROBLEMS

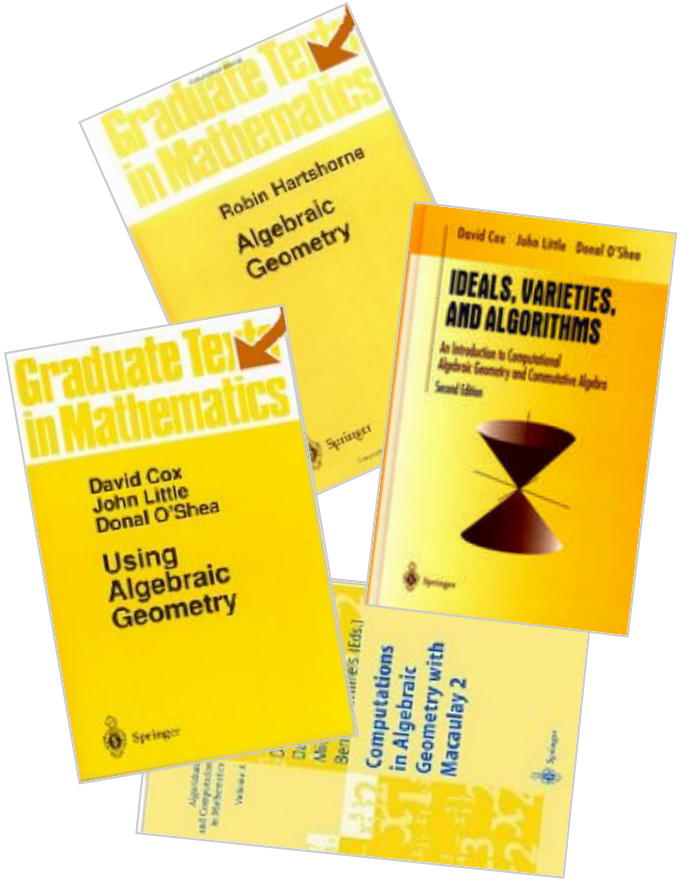
Minimal problem:

use as few data as to generate a system of algebraic equations with a finite number of solutions

1. Problem formulation $\rightarrow$ algebraic equations

2. Solving algebraic equations

SOLVING ALGEBRAIC EQUATIONS



!?

Quotient ring

1+1=?

action matrix

field

ideal

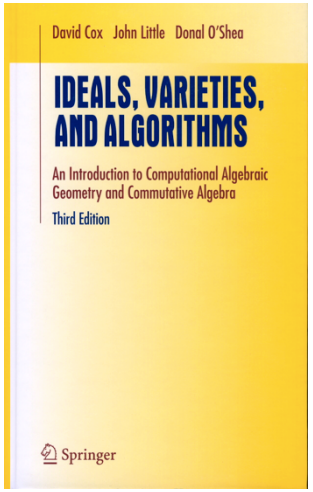
$xy^2+7y+..$

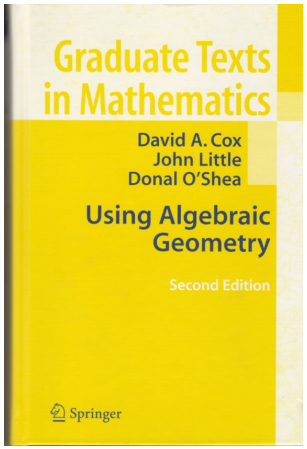
LOOKS AS "the MATHEMATICS" !!!

LET'S TAKE AN ENGINEERING APPROACH

pajdla@cmp.felk.cvut.cz

LITERATURE





Mathematics

1. Cox et al. Ideal Varieties and Algorithm. Springer 2015.
   (An Introduction to Algebraic Geometry … New edition!)
2. Cox et al. Using Algebraic Geometry. Springer 1998.
   (More advanced Algebraic Geometry)
3. F. Kubler, P. Renner, K. Schmedders.
   Computing All Solutions to Polynomial Equations in Economics
   (Very lightweight intro to polynomial system solving).

Minimal problem papers

cmp.felk.cvut.cz/minimal

SINGLE UNKNOWN → EIGENVALUES

1 equation, 1 variable → companion matrix → eigenvalues

$$f(x) = x^3 + 4\,x^2 + x - 6 \;= -6 + 1\,x + 4\,x^2 + 1\,x^3$$

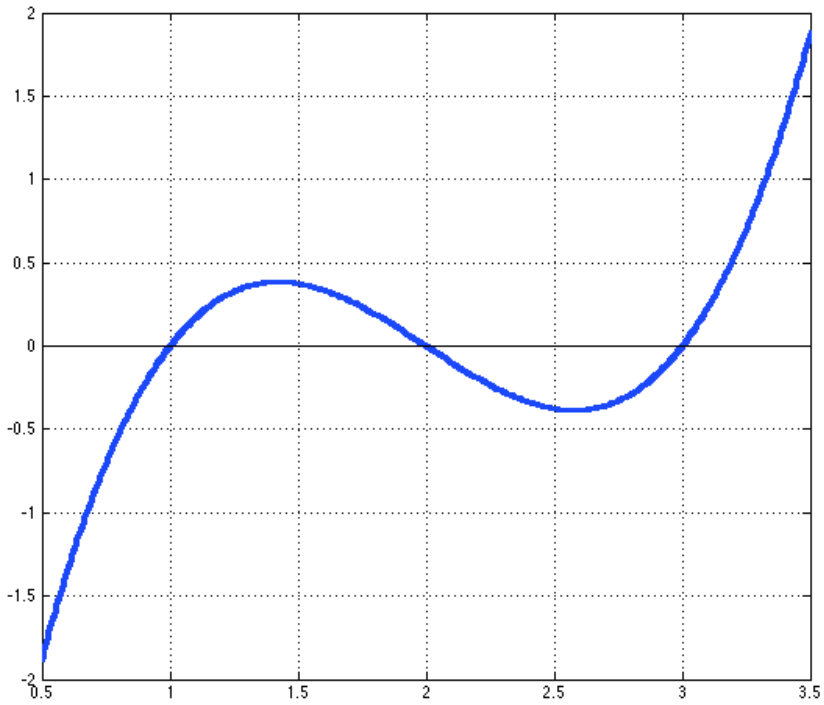
$$\mathtt{M}_x = \begin{bmatrix} 0 & 0 & 6 \\ 1 & 0 & -1 \\ 0 & 1 & -4 \end{bmatrix}$$

... a simple rule

```
>> e=eig(M_x)
```

$$\mathtt{e} = \begin{bmatrix} 1 \\ -2 \\ -3 \end{bmatrix} \qquad x_1 = 1,\; x_2 = -2,\; x_3 = -3$$

Towards the MULTIPLICATION MATRIX $\mathtt{M}_x$



a polynomial

$$\begin{aligned} f(x) &= (x-1)\,(x-2)\,(x-3) \\ &= x^3 - 6\,x^2 + 11\,x - 6 \end{aligned}$$

with roots

$$x_1 = 1,\ x_2 = 2,\ x_3 = 3$$

i.e.

$$f(x):\ \ f(x_1) = f(x_2) = f(x_3) = 0$$

REMAINDERS are a linear space

$r(x) = h(x) \bmod f(x)$ ... remainder on division by $f(x)$

$$h(x) = q(x)\, f(x) + r(x)$$

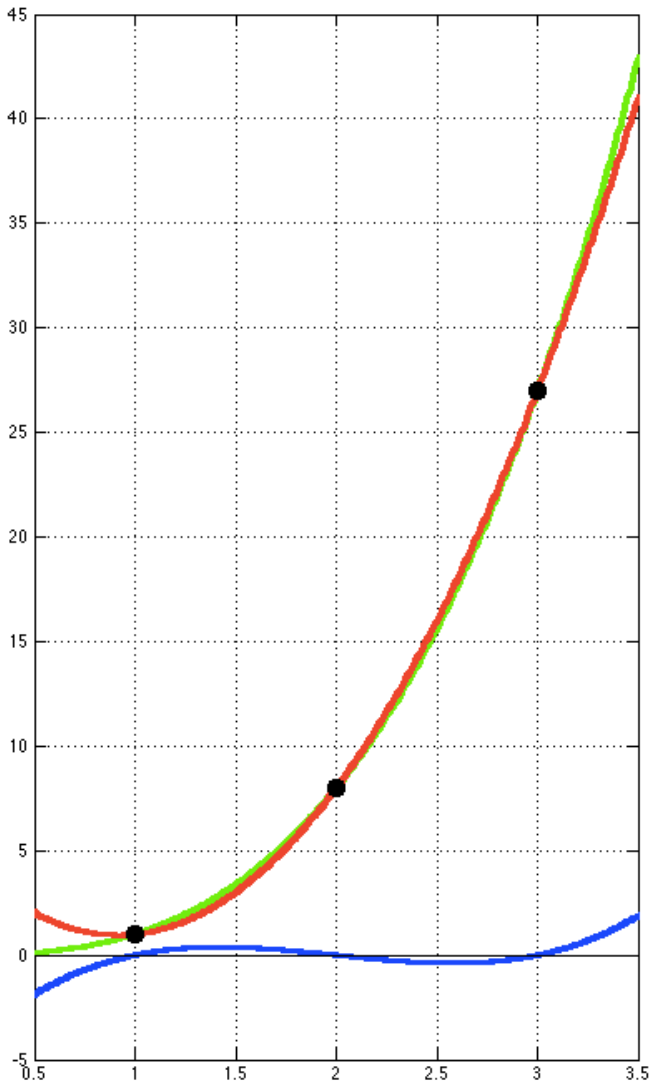
such that $\deg r(x) < \deg f(x)$ or $r(x) = 0$

- remainders have low degrees
- can be represented by finite vectors

e.g. for $f(x) = x^3 - 6\,x^2 + 11\,x - 6$ deg of $r(x)$ is smaller than 3:

$$r(x) = a_2\, x^2 + a_1\, x + a_0 \equiv \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} \in \mathbb{R}^3$$

REMAINDERS evaluate



Evaluate a general polynomial $h(x)$ on

roots of $f(x)$: $x_1 = 1,\ x_2 = 2,\ x_3 = 3$

using the the reminder $r(x)$

$$\begin{aligned} h(x_i) &= q(x_i)\, f(x_i) + r(x_i) \\ &= q(x_i)\, 0 + r(x_i) = r(x_i) \end{aligned}$$

$$h(x_i) = r(x_i) \quad i = 1, 2, 3$$

Example

$h(x) = x^3$ $\qquad$ $f(x) = x^3 - 6\,x^2 + 11\,x - 6$

$$h(x) = 1\, f(x) + (6\,x^2 - 11\,x + 6)$$

$$r(x) = 6\,x^2 - 11\,x + 6$$

MAPPING by MULTIPLICATION

Consider a mapping $\mathtt{M}$ of polynomials to polynomials generated by multiplication by x

$$h(x) \rightarrow x\,h(x)$$

It generates a mapping on reminders on division by $f(x)$

$$h(x) \bmod f(x) \xrightarrow{\mathtt{M}} (x\,h(x)) \bmod f(x)$$

$$a_2x^2 + a_1x_+a_0 \;\rightarrow\; b_2x^2 + b_1x + b_0$$

… can be seen as a mapping in the linear space of vectors of coefficients

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} \xrightarrow{\mathtt{M}} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} \quad \ldots \quad \mathtt{M}\colon \mathbb{R}^3 \rightarrow \mathbb{R}^3$$

… IS LINEAR

$\mathtt{M}\colon \mathbb{R}^3 \to \mathbb{R}^3 \qquad h(x) \bmod f(x) \xrightarrow{\mathtt{M}} (x\,h(x)) \bmod f(x)$

is a LINEAR MAPPING! … can be represented by a matrix

Example $\quad f(x) = x^3 - 6\,x^2 + 11\,x - 6$

$$\begin{aligned}
\mathtt{M}(1) &= x\,1 = x \\
\mathtt{M}(x) &= x\,x = x^2 \\
\mathtt{M}(x^2) &= x^3 \bmod f(x) \\
&= 6\,x^2 - 11\,x + 6
\end{aligned}$$

$$\begin{bmatrix}1\\0\\0\end{bmatrix} \xrightarrow{\mathtt{M}} \begin{bmatrix}0\\1\\0\end{bmatrix}, \quad \begin{bmatrix}0\\1\\0\end{bmatrix} \xrightarrow{\mathtt{M}} \begin{bmatrix}0\\0\\1\end{bmatrix}, \quad \begin{bmatrix}0\\0\\1\end{bmatrix} \xrightarrow{\mathtt{M}} \begin{bmatrix}6\\-11\\6\end{bmatrix}$$

$$\begin{bmatrix}0 & 0 & 6\\1 & 0 & -11\\0 & 1 & 6\end{bmatrix} = \begin{bmatrix}0 & 0 & 6\\1 & 0 & -11\\0 & 1 & 6\end{bmatrix} \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

$$\begin{bmatrix}a_0\\a_1\\a_2\end{bmatrix} \xrightarrow{\mathtt{M}} \begin{bmatrix}b_0\\b_1\\b_2\end{bmatrix}$$

$$\begin{bmatrix}b_0\\b_1\\b_2\end{bmatrix} = \mathtt{M}_x \begin{bmatrix}a_0\\a_1\\a_2\end{bmatrix}$$

pajdla@cmp.felk.cvut.cz

FINIS CORONAT OPUS

Putting it all together

Evaluate reminder

$$r(x_i) = a_2 x_i^2 + a_1 x_i + a_0$$

and its image

$$x_i r(x_i) = b_2 x_i^2 + b_1 x_i + b_0$$

$$(x_i a_2)\, x_i^2 + (x_i a_1)\, x_i + (x_i a_0) = b_2 x_i^2 + b_1 x_i + b_0$$

For all generic roots →

- use monic
- use LA

$$x_i \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} x_i a_0 \\ x_i a_1 \\ x_i a_2 \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} = \mathtt{M}_x \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$$

$$\mathtt{M}_x \mathtt{a} = x_i \mathtt{a}$$

Roots of $f(x)$ are eigenvalues of $\mathtt{M}_x$

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SOLVING ALGEBRAIC EQUATIONS & GB

→ generalize remainder on division to
m equations & n vars

$$h(x,y,...) = q_1(x,y,\ldots)\, f_1(x,y,...) + q_2(x,y,...)\, f_2(x,y,...) + \cdots + r(x,y,...)$$

divisors remainder

For general $f_i(x,y,...)$ reminder $r(x,y,...)$ not well defined

… depends on the ordering of divisors!

→ Groebner basis … divisors for which remainder well defined
→ does not depend on the ordering of divisors

SOLVING ALGEBRAIC EQUATIONS & GB

Linear equations are algebraic equations

linear equations

$$\begin{aligned} 4\,x + 8\,y + 7\,z &= 0 \\ 6\,x + 3\,y + 2\,z &= 0 \\ 7\,x + 7\,y + 1\,z &= 0 \end{aligned}$$

Gaussian elimination

$$\begin{bmatrix} 4 & 8 & 7 \\ 6 & 3 & 2 \\ 7 & 7 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\begin{bmatrix} 4 & 8 & 7 \\ 0 & 36 & 34 \\ 0 & 0 & -167 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

Groebner basis

$$\begin{aligned} 4\,x + 8\,y + 7\,z &= 0 \\ 36\,y + 34\,z &= 0 \\ 167\,z &= 0 \end{aligned}$$

SOLVING ALGEBRAIC EQUATIONS & GB

m equations, n variables (an example for m = 2 & n = 2)

$$
\begin{aligned}
0 &= 25\,x\,y - 20\,y - 15\,x + 12 \\
0 &= y^2 + x^2 - 1
\end{aligned}
$$

$\rightarrow$ Groebner basis

(polynomials with the same solutions but easy to solve

lexicographical ordering of monomials)

$$
\begin{aligned}
f_1 \;\rightarrow\; 0 &= 25\,x\,y - 20\,y - 15\,x + 12 \\
f_2 \;\rightarrow\; 0 &= y^2 + x^2 - 1 \\
0 &= 125\,y^3 - 75\,y^2 + 27 - 45\,y \qquad \leftarrow \text{ how to get it ?} \\
&= (-5\,x - 4)\,f_1 + (125\,y - 75)\,f_2
\end{aligned}
$$

... a polynomial combination of f_1, f_2

MATRIX FORM OF POLYNOMIALS (F4-Like Approach)

$$
\begin{aligned}
f_1 &= 25\,x\,y - 20\,y - 15\,x + 12 \\
f_2 &= y^2 + x^2 - 1 \\
f_3 &= (-5\,x - 4)\,f_1 + (125\,y - 75)\,f_2
\end{aligned}
$$

$$
\begin{array}{l}
f_1 \rightarrow \\
f_2 \rightarrow
\end{array}
\begin{bmatrix}
0 & 25 & -15 & 0 & -20 & 12 \\
1 & 0 & 0 & 1 & 0 & -1
\end{bmatrix}
\begin{bmatrix}
x^2 \\ x\,y \\ x \\ y^2 \\ y \\ 1
\end{bmatrix}
$$

ADDING MULTIPLES

$$
\begin{aligned}
f_1 &= 25\,x\,y - 20\,y - 15\,x + 12 \\
f_2 &= y^2 + x^2 - 1 \\
f_3 &= (-5\,x - 4)\,f_1 + (125\,y - 75)\,f_2 = a\,\boxed{x\,f_1} + b\,\boxed{y\,f_2} + c
\end{aligned}
$$

$$
\begin{array}{rcl}
f_1 & \rightarrow & \\
f_2 & \rightarrow & \\
x\,f_1 & \rightarrow & \\
y\,f_2 & \rightarrow &
\end{array}
\begin{bmatrix}
0 & 0 & 25 & -15 & 0 & 0 & -20 & 12 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & -1 \\
25 & -15 & -20 & 12 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & -1 & 0
\end{bmatrix}
\begin{bmatrix}
\boxed{x^2\,y} \\
x^2 \\
x\,y \\
x \\
\boxed{y^3} \\
y^2 \\
y \\
1
\end{bmatrix}
$$

GAUSSIAN ELIMINATION

$$
\begin{aligned}
f_1 &= 25\,x\,y - 20\,y - 15\,x + 12 \\
f_2 &= y^2 + x^2 - 1 \\
f_3 &= (-5\,x - 4)\,f_1 + (125\,y - 75)\,f_2 = a\,x\,f_1 + b\,y\,f_2 + c \\
&= 125\,y^3 - 75\,y^2 - 45\,y + 27
\end{aligned}
$$

$$
\begin{bmatrix}
5 & -3 & -4 & \frac{12}{5} & 0 & 0 & 0 & 0 \\
0 & 5 & 0 & 0 & 0 & 5 & 0 & -5 \\
0 & 0 & 125 & -75 & 0 & 0 & -100 & 60 \\
0 & 0 & 0 & 0 & 125 & -75 & -45 & 27
\end{bmatrix}
\begin{bmatrix}
x^2\,y \\ x^2 \\ x\,y \\ x \\ y^3 \\ y^2 \\ y \\ 1
\end{bmatrix}
$$

Gaussian elimination

SOLVING POLYNOMIAL EQUATIONS BY CONSTRUCTING GROEBNER BASIS

A generalization of the Gaussian elimination

multiplication by scalars

↓

multiplication by scalars & variables

PRACTICAL CASES – (10 eqns, 10 vars, deg 3)

m equations, n variables

$$
\begin{aligned}
f_1(x,y) &= 25\,x\,y - 15\,x - 20\,y + 12 \\
f_2(x,y) &= x^2 + y^2 - 1 \\
&\vdots
\end{aligned}
$$

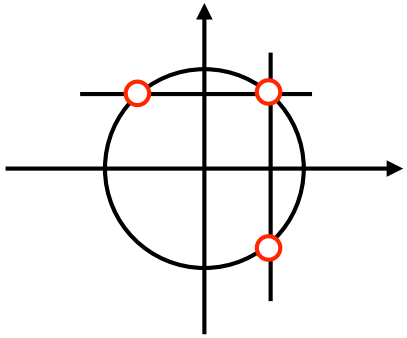
$\rightarrow$ Groebner basis $\rightarrow$ generalized companion matrix (multiplication in $\mathbb{Q}[x_1, \ldots]/I(f_1, \ldots)$)

(e.g. in grevlex ordering)

$$
\mathtt{M}_{x+y} = \begin{bmatrix} 0 & 125 & 0 & 125 \\ -60 & 100 & 125 & 75 \\ -63 & 45 & 175 & 45 \\ 65 & 100 & -125 & 75 \end{bmatrix}
$$

SOLVING ALGEBRAIC EQUATIONS

m equations, n variables



$$
\begin{aligned}
f_1(x,y) &= 25\,x\,y - 15\,x - 20\,y + 12 \\
f_2(x,y) &= x^2 + y^2 - 1
\end{aligned}
$$

→ Groebner basis → generalized companion matrix → eigenvectors

$$
\mathrm{v} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ -\frac{4}{5} & \frac{4}{5} & \frac{4}{5} & \frac{4}{5} \\ \frac{3}{5} & -\frac{3}{5} & \frac{3}{5} & \frac{3}{5} \\ \frac{16}{25} & \frac{16}{25} & \frac{16}{25} & \frac{16}{25} \end{bmatrix}
$$

$$
\begin{aligned}
x_1 &= -\frac{4}{5}, \quad y_1 = \frac{3}{5} \\
x_2 &= \frac{4}{5}, \quad y_3 = -\frac{3}{5} \\
x_3 &= \frac{4}{5}, \quad y_3 = \frac{3}{5}
\end{aligned}
$$

THE DIFFICULT PART

Equations $\rightarrow$ Groebner basis

$\uparrow$

no simple rule:

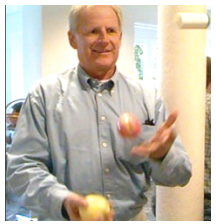
1. NP-complete in m, n (~ 3-coloring of graphs)
   (takes very long time to compute)

2. EXPSPACE-complete problem
   (needs huge space to remember intermediate results)

HISTORY

1888 David Hilbert: Finitness theorem

Every ideal has a finite generating set



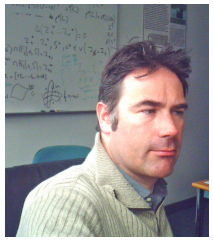
1965 Bruno Burchberger: Groebner bases

Computational procedure for solving systems of polynomial equations

(Extremely simple: 20 lines of Maple code!)

1998 Hans Stetter: Multiplication matrix

A stable numerical procedure via eigenvectors



1999 Jean-Charles Faugere: F4 algorithm

An efficient computational tool for cryptography

GB COMPUTATION ALGORITHMS

1. "Standard" (Hironaka 1964) and
   Groebner (Burchberger 1965) bases

2. 1965: Buchberger's algorithm

   – a generalization of the Gauss-Jordan elimination
   – extremely simple: 7 (+ 13 for the rem division) lines of code
   – not efficient
   – not good for numerical approximations

3. 1999 (2005): F4 (F5) algorithm (J.-C. Faugere)

   – more efficient, more robust, more complex

COMPUTING GB MAY BE VERY HARD

Example:

4 polynomials, 3 variables, degree ≤ 6, small integer coeffs

$$
\begin{aligned}
f_1 &= 8\,x^2\,y^2 + 5\,x\,y^3 + 3\,x^3\,z + x^2\,y\,z \\
f_2 &= x^5 + 2\,y^3\,z^2 + 13\,y^2\,z^3 + 5\,y\,z^4 \\
f_3 &= 8\,x^3 + 12\,y^3 + x\,z^2 + 3 \\
f_4 &= 7\,x^2\,y^4 + 18\,x\,y^3\,z^2 + y^3\,z^3
\end{aligned}
$$

have extremely simple Groebner basis

$$
\begin{aligned}
g_1 &= x \\
g_2 &= y^3 + \frac{1}{4} \\
g_3 &= z^2
\end{aligned}
$$

H O W E V E R

when computed by the Buchberger's algorithm over the rational numbers w.r.t. the grevlex ordering x > y > z, the following polynomial appears during the computation:

$$y^3 - \underbrace{17359065042904512907647471 82\ldots}_{\sim\ 80{,}000\ \text{digits}}$$

W H Y ?

General algorithms can construct all GBs but often generate many complicated polynomials

Complexity

Specific algorithm

$$y^3 + 1$$

$$\begin{aligned} f_1 &= 8\,x^2\,y^2 + 5\,x\,y^3 + 3\,x^3\,z + x^2\,y\,z \\ f_2 &= x^5 + 2\,y^3\,z^2 + 13\,y^2\,z^3 + 5\,y\,z^4 \\ f_3 &= 8\,x^3 + 12\,y^3 + x\,z^2 + 3 \\ f_4 &= 7\,x^2\,y^4 + 18\,x\,y^3\,z^2 + y^3\,z^3 \end{aligned}$$

Input

$$\begin{aligned} g_1 &= x \\ g_2 &= y^3 + 10 \\ g_3 &= z^2 \end{aligned}$$

Groebner basis

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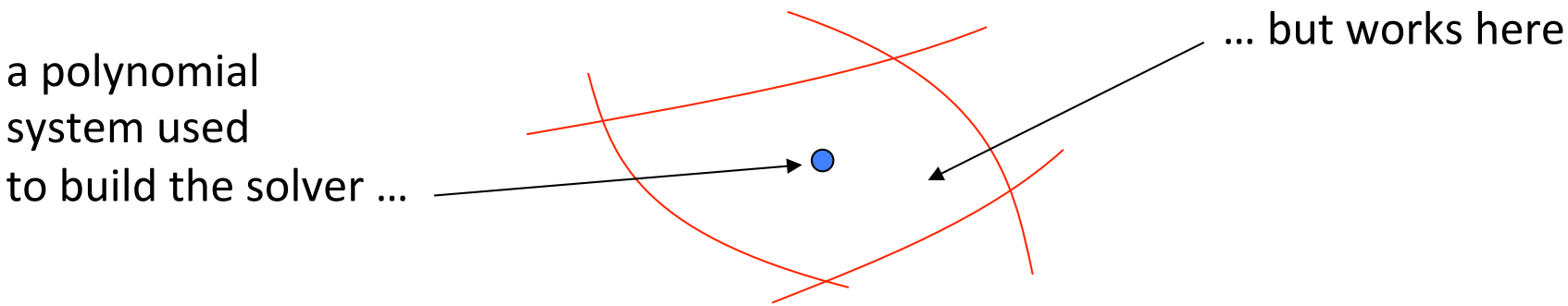
SPECIFIC GB CONSTRUCTION ALG'S

1. Find a short path towards the GB which is independent from the actual coefficients, implement it efficiently

   → Solver is made out a single concrete system and then used on other systems

   → This works around generic solutions



2. Use floating-point arithmetics to do the manipulations to avoid huge coefficients.

COMPUTATION

Macaulay2 program over the rational field $\mathbb{Q}$

```
R = QQ[x,y,z, MonomialOrder=>GRevLex];
I = ideal(8*x^2*y^2 +  5*x*y^3     +  3*x^3*z   +   x^2*y*z,
            x^5     +  2*y^3*z^2   + 13*y^2*z^3 + 5*y*z^4,
          8*x^3     + 12*y^3       +    x*z^2   + 3,
          7*x^2*y^4 + 18*x*y^3*z^2 +    y^3*z^3)
G = gens gb I
```

will run very long.

The problem is in remembering very long coefficients.

COMPUTATION

Macaulay2 program over the finite field $\mathbb{Z}/13$

```
R = ZZ/13[x,y,z, MonomialOrder=>GRevLex];
I = ideal(8*x^2*y^2 +  5*x*y^3     +  3*x^3*z   +   x^2*y*z,
            x^5     +  2*y^3*z^2   + 13*y^2*z^3 + 5*y*z^4,
          8*x^3     + 12*y^3       +    x*z^2   + 3,
          7*x^2*y^4 + 18*x*y^3*z^2 +    y^3*z^3)
G = gens gb I
```

returns a very similar basis in a fraction of second

$$\begin{aligned} g_1 &= x \\ g_2 &= y^3 + 10 \\ g_3 &= z^2 \end{aligned} \qquad \left(\begin{aligned} g_1 &= x \\ g_2 &= y^3 + \tfrac{1}{4} \\ g_3 &= z^2 \end{aligned} \right)$$

SHORT PATH TO GB

1. Use a computer algebra system (Macaulay2) to compute the Groebner basis in a finite field (fast!) for random coefficients and remember the path:

```
R = ZZ/P[x,y,z, MonomialOrder=>GRevLex];
```

↑

"lucky" prime number (always exists)

try P = 1, 2, 3, 5, 7, ..., 30011, 30013, 30029, ...

untill the result stabilizes (always does)

2. Remember the "path of construction" and hard-code it.

FLOATING POINT ARITHMETICS

Strictly speaking, polynomial manipulations must be done in exact arithmetics

$$
\begin{aligned}
f_1 &= 25\,x\,y + 25\,x + 12 \\
f_2 &= x^2\,y + x^2 + 3
\end{aligned}
$$

$$
\begin{aligned}
x\,f_1 - 25\,f_2 &= x\,(\cancel{25}\,x\,y + \cancel{25}\,x + 12) - 25(\cancel{x^2}\,y + \cancel{x^2} + 3) \\
&= 12\,x - 75
\end{aligned}
$$

Double canceling may fail when rounding occurs

in floating point arithmetics

SOLVER GENERATOR

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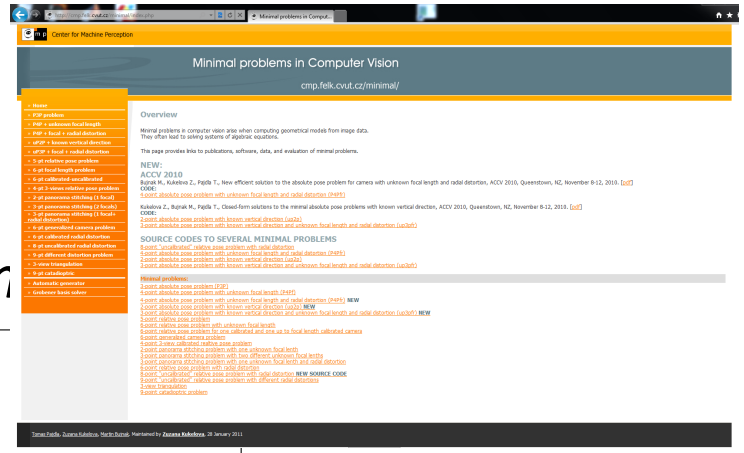
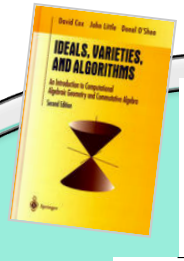
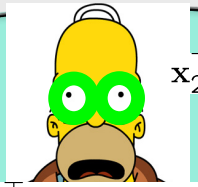


Z. Kukelova, M. Bujnak, T. Pajdla,
Automatic Generator of Minimal Problem
ECCV 2008





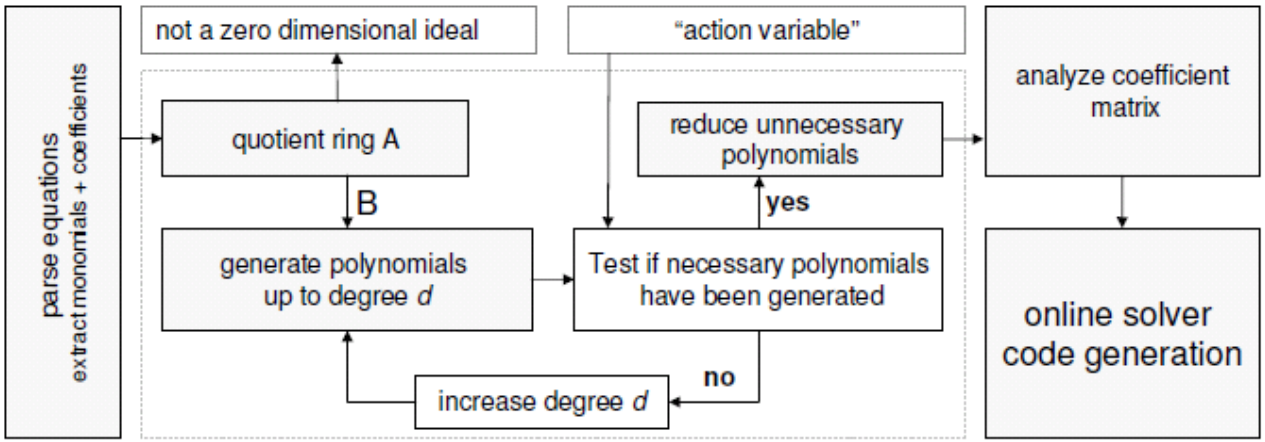
$\mathbf{x}_2^\top \mathbf{F}\, \mathbf{x}_1 =$

$\det \mathbf{F} =$

$2\mathbf{F}\mathbf{F}^\top\mathbf{F} - \mathrm{trace}(\mathbf{F}\mathbf{F}^\top)\mathbf{F} =$

```
% symbolic variables
syms a0 a1 a2 a3 a4 b0 b1 b2 b3 b4;

syms x y;

% two equations of an ellipse and a hyperbola
eq(1) = a0*x^2 + a1*x + a2*y^2 + a3*y +a4;
eq(2) = b0*x^2 + b1*x - b2*y^2 + b3*y +b4;

% known parameters
known = {'a0','a1','a2','a3','a4','b0','b1','b2','b3','b4'}

%two unknowns
unknown = {'x','y'}

% call code generator
[res export] = gbs_CreateCode('example', eq, known, unknown);
```

Offline



Online solver

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Principles of Solving Minimal Problems

T o m a s P a j d l a

The Art of Solving Minimal Problems
ICCV 2015 Tutorial

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SINGLE UNKNOWN → EIGENVALUES

Roots may be (often are) complex

1. Often, due to relaxing originally semi-algebraic problem, (i.e. including inequalities or only interest in real solutions) to an algebraic problem.

2. Sometimes many more complex than real solutions
   → waste of effort on computing them

Remedy

1. Use bracketing to look for real roots only

\+ Easy to stop as soon as sufficient accuracy reached

STURM Theorem

1. A polynomial $f(z) = 0$ in a single variable z

2. $g(z) = \frac{f(z)}{\text{gcd}(f(z), f'(z))}$ has no multiple roots!

3. Sturm sequence

$$
\begin{aligned}
g_0(z) &= g(z) \\
g_1(z) &= g'(z) \\
g_2(z) &= -\text{rem}(g_0(z), g_1(z)) \\
g_3(z) &= -\text{rem}(g_1(z), g_2(z)) \\
&\vdots \\
g_n(z) &= \text{constant}
\end{aligned}
$$

STURM Theorem

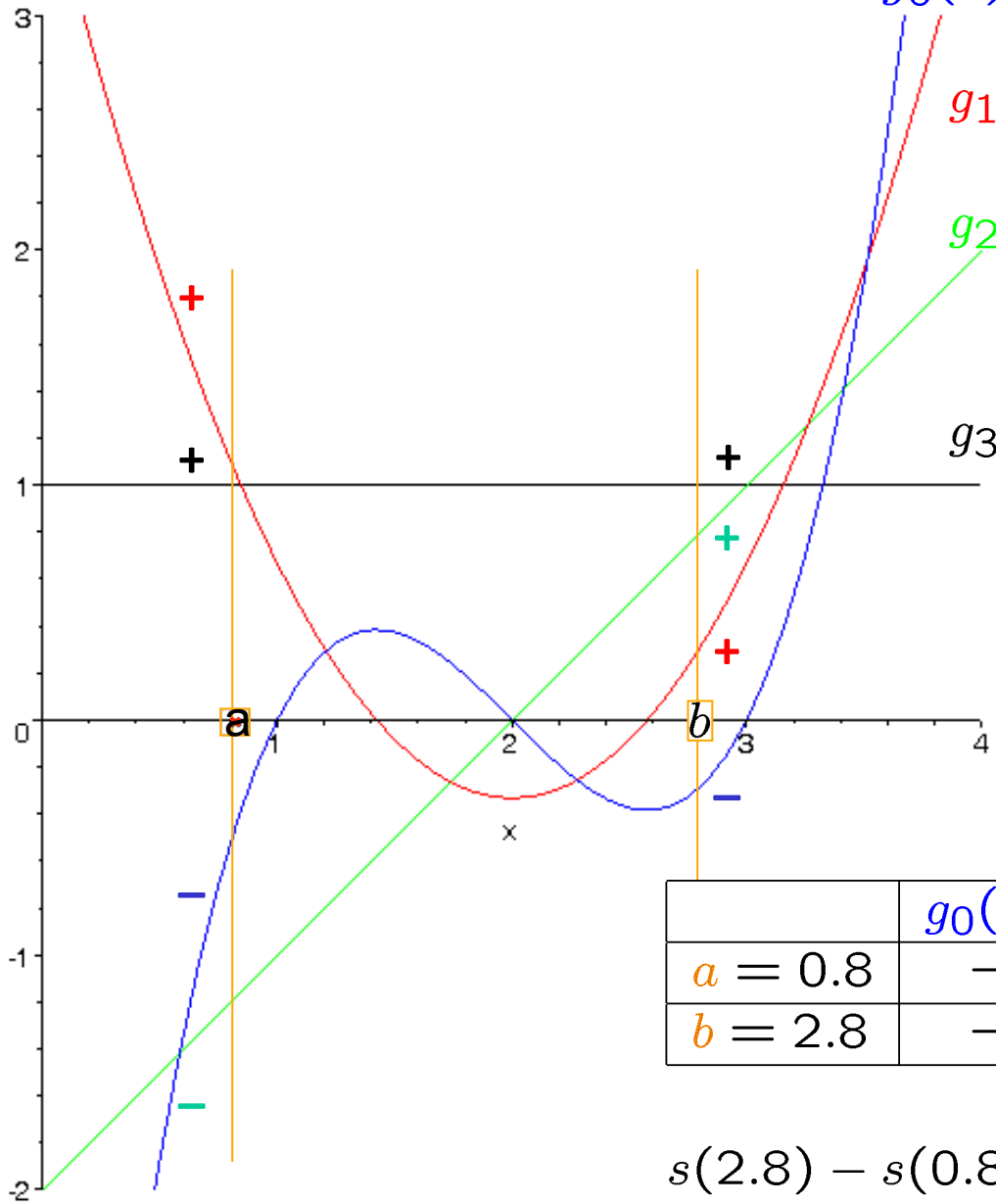
Theorem [Sturm]

- $(a, b]$ ... *an interval in* $\mathbb{R}$
- s_a ... *the number of sign changes in* $g_0(a), g_1(a), \ldots g_n(a)$
- s_b ... *the number of sign changes in* $g_0(b), g_1(b), \ldots g_n(b)$

$\Rightarrow$ *the number of real roots in* $(a, b] = s_b - s_a$

If there are roots in an interval ... split it ... test again

STURM Theorem



$$g_0(z) = f(z) = (z-1)\,(z-2)\,(z-3)$$

$$g_1(z) = f'(z) = z^2 - 4\,z + \frac{11}{3}$$

$$g_2(z) = -\mathrm{rem}(g_0(z), g_1(z)) = z - 2$$

$$g_3(z) = -\mathrm{rem}(g_1(z), g_2(z)) = 1$$

	$g_0(z)$	$g_1(z)$	$g_2(z)$	$g_3(z)$	$s(z)$
$a = 0.8$	$-$	$+$	$-$	$+$	3
$b = 2.8$	$-$	$+$	$+$	$+$	1

$$s(2.8) - s(0.8) = 3 - 1 = 2$$

pajdla@cmp.felk.cvut.cz

STURM Theorem

+ only real roots are computed
+ search only roots with meaningful values
+ precision can be controlled

$\rightarrow$ Often faster for many computer vision problems

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Principles of Solving Minimal Problems

T o m a s P a j d l a

The Art of Solving Minimal Problems
ICCV 2015 Tutorial

Czech Technical University Prague
Center for Machine Perception