Epipolar Geometry of some Non-classical Cameras

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Abstract

Classical multiview geometry assumes pinhole (central) cameras. All rays of a central camera intersect in a projection center. In this work we introduce oblique cameras. Their rays do not intersect at all. For such cameras in certain configurations we elaborate a theory that generalizes classical epipolar geometry. Subsets, in which the correspondences can be solved separately from the rest, are called visibility closures. We give a configuration of rays in projective space that forms two different oblique cameras with visibility closures being double ruled quadrics. We also show that other arrangements of non-central cameras, which appeared in the literature, have double ruled visibility closures.

1 Introduction

Considerable attention has been paid to multiview geometry of pinhole (i.e. central) cameras. A complete theory as well as computational techniques were elaborated in order to reconstruct three-dimensional models of scenes form images taken by central cameras [3].

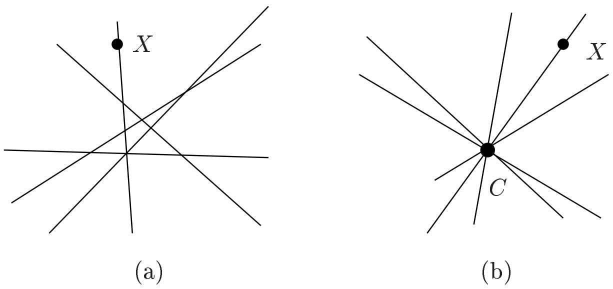
A motivation for studying generalized cameras stems from the literature describing the images taken along rays that do not all intersect in one point. For instance, the rays reflected from a spherical mirror that is imaged by a central camera do not intersect in a point but on a line. Derrien and Konolige [1] investigated how much such a camera differs from a pinhole one and proposed camera-mirror arrangements that well approximate a central projection. Rademacher and Bischop [10] introduced multiple-center-of-projection images to facilitate image-based rendering. Gupta and Hartley [2] analyze linear pushbroom cameras which are formed by a pencil of rays swept in constant speed along a line which is perpendicular to the pencil. Peleg, Pritch, and Ban-Ezra [9] propose a realization of a concentric panorama using a special shaped mirror such that stereo panoramic images of moving scenes can be acquired.

In this paper we indicate that certain arrangements of mutually oblique rays of a non-central camera allow for a generalization of epipolar geometry known from classical two-view geometry of pinhole cameras. The generalization introduced here leads to a double ruled surfaces that replace epipolar planes, thus generalizing epipolar planes to second order surfaces. The geometry of two non-central cameras with surfaces generalizing epipolar planes that are spanned by two different one-parametric sets of rays will further be called *epilinear geometry* and the surfaces that generalize epipolar planes will be called *epilinear surfaces*.

The structure of the paper is the following. Notation and concepts often used through the paper are defined in Section 2. Section 3 introduces the notion of visibility closure to interpret the classical epipolar geometry of two central cameras in a new way, which allows for a generalization. Section 5 describes a simple non-central – oblique – camera. In Section 5 the geometry of two oblique cameras satisfying natural axioms is derived. Both geometrical as well as algebraic characterizations of epipolar surfaces are given. Section 3 shows two examples of non-central camera geometries that have double ruled epilinear surfaces. Section 7 summarizes the work and outlines directions of future research.

2 Notation and concepts

Let $expA$ denote the set of all subsets of a set A. A three-dimensional real projective space will be called *space* and denoted by P_3 through the paper. Space P_3 consists of a set of points, a set of lines, and an incidence relation "$\circ$" satisfying the axioms of three-dimensional projective space [6]. We understand the lines as some subsets of



(a) (b)

Fig. 1. (a) A camera is a set of rays in P_3. (b) A central camera is a set of all rays incident to one point – the projection center C

points and therefore "$\circ$" equals "$\in$". We say that two lines intersect if there is a point incident to both. We say that a point lies on a line iff it is incident with the line.

By a camera we understand a set of lines in P_3, Figure 1(a). Sometimes we also refer to the lines of a camera as to rays. In this paper the term ray will mean a non-oriented line and it will be used whenever we want to stress that a line is a line of a camera.

Our notion of a camera is more general than the usual notion of a camera adopted in computer vision community. In particular, we do not, for a general camera, impose any further constraint on the rays of the camera. By a pinhole (central) camera, Figure 1(b), we understand a set of rays in P_3 that are all incident to one point, the center of projection. By *imaging* we mean a mapping that assigns rays from a camera to points in P_3. Thus, for each point X in P_3, other than the center of a central camera, a unique ray from the central camera is assigned to it by choosing the ray that is incident with the center of the projection and with X. No ray can be assigned to the center of projection this way and therefore it has no image. Central cameras do not image exactly one point (the projection center) and image all the other points in P_3 once and only once. We say that point $X \neq C$ is seen by camera $\mathcal{C}$ iff there is a ray $r \in \mathcal{C}$ such that $X \circ r$. All points, except for the center, incident to a ray $r \in \mathcal{C}$ are seen by $\mathcal{C}$. We say that a point is seen once by a camera if there exist just one ray from the camera that is incident with the point.

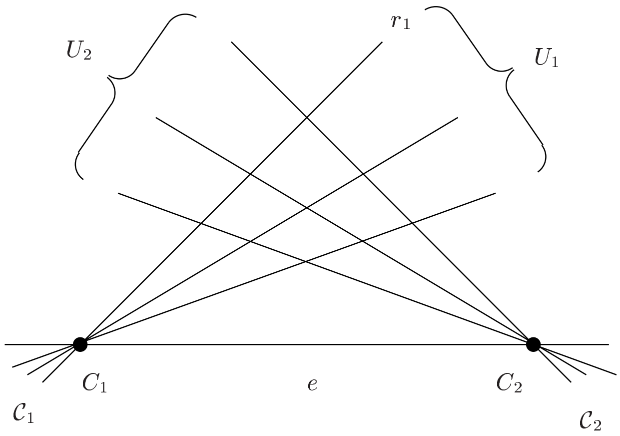


Fig. 2. Epipolar geometry of two central cameras, see text

Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two cameras, i.e. two sets of rays in P_3. We say that a ray r from camera $\mathcal{C}_1$ is visible from camera $\mathcal{C}_2$ iff each point of r lies on a ray from $\mathcal{C}_2$.

3 Visibility closures

Figure 2 shows a diagram of epipolar geometry of two central cameras. The first resp. the second camera, denoted $\mathcal{C}_1$ resp. $\mathcal{C}_2$, is formed by the set of all lines in P_3 passing through point C_1 resp. C_2. Let us study the visibility of rays from one camera by the other camera.

Let us choose arbitrary line $r_1 \in \mathcal{C}_1$, $r_1 \neq e$. The set U_2 of all lines from $\mathcal{C}_2$ intersecting r_1 spans a (epipolar) plane. Symmetrically, the set $U_1 \subset \mathcal{C}_1$ of lines intersecting some $r_2 \in U_2$, $r_2 \neq e$, spans also a plane. Such planes for all $r_2 \in U_2$ are identical with the plane spanned by U_2 because both C_1 and C_2 lie in U_2. An epipolar plane can be thus seen as the set of points that are double-covered by set of lines $U = U_1 \cup U_2$, $U_1 \subset \mathcal{C}_1$ and $U_2 \subset \mathcal{C}_2$, such that each line from U_1 is visible from U_2 and vice versa. We define *visibility closures* to formalize the above concept.

Definition 1 (Visibility closure). *We say that set $U = U_1 \cup U_2$, $U_1 \subset \mathcal{C}_1$, $U_2 \subset \mathcal{C}_2$ is a visibility closure of rays in cameras $\mathcal{C}_1$, $\mathcal{C}_2$ iff*

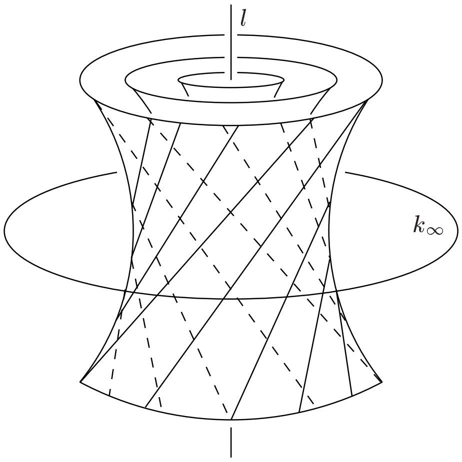


Fig. 3. Two set of lines — full and dashed — that form oblique cameras satisfying Assumption 2

it holds that

$$\forall k \in U_1, \forall X \in P_3, X \circ k: \ \exists l \in U_2 \text{ such that } X \circ l \qquad (1)$$
$$\forall k \in U_2, \forall X \in P_3, X \circ k: \ \exists l \in U_1 \text{ such that } X \circ l$$

The empty set is a visibility closure.

Two pencils of rays passing through C_1 resp. C_2 and spanning the epipolar plane containing ray r can be seen as the visibility closure of r.

Let us show that the notion of visibility closure is well-defined even for more general arrangements of rays than the one provided by two classical central cameras. It seems that Definition 1 is too free to allow for well-defined closures but we can show that our visibility closure is the set theoretical closure [4] under the following assumption.

Assumption 1 (Unique visibility) *Let us assume that it holds for our cameras $\mathcal{C}_1$, $\mathcal{C}_2$ that all points in space that are seen by one of the cameras are seen exactly once by both of them.*

Now, we will define the operation that creates visibility "closures" from sets of rays and show that it is the set theoretical closure.

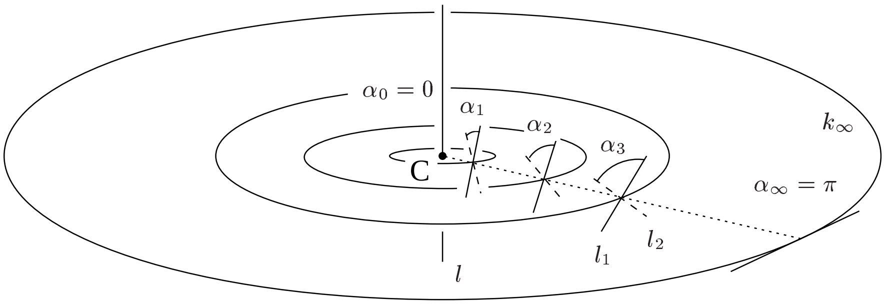


Fig. 4. The arrangement of lines from Figure 3 can be generated by rotating two lines along a set of concentric circle

Definition 2. *Let $\mathcal{C}_1$, $\mathcal{C}_2$ be two cameras satisfying Assumption 1. Let us define the operation* $\overline{\quad}$ *: $exp(\mathcal{C}_1 \cup \mathcal{C}_2) \rightarrow exp(\mathcal{C}_1 \cup \mathcal{C}_2)$ that assigns to a set of lines $L \subset \mathcal{C}_1 \cup \mathcal{C}_2$ the smallest (ordered by set inclusion) visibility closure $U \in exp(\mathcal{C}_1 \cup \mathcal{C}_2)$ containing L. Let furthermore $\overline{\emptyset} = \emptyset$.*

Theorem 1. *Visibility closure is the set theoretical closure.*

The above theorem justifies the choice of the name "visibility closure". Any visibility closure A can be obtained as the set theoretical closure $\overline{\quad}$ of some set of lines in $\mathcal{C}_1 \cup \mathcal{C}_2$, e.g. as $A := \overline{A}$.

Let us study the structure of visibility closures of two central cameras. Firstly, there is one one-line closure consisting of the line joining camera centers C_1, C_2 in Figure 2. The line C_1C_2 contains exactly those points in space that project into epipoles and that cannot be reconstructed by intersection of rays from cameras $\mathcal{C}_1$, $\mathcal{C}_2$. Secondly, there is a fan of planar closures – epipolar planes – all intersecting in line C_1C_2. Finally, the maximal closure is formed by $\mathcal{C}_1 \cup \mathcal{C}_2$. The closures are partially ordered by inclusion: line $C_1C_2 \subset$ rays spanning epipolar planes $\subset \mathcal{C}_1 \cup \mathcal{C}_2$.

4 Oblique cameras

The geometry of central cameras is characterized by requiring that all the rays intersect in one point. When generalizing to non-central cameras we have a plethora possible constraints to impose on the

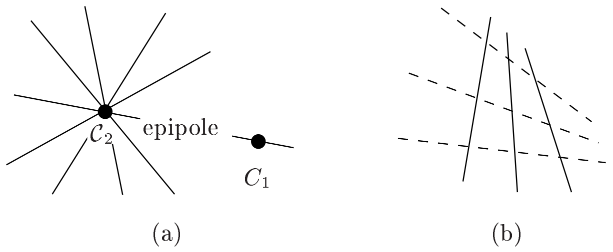


Fig. 5. (a) The epipole (ray) in the second camera is the image of the set of points that have no image in the first camera. There is only one such point for central cameras – the projection center C_1. (b) All points in P_3 are imaged by oblique cameras and therefore there is no analogy to the epipole

rays of a camera. We may require all the rays intersect on a circle like it is in the case of a circular panoramas [8], or on a line as it is for a pushbroom camera [2] or a catadioptric camera with a spherical mirror [1].

In this work, let us concentrate on cameras that *image all points in space once and only once* . Let us define such cameras.

Definition 3 (Oblique camera). *We say that set* $\mathcal{C}$ *is an* oblique camera *if*

$$\forall X \in P_3 \; \exists! l \in \mathcal{C} \text{ such that } X \circ l \; . \tag{2}$$

Even such a general requirement is sufficient to provide our camera with a geometrical structure that justifies the name "oblique camera".

Lemma 1. *Two rays of an oblique camera are either identical or oblique.*

Oblique cameras are exactly opposite to central cameras. While all rays of a central camera intersect in a projection center, there are no intersecting rays in an oblique camera.

The structure of an oblique camera is not completely fixed by the above requirement and it is also not clear whether there is any such camera. In what follows we will assume to have two different oblique cameras $\mathcal{C}_1 \neq \mathcal{C}_2$ and study their possible visibility closures.

5 Epilinear geometry of oblique cameras

We will show that by adopting the following constraint on the relationship between the rays of the two oblique cameras, interesting visibility closures are obtained.

Assumption 2 (Configuration of two oblique cameras) *Let us assume that the rays of oblique cameras $\mathcal{C}_1 \neq \mathcal{C}_2$ are in such a configuration that for each three distinct rays l_1, l_2, l_3 from $\mathcal{C}_1$ resp. $\mathcal{C}_2$ all rays intersecting l_1, l_2, l_3 simultaneously are in $\mathcal{C}_2$ resp. $\mathcal{C}_1$.*

Let us show the structure of visibility closures for oblique cameras satisfying Assumption 2.

5.1 Structure of visibility closures

Visibility closures of oblique cameras satisfying Assumption 2 are, as sets of lines in P_3, subsets of the set of all lines in P_3. Therefore, it will be useful to restate the following classical geometrical theorem [5].

Lemma 2 (Double ruled surfaces in P_3). *Let k_1, k_2, k_3 be three mutually oblique lines in P_3. Then the set of all points that lie on all lines intersecting k_1, k_2, k_3 form a double ruled quadric. Lines k_1, k_2, k_3 lie in one ruling while the lines intersecting them lie in the other. Both rulings are generated by any three distinct lines form the other ruling. Every line from one ruling intersects all the lines from the other ruling. The double ruled quadrics and planes are the only one double ruled surfaces in three-dimensional real projective space. The double ruled quadrics are either a hyperboloid of one sheet or a hyperbolic paraboloid in Euclidean three-dimensional space.*

Proof. See [5]. □

Applying Lemma 2 to oblique cameras in the configuration satisfying Assumption 2 yields the following theorem.

Theorem 2. *The visibility closure of a ray in two oblique cameras satisfying Assumption 2 is either a ray or a double ruled quadric.*

The following theorem characterizes the structure of visibility closures of oblique cameras.

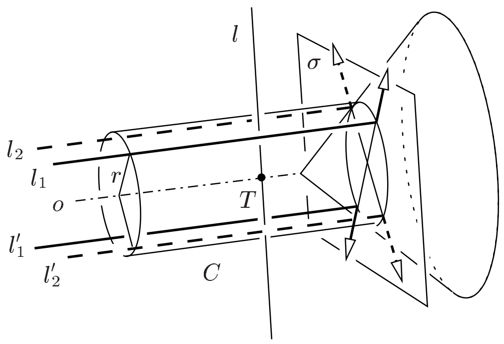


Fig. 6. Two sets of rays forming hyperboloidal closures can be realized by rotating catadioptric camera consisting from a telecentric optics and a conical mirror, see text

Theorem 3. *The visibility closure of rays in a pair of oblique cameras satisfying Assumption 2 is a union of mutually disjoint lines and double ruled quadrics.*

5.2 Epilinear geometry

Let $U = U_1 \cup U_2$, where $U_1 \in \mathcal{C}_1$ and $U_2 \in \mathcal{C}_2$ be a visibility closure. Wee see that there are two kinds of closures in U.

The closures of the first kind are lines that are themselves visibility closures. No point on such lines can be reconstructed as an intersection of a line from U_1 with a line from U_2. The line l, which is a visibility closure, is in U_1 as well as in U_2. Since there is only one line in each U_i that passes through a point in space, there is no line in U_1, other than l, that would intersect a line from U_2 in a point on l.

The closures of the second kind are double ruled quadrics. Their each point can be reconstructed as an intersection of a line form U_1 with a line from U_2.

Analogically to the classical epipolar plane, the set of rays emanating from one oblique camera and spanning a non-degenerate visibility closure (a double ruled quadric) is in a one-to-one correspondence with a line. Thus solving for a correspondence inside one

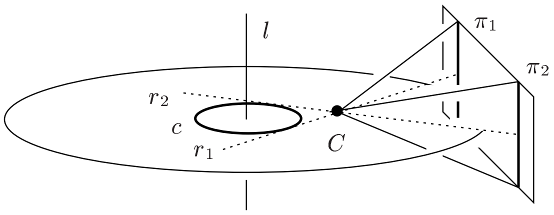


Fig. 7. Concentric symmetric panoramas are obtained by rotating two line cameras with their projection centers at the same point C. The line cameras (or two columns of a standard camera) are oriented so that the planes spanned by the rays are tangent to circle c

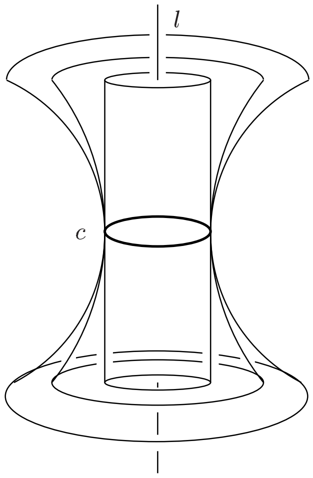


Fig. 8. Concentric panoramas have visibility closures in the form of hyperboloids of one sheet. They all meet in circle c.

quadric visibility closure of two oblique cameras, i.e. finding correspondences between the ray sets U_1 and U_2 in one U, amounts to finding a correspondence between points of two lines. It can be done exactly the same way as it is done for epipolar lines in a pair of central images.

While the notion of epipolar line in central images has a symmetrical notion in oblique images, there is no equivalent notion for epipoles. Let us have two central cameras $\mathcal{C}_1$, $\mathcal{C}_2$. An epipole in $\mathcal{C}_2$ is the image of a center C_1 of the camera $\mathcal{C}_1$, i.e. the image of a point in space that is not imaged by $\mathcal{C}_1$, see Figure 5(a). Since all points are imaged by any oblique camera, there are no equivalents to epipoles.

6 Some cameras with double ruled quadric visibility closures

Theorem 3 characterizes nonempty visibility closures. Is there any arrangement of lines in P_3 such that they might form two oblique cameras satisfying Assumption 2? Figure 3 shows one such example. The figure shows two lines l and k_∞ and a set of rotational hyperboloids of one sheet having axis l and filling the space between k_∞ and l. Line k_∞ lies in the plane at infinity and therefore it is, in a Euclidean space of the figure, depicted as a circle. One oblique camera is e.g. formed by the set of all rulings depicted by full lines while the other is depicted by dashed lines. Both cameras contain the lines l and k_∞, which are the only line visibility closures of the arrangement.

6.1 Oblique cameras with quadric visibility closures

The arrangement of lines depicted on Figure 3 can be realized by rotating two intersecting lines l_1, l_2 along concentric circles lying in a plane perpendicular to the line l as shown in Figure 4. The center of circles coincides with the point C where l intersects the plane. The angle α between l_i is zero for the zero radius, thus forming one line l, and π for the "infinite" radius, thus forming line k_∞. If the angle is not zero or π, the set of all lines obtained by rotating l_1, l_2 forms a hyperboloid of one sheet with one ruling generated by rotated l_1 and the other by rotated l_2.

The following camera has been proposed by Nayar and Karmarkar in [7] in order to obtain a complete spherical mosaics. Let us have a conical mirror observed by a telecentric lens [11] in the direction of the mirror rotation axis o as shown in Figure 6. Let furthermore the mirror be such that all the parallel rays going through the telecentric lens are reflected to rays that are perpendicular to the mirror axis thus spanning plane σ perpendicular to o. As the radius r of the cylinder grows, the plane moves away from the tip of the conical mirror. If one had an infinitely large mirror as well as the telecentric lens, the moving plane would fill the whole half-space.

On each cylinder, e.g. C, of parallel rays we can select four rays, e.g. l_1, l'_1 and l_2, l'_2 in Figure 6, such that l_1, l'_1 as well as l_2, l'_2 reflect into two rays lying on the same line in σ. The angle between the lines in σ can be made arbitrary by the choice of the rays on C. Thus by selecting certain four rays from each cylinder, one can obtain the set of ray couples lying in planes perpendicular to o.

A subset of primitive visibility closures is then obtained by rotating the mirror around the line l intersecting the axis o at the point T. There are only those closures that intersect the volume swept by the rotating mirror. Each closure consists of that part of a rotational hyperboloid of one sheet that is not contained in the swept volume because the camera does not see inside of the mirror.

6.2 Concentric symmetric panoramas

Concentric symmetric panoramas [8] are formed by rotating a line camera along a circle γ, see Figure 7. By that, a set of rays that can be partitioned into double ruled hyperboloids of one sheet is obtained as shown in Figure 8. All the hyperboloids contain circle c. Therefore, the rays of symmetric panoramas are not oblique cameras as more than two rays from the same cameras intersect in points of circle c. However, it is very likely that concentric panoramas could be modeled by more general theory obtained by relaxing the requirement of point projection uniqueness.

6.3 Pushbroom cameras

Pushbroom cameras [2] are formed by sweeping a line camera along a line. Let us call that line the *center line* and denote it by l. All the rays of a pushbroom camera intersect and are perpendicular to line l. Two pushbroom cameras with centerlines l_1, l_2 arranged such that l_1 intersects l_2 have nontrivial visibility closures corresponding to hyperbolic paraboloids as shown in Figure 9. All the hyperbolic paraboloids intersect in l_1 and l_2. Lines l_1 and l_2 lie in all paraboloids. Again, a pushbroom camera is similar to but not exactly an oblique camera as many rays from it intersect on its central line.

7 Summary and Conclusions

We have generalized the notion of a camera by replacing the requirement that all the rays of a camera intersect in one point. Instead, we introduced oblique cameras, for which it holds that no two rays from one camera intersect.

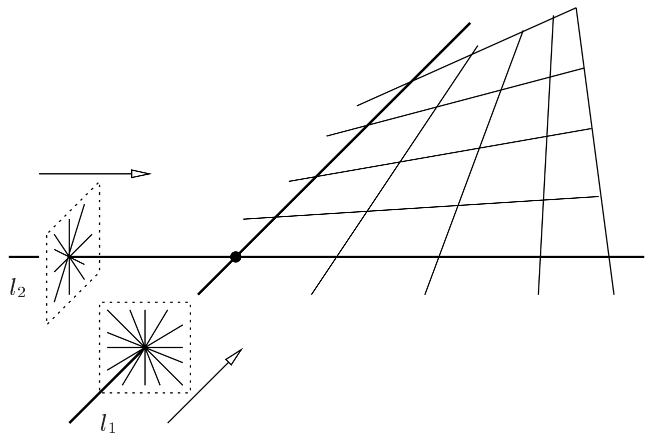


Fig. 9. Pushbroom cameras with intersecting central lines l_1, l_2 have visibility closures in the form of a hyperbolic paraboloids. All closures intersect l_1, l_2.

We show that there is an arrangement of rays in P_3 that allows for two oblique cameras with nontrivial one-parametric visibility closures and propose a technical realization of an oblique camera pair having a subset of such visibility closures. Furthermore, we analyze existing non-central cameras, i.e. circular panoramas and pushbroom cameras, and show that they have double ruled visibility closures even though they are not oblique cameras.

Acknowledgements

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