

# Classifier performance evaluation

Václav Hlaváč

Czech Technical University in Prague, Faculty of Electrical Engineering

Department of Cybernetics, Center for Machine Perception

121 35 Praha 2, Karlovo náměstí 13, Czech Republic

<http://cmp.felk.cvut.cz/~hlavac>, [hlavac@fel.cvut.cz](mailto:hlavac@fel.cvut.cz)

---

## LECTURE PLAN

- ◆ Classifier performance as statistical hypothesis testing.
- ◆ Criterion functions.
- ◆ Confusion matrix, characteristics.
- ◆ Receiver operation curve (ROC).

## APPLICATION DOMAINS

- ◆ Classifiers (our main concern in this lecture), 1940s – radars, 1970s – medical diagnostics, 1990s – data mining.
- ◆ Regression.
- ◆ Estimation of probability densities.

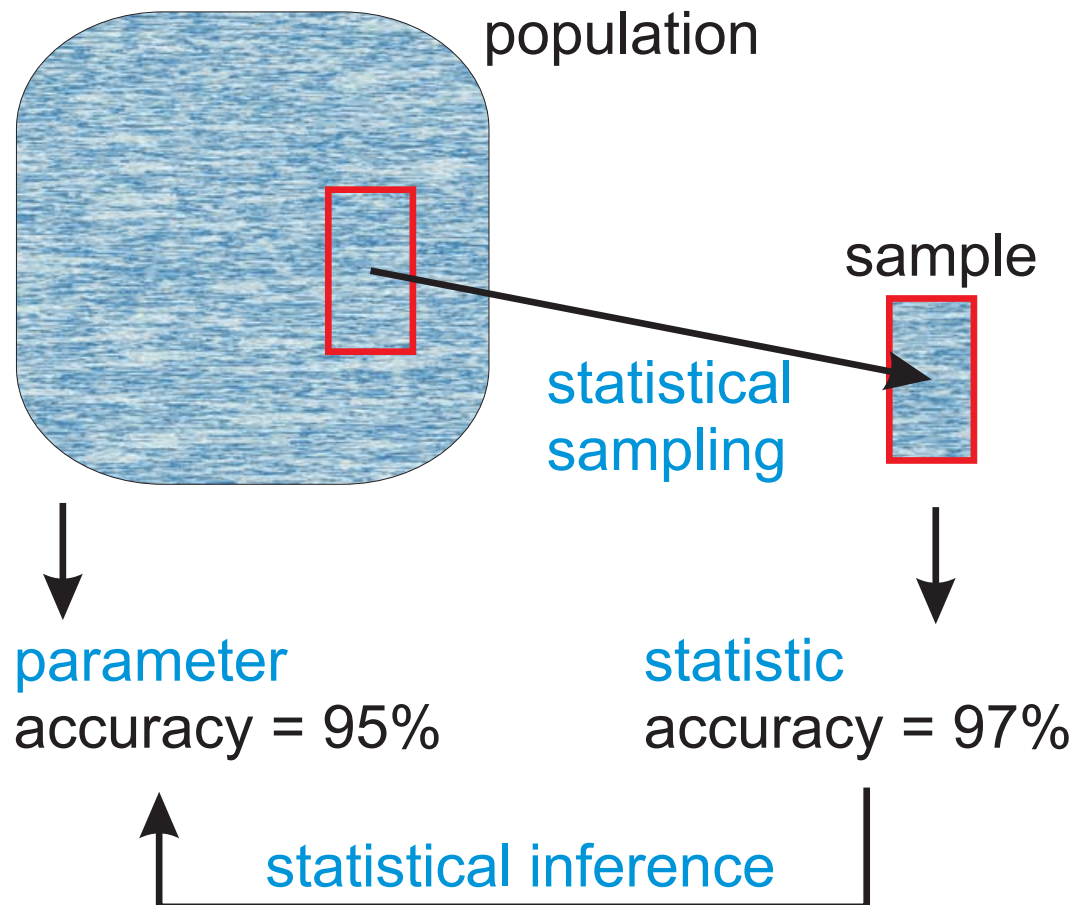
# Classifier evaluation

How much should we believe in what was learned?

- ◆ Classifiers (both supervised and unsupervised) are learned on a finite training set.
- ◆ A classifier learns on the training set and has to be tested experimentally on a different test set.
- ◆ The classifier performs on different data in the run mode that on which it has learned.
- ◆ Experimental performance on test data is a proxy for performance on unseed data (to test the ability to generalize).
- ◆ The criterion function for assessing classifier performance is needed, e.g., accuracy, expected Bayesian risk (to be discussed later).

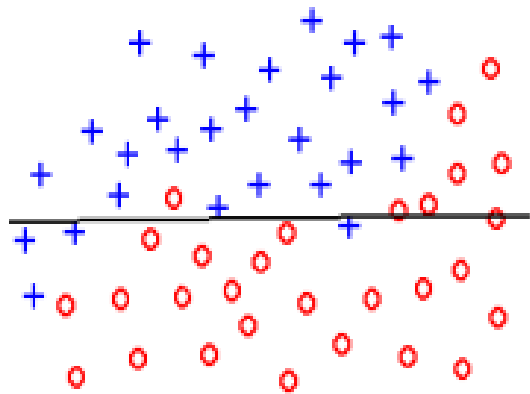
# Evaluation is an instance of hypothesis testing

- ◆ Evaluation has to be treated as hypothesis testing in statistics.
- ◆ The value of the population parameter has to be statistically inferred based on the sample statistics (i.e., training set in pattern recognition).

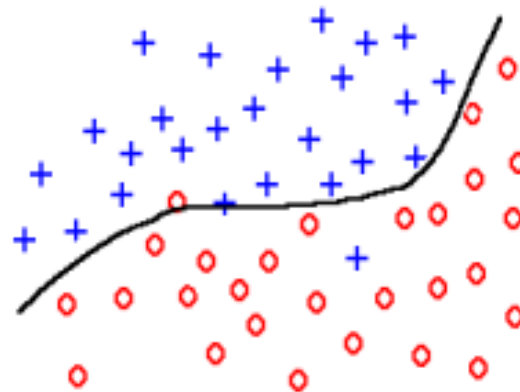


# Danger of overfitting

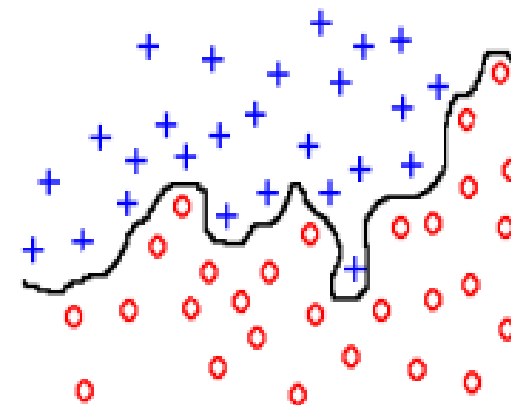
- ◆ Learning the training data too precisely usually leads to poor classification results on new data.
- ◆ Classifier has to have the ability to generalize.



underfit



fit



overfit

## Training vs. test data

**Problem:** Finite data are available only and have to be used both for training and testing.

- ◆ More training data gives better generalization.
- ◆ More test data gives better estimate for classification error probability.
- ◆ Never evaluate performance on training data. The conclusion would be optimistically biased.

---

Partitioning of available finite set of data to training / test sets.

- ◆ Hold out.
- ◆ Cross validation.
- ◆ Bootstrap.

Once evaluation is finished, all the available data can be used to train the final classifier.

# Criterion function to assess classifier performance

- ◆ Accuracy, error rate.
  - *Accuracy* is the percent of correct classifications.
  - *Error rate* = is the percent of incorrect classifications.
  - $Accuracy = 1 - Error\ rate$ .
  - Problems with accuracy:
    - Assumes equal costs for misclassification.
    - Assumes relatively uniform class distribution (cf. 0.5% patients of certain disease in the population).
- ◆ Other characteristics derived from the confusion matrix (to be explained later).
- ◆ Expected Bayesian risk (to be explained later).

# Confusion matrix, two classes only

		predicted	
		negative	positive
actual examples	negative	<i>a</i> TN - True Negative correct rejections	<i>b</i> FP - False Positive false alarms type I error
	positive	<i>c</i> FN - False Negative misses, type II error overlooked danger	<i>d</i> TP - True Positive hits

R. Kohavi, F. Provost: Glossary of terms, Machine Learning, Vol. 30, No. 2/3, 1998, pp. 271-274.

Performance measures calculated from the confusion matrix entries:

- ◆ Accuracy =  $(a + d) / (a + b + c + d)$   
=  $(TN + TP) / total$
- ◆ **True positive rate**, recall, sensitivity =  $d / (c + d) = TP / actual\ positive$
- ◆ Specificity, true negative rate =  $a / (a + b) = TN / actual\ negative$
- ◆ Precision, predicted positive value =  $d / (b + d) = TP / predicted\ positive$
- ◆ **False positive rate** =  $b / (a + b) = FP / actual\ negative = 1 - specificity$
- ◆ False negative rate =  $c / (c + d) = FN / actual\ positive$

# Confusion matrix, # of classes > 2

- ◆ The **toy example** (courtesy Stockman) shows predicted and true class labels of optical character recognition for numerals 0-9. Empirical performance is given in percents.
- ◆ The classifier allows the reject option, class label  $R$ .
- ◆ Notice, e.g., unavoidable confusion between 4 and 9.

	class $j$ predicted by a classifier										
true class $i$	'0'	'1'	'2'	'3'	'4'	'5'	'6'	'7'	'8'	'9'	'R'
'0'	97	0	0	0	0	0	1	0	0	1	1
'1'	0	98	0	0	1	0	0	1	0	0	0
'2'	0	0	96	1	0	1	0	1	0	0	1
'3'	0	0	2	95	0	1	0	0	1	0	1
'4'	0	0	0	0	98	0	0	0	0	2	0
'5'	0	0	0	1	0	97	0	0	0	0	2
'6'	1	0	0	0	0	1	98	0	0	0	0
'7'	0	0	1	0	0	0	0	98	0	0	1
'8'	0	0	0	1	0	0	1	0	96	1	1
'9'	1	0	0	0	3	1	0	0	0	95	0

## Unequal costs of decisions

- ◆ Examples:

*Medical diagnosis:* The cost of falsely indicated breast cancer in population screening is smaller than the cost of missing a true disease.

*Defense against ballistic missiles:* The cost of missing a real attack is much higher than the cost of false alarm.

- ◆ Bayesian risk is able to represent unequal costs of decisions.

- ◆ We will show that there is a tradeoff between apriori probability of the class and the induced cost.

## Criterion, Bayesian risk

For set of observations  $X$ , set of hidden states  $K$  and decisions  $D$ , statistical model given by the joint probability  $p_{XK}: X \times K \rightarrow \mathbb{R}$  and the penalty function  $W: K \times D \rightarrow \mathbb{R}$  and a decision strategy  $Q: X \rightarrow D$  the **Bayesian risk** is given as

$$R(Q) = \sum_{x \in X} \sum_{k \in K} p_{XK}(x, k) W(k, Q(x)) .$$

- ◆ It is difficult to fulfil the assumption that a statistical model  $p_{XK}$  is known in many practical tasks.
- ◆ If the Bayesian risk would be calculated on the finite (training) set then it would be too optimistic.
- ◆ Two substitutions for Bayesian risk are used in practical tasks:
  - Expected risk.
  - Structural risk.

## Two paradigms for learning classifiers

- ◆ Choose a class  $Q$  of decision functions (classifiers)  $q: X \rightarrow K$ .
- ◆ Find  $q^* \in Q$  by minimizing some criterion function on the training set that approximates the risk  $R(q)$  (which cannot be computed).
- ◆ Learning paradigm is defined by the criterion function:
  1. **Expected risk minimization** in which the true risk is approximated by the error rate on the training set,

$$R_{\text{emp}}(q(x, \Theta)) = \frac{1}{L} \sum_{i=1}^L W(k_i, q(x_i, \Theta)) ,$$

$$\Theta^* = \underset{\Theta}{\operatorname{argmin}} R_{\text{emp}}(q(x, \Theta)) .$$

*Examples: Perceptron, Neural nets (Back-propagation), etc.*

2. **Structural risk minimization** which introduces the guaranteed risk  $J(Q)$ ,  $R(Q) < J(Q)$ .

*Example: SVM (Support Vector Machines).*

## Problem of unknown class distribution and costs

*We already know:* The class distribution and the costs of each error determine the goodness of classifiers.

*Additional problem:*

- ◆ In many circumstances, until the application time, we do not know the class distribution and/or it is difficult to estimate the cost matrix. E.g., an email spam filter.
- ◆ Statistical models have to be learned before.

*Possible solution:* Incremental learning.

# Unbalanced problems and data

- ◆ Classes have often unequal frequency.
  - Medical diagnosis: 95 % healthy, 5% disease.
  - e-Commerce: 99 % do not buy, 1 % buy.
  - Security: 99.999 % of citizens are not terrorists.
- ◆ Similar situation for multiclass classifiers.
- ◆ Majority class classifier can be 99 % correct but useless.

*Example: OCR, 99 % correct, error at the every second line. This is why OCR is not widely used.*

---
- ◆ How should we train classifiers and evaluated them for unbalanced problems?

# Balancing unbalanced data

## Two class problems:

- ◆ Build a balanced training set, use it for classifier training.
  - Randomly select desired number of minority class instances.
  - Add equal number of randomly selected majority class instances.
- ◆ Build a balanced test set (different from training set, of course) and test the classifier using it.

## Multiclass problems:

- ◆ Generalize 'balancing' to multiple classes.
- ◆ Ensure that each class is represented with approximately equal proportions in training and test datasets.

# Thoughts about balancing

*Natural hesitation:* Balancing the training set changes the underlying statistical problem. Can we change it?

*Hesitation is justified.* The statistical problem is indeed changed.

*Good news:*

- ◆ Balancing can be seen as the change of the penalty function. In the balanced training set, the majority class patterns occur less often which means that the penalty assigned to them is lowered proportionally to their relative frequency.

$$R(q^*) = \min_{q \in D} \sum_{x \in X} \sum_{k \in K} p_{XK}(x, k) W(k, q(x))$$

$$R(q^*) = \min_{q(x) \in D} \sum_{x \in X} \sum_{k \in K} p(x) p_{K|X}(k|x) W(k, q(x))$$

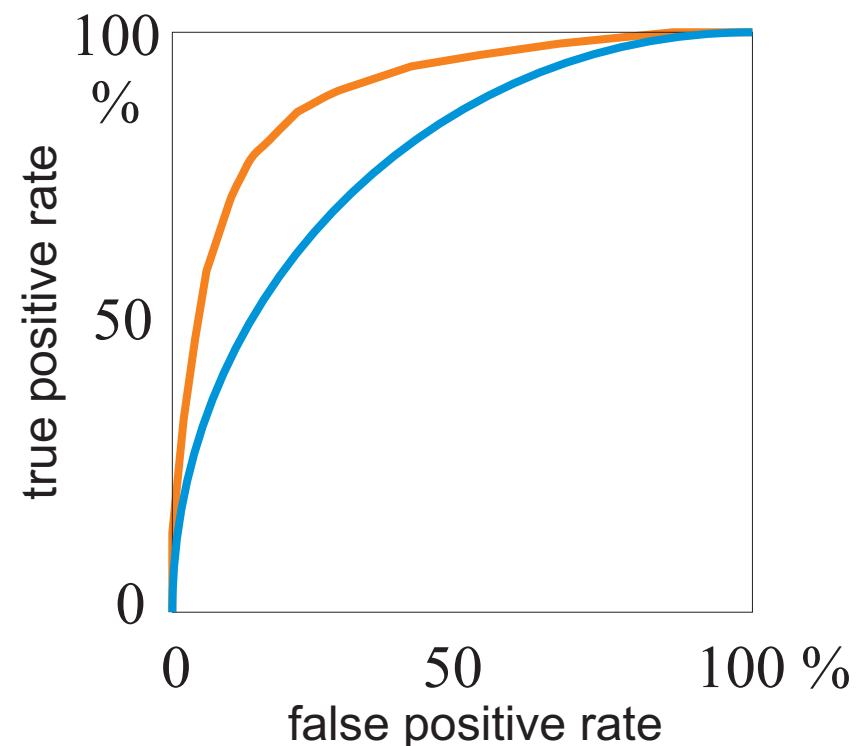
- ◆ This modified problem is what the end user usually wants as she/he is interested in a good performance for the minority class.

# Scalar characteristics are not good for evaluating performance

- ◆ Scalar characteristics as the accuracy, expected cost, area under ROC curve (AUC, to be explained soon) do not provide enough information.
- ◆ We are interested in:
  - How are errors distributed across the classes ?
  - How will each classifier perform in different testing conditions (costs or class ratios other than those measured in the experiment)?
- ◆ Two numbers – true positive rate and false positive rate – are much more informative than the single number.
- ◆ These two numbers are better visualized by a curve, e.g., by a Receiver Operating Characteristic (ROC), which informs about:
  - Performance for all possible misclassification costs.
  - Performance for all possible class ratios.
  - Under what conditions the classifier  $c_1$  outperforms the classifier  $c_2$ ?

# ROC – Receiver Operating Characteristic

- ◆ Called also often ROC curve.
- ◆ Originates in WWII processing of radar signals.
- ◆ Useful for the evaluation of dichotomic classifiers performance.
- ◆ Characterizes degree of overlap of classes for a single feature.
- ◆ Decision is based on a single threshold  $\Theta$  (called also operating point).
- ◆ Generally, false alarms go up with attempts to detect higher percentages of true objects.
- ◆ A graphical plot showing (hit rate, false alarm rate) pairs.
- ◆ Different ROC curves correspond to different classifiers. The single curve is the result of changing threshold  $\Theta$ .



# A model problem

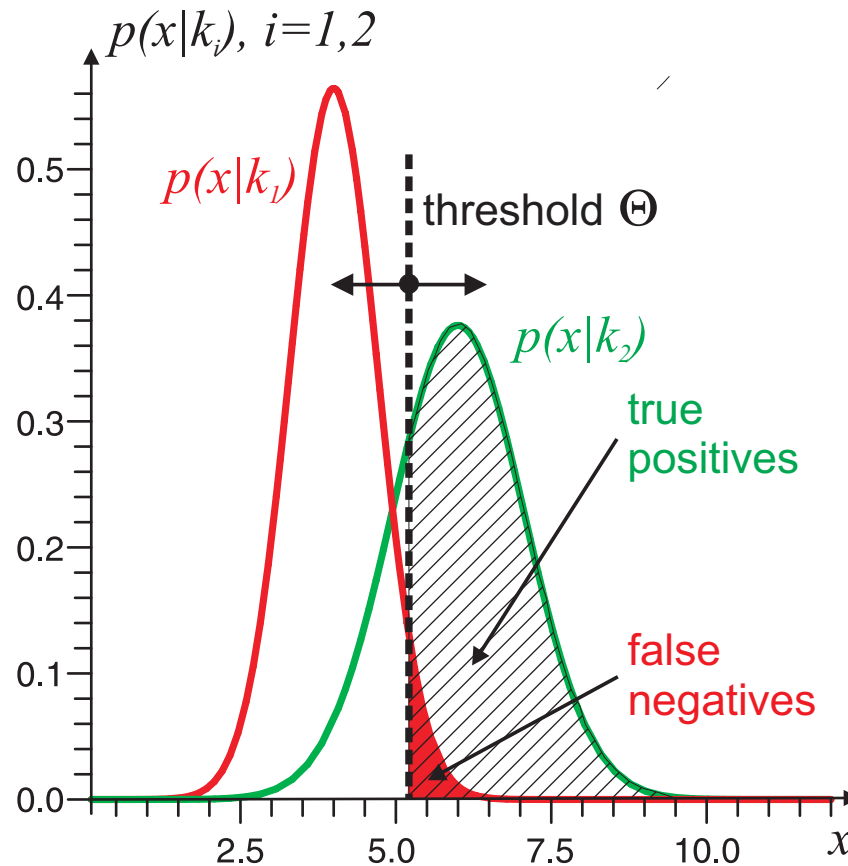
## Receiver of weak radar signals

- ◆ Suppose a receiver detecting a single weak pulse (e.g., a radar reflection from a plane, a dim flash of light).
- ◆ A dichotomic decision, two hidden states  $k_1, k_2$ :
  - $k_1$  – a plane is not present (true negative) or
  - $k_2$  – a plane is present (true positive).
- ◆ Assume a simple statistical model – two Gaussians.
- ◆ Internal signal of the receiver, voltage  $x$  with the mean  $\mu_1$  when the plane (external signal) is not present and  $\mu_2$  when the plane is present.
- ◆ Random variables due to random noise in the receiver and outside of it.  
 $p(x|k_i) = N(\mu_i, \sigma_i^2), i = 1, 2.$

## Four probabilities involved

- ◆ Let suppose that the involved probability distributions are Gaussians and the correct decision of the receiver are known.
- ◆ The mean values  $\mu_1, \mu_2$ , standard deviations  $\sigma_1, \sigma_1$ , and the threshold  $\Theta$  are not known too.
- ◆ There are four conditional probabilities involved:
  - Hit (true positive)  $p(x > \Theta | x \in k_2)$ .
  - False alarm (false positive)  $p(x > \Theta | x \in k_1)$ .
  - Miss, overlooked danger (false negative)  $p(x < \Theta | x \in k_2)$ .
  - Correct rejection (true negative)  $p(x < \Theta | x \in k_1)$ .

# Radar receiver example, graphically



Any decision threshold  $\Theta$  on the voltage  $x$ ,  $x > \Theta$ , determines:

**Hits** – their probability is the red filled area under the curve  $p(x|k_2)$ .

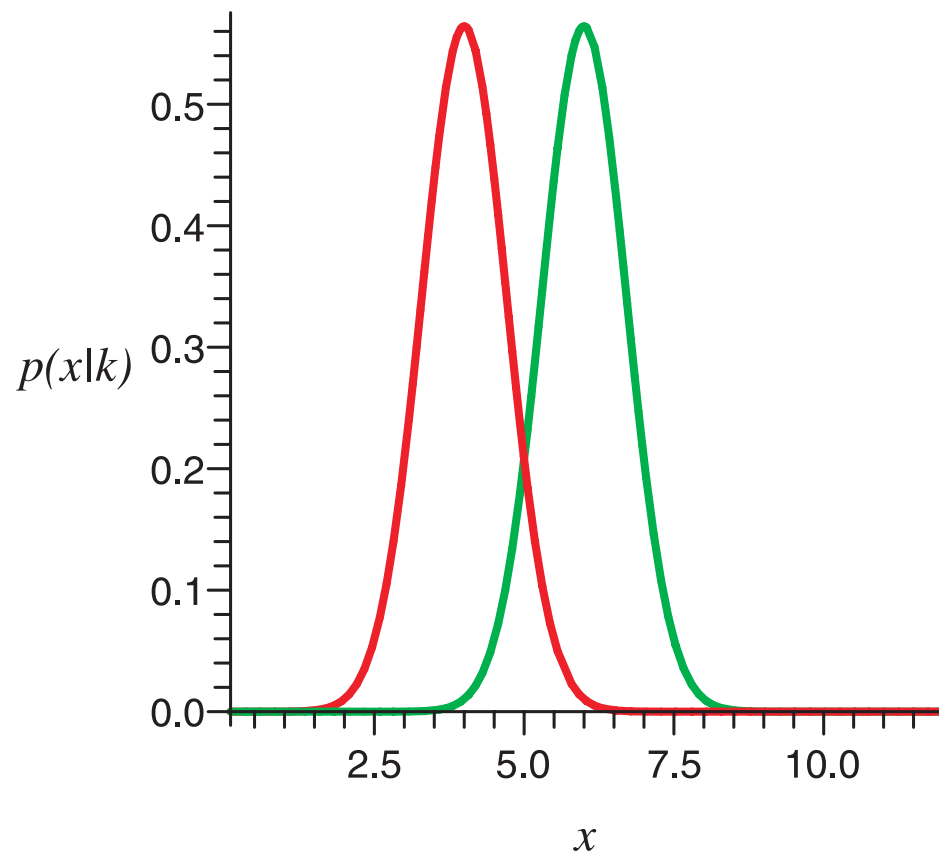
**False alarms** – their probability is the hatched area under the curve  $p(x|k_1)$ .

# ROC – Example A

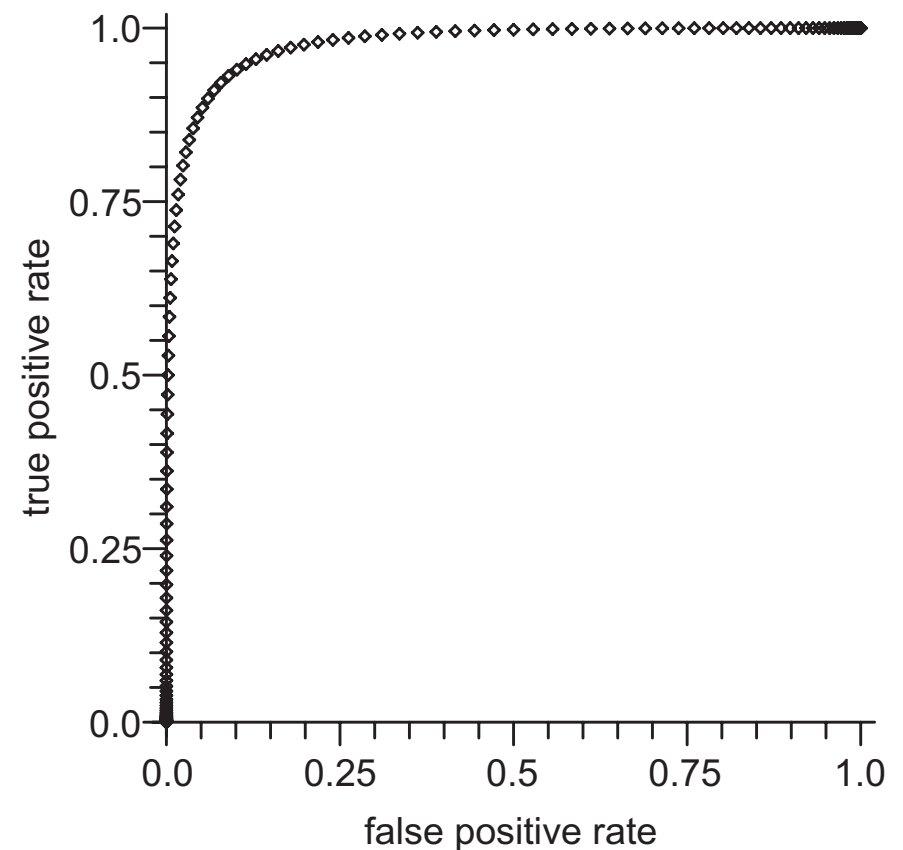
## Two gaussians with equal variances

- ◆ Two Gaussians,  $\mu_1 = 4.0$ ,  $\mu_2 = 6.0$ ,  $\sigma_1 = \sigma_2 = 1.0$
- ◆ Less overlap, better discriminability.
- ◆ In this special case, ROC is convex.

Stochastic model, Gaussians, equal variances



ROC curve

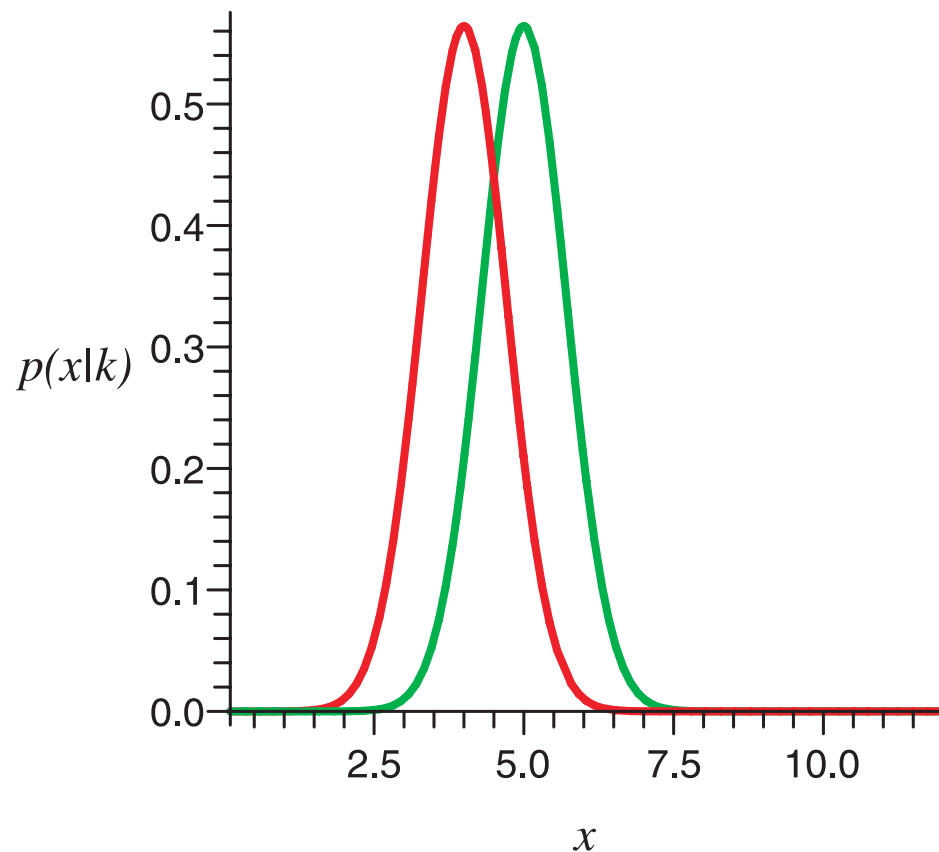


# ROC – Example B

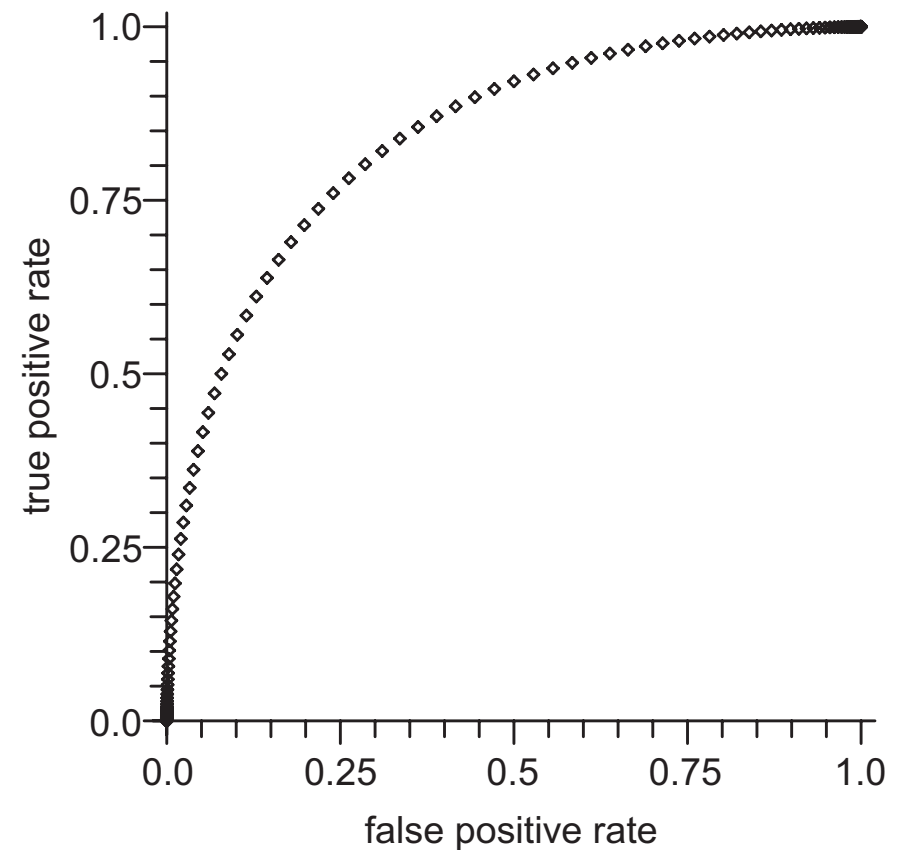
## Two gaussians with equal variances

- ◆ Two Gaussians,  $\mu_1 = 4.0$ ,  $\mu_2 = 5.0$ ,  $\sigma_1 = \sigma_2 = 1.0$
- ◆ More overlap, worse discriminability.
- ◆ In this special case, ROC is convex.

Stochastic model, Gaussians, equal variances



ROC curve

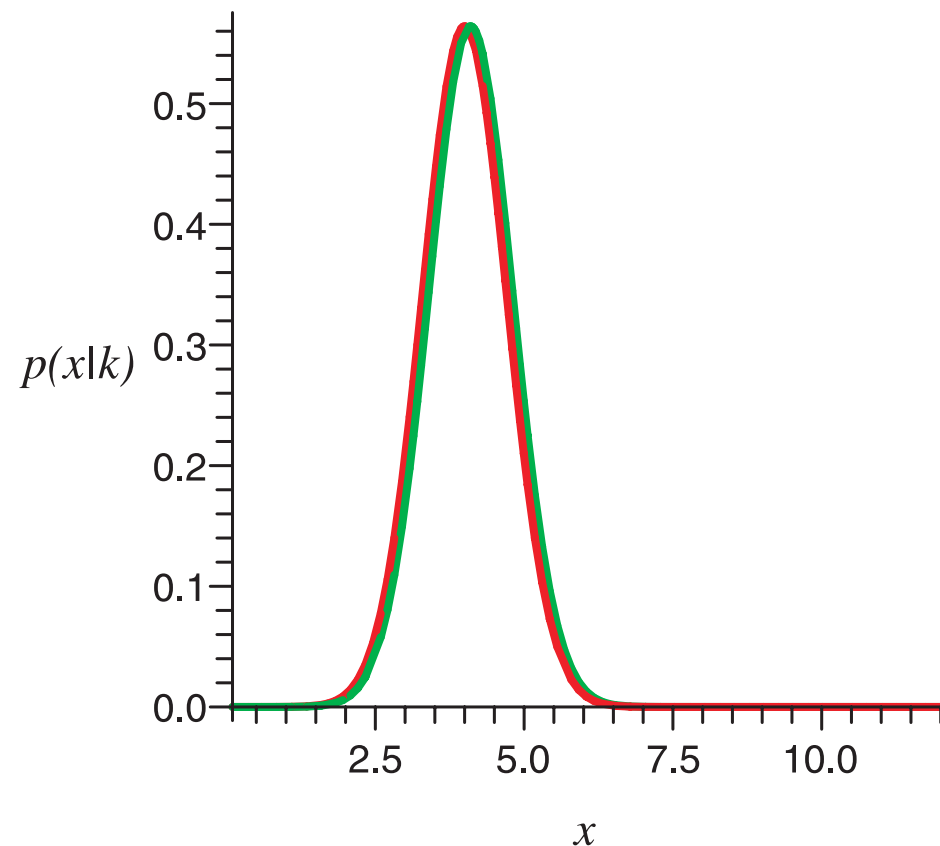


# ROC – Example C

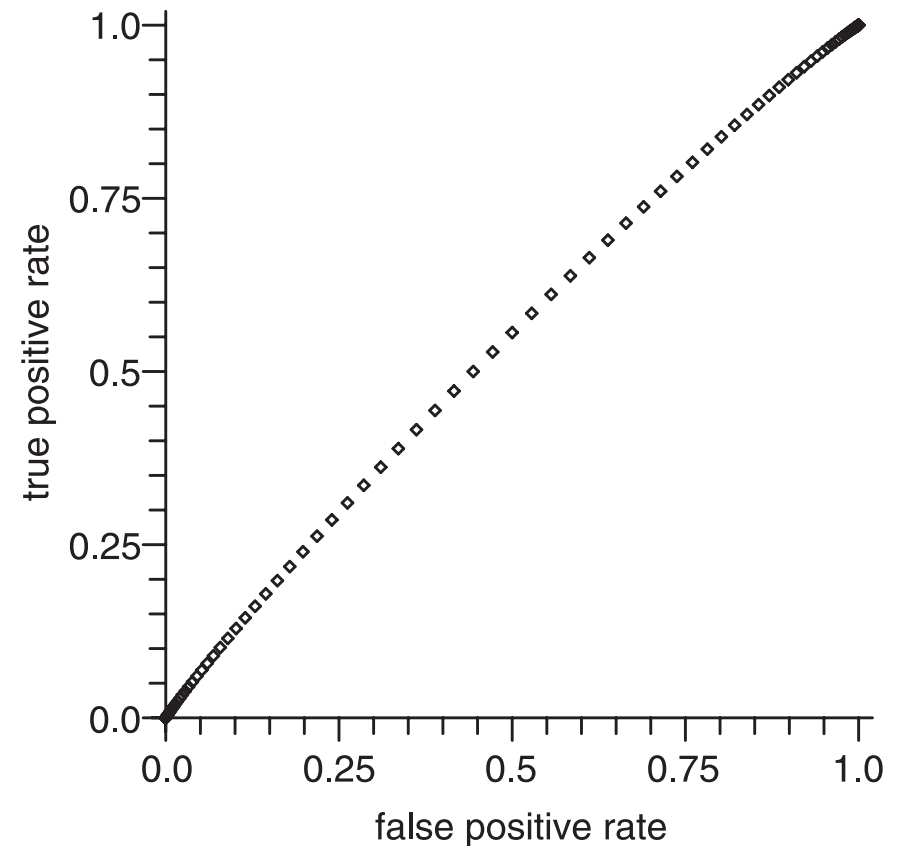
## Two gaussians with equal variances

- ◆ Two Gaussians,  $\mu_1 = 4.0$ ,  $\mu_2 = 4.1$ ,  $\sigma_1 = \sigma_2 = 1.0$
- ◆ Almost total overlap, almost no discriminability.
- ◆ In this special case, ROC is convex.

Stochastic model, Gaussians, equal variances



ROC curve



## ROC and the likelihood ratio

- ◆ Under the assumption of the Gaussian signal corrupted by the Gaussian noise, the slope of the ROC curve equals to the likelihood ratio

$$L(x) = \frac{p(x|\text{noise})}{p(x|\text{signal})}.$$

- ◆ In the even more special case, when standard deviations  $\sigma_1 = \sigma_2$  then
  - $L(x)$  increases monotonically with  $x$ . Consequently, ROC becomes a convex curve.
  - The optimal threshold  $\Theta$  becomes

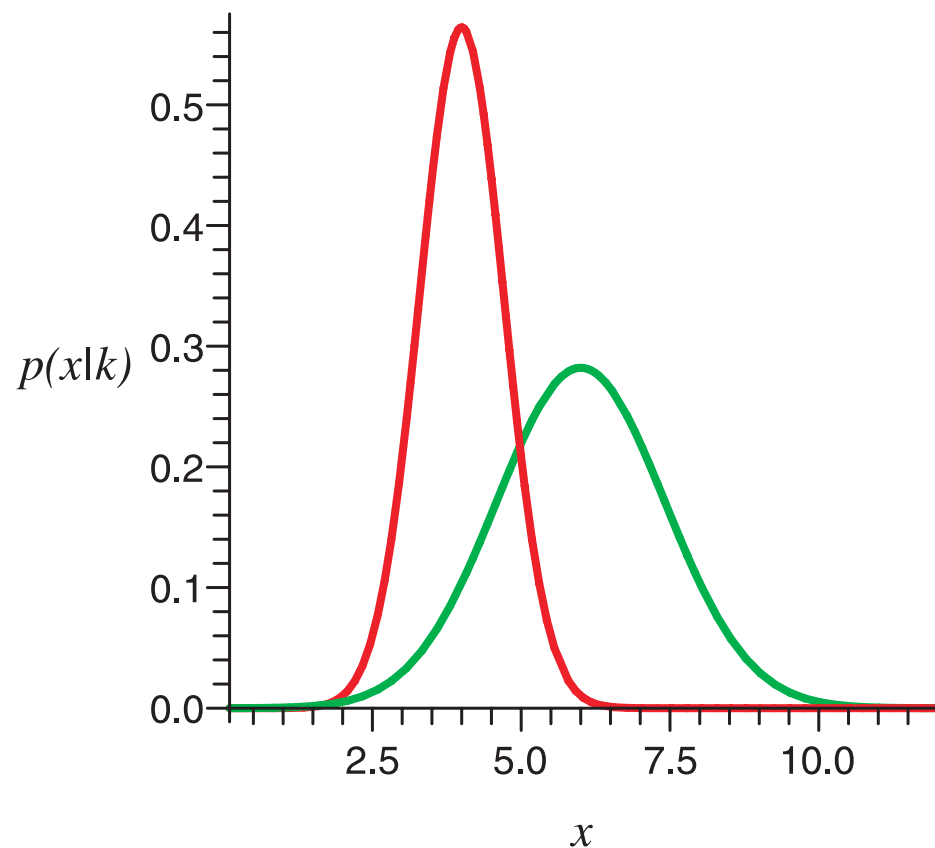
$$\Theta = \frac{p(\text{noise})}{p(\text{signal})}.$$

# ROC – Example D

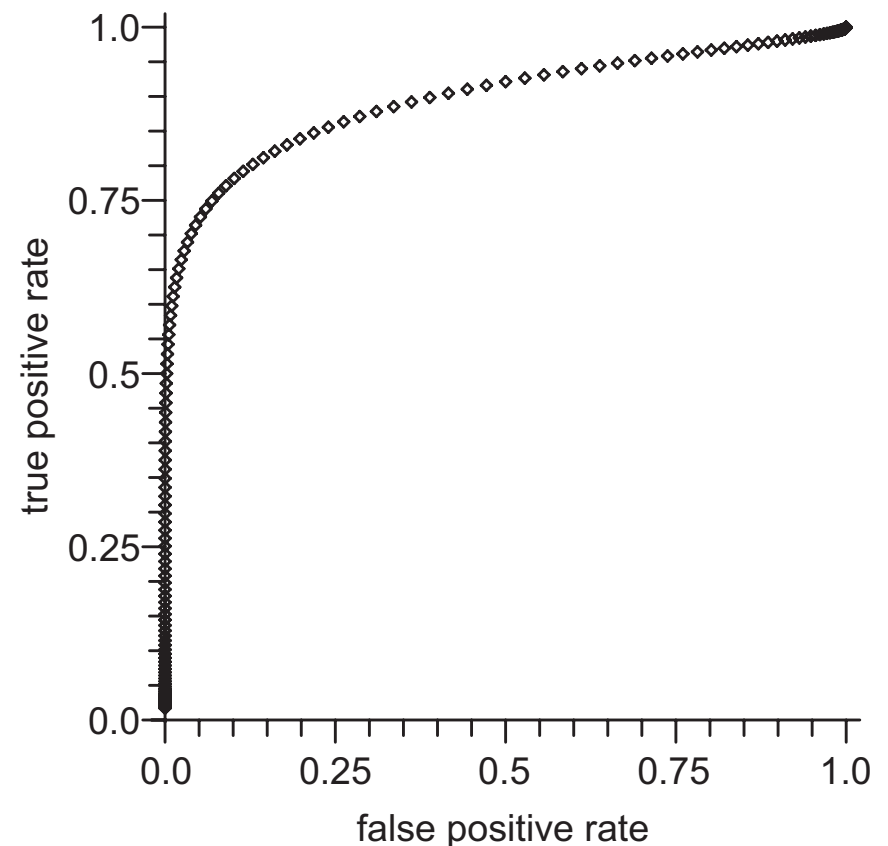
## Two Gaussians with different variances

- ◆ Two Gaussians,  $\mu_1 = 4.0$ ,  $\mu_2 = 6.0$ ,  $\sigma_1 = 1.0$ ,  $\sigma_2 = 2.0$
- ◆ Less overlap, better discriminability.
- ◆ In general, ROC is not convex.

Stochastic model, Gaussians, different variances



ROC curve

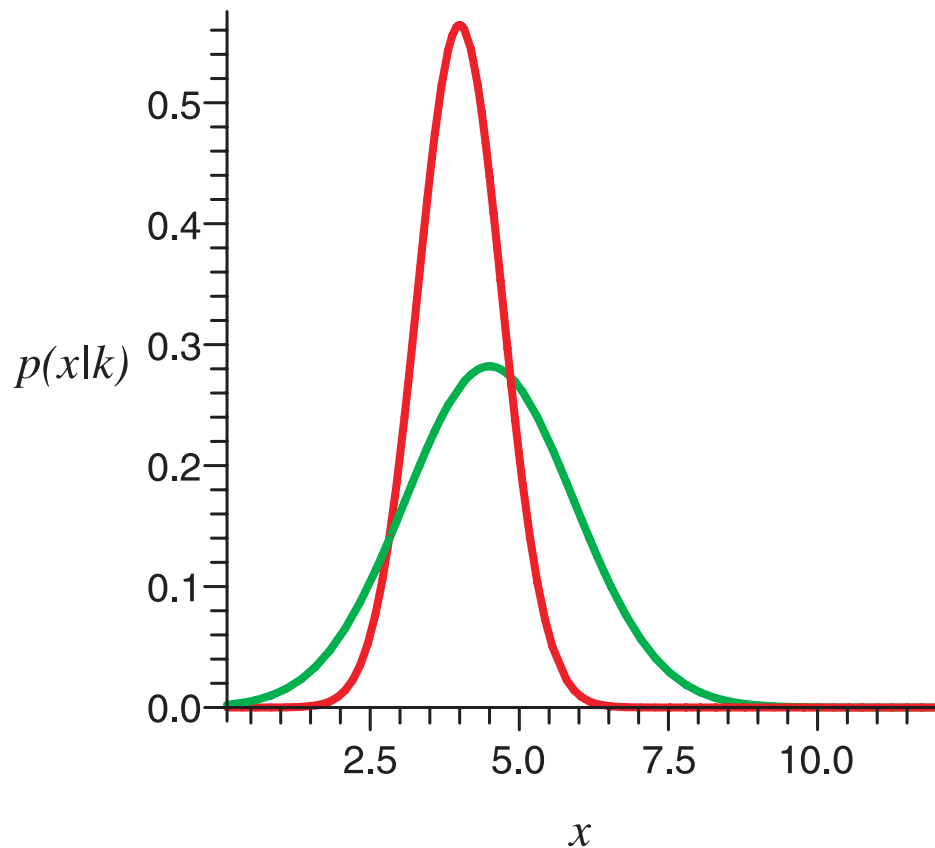


# ROC – Example E

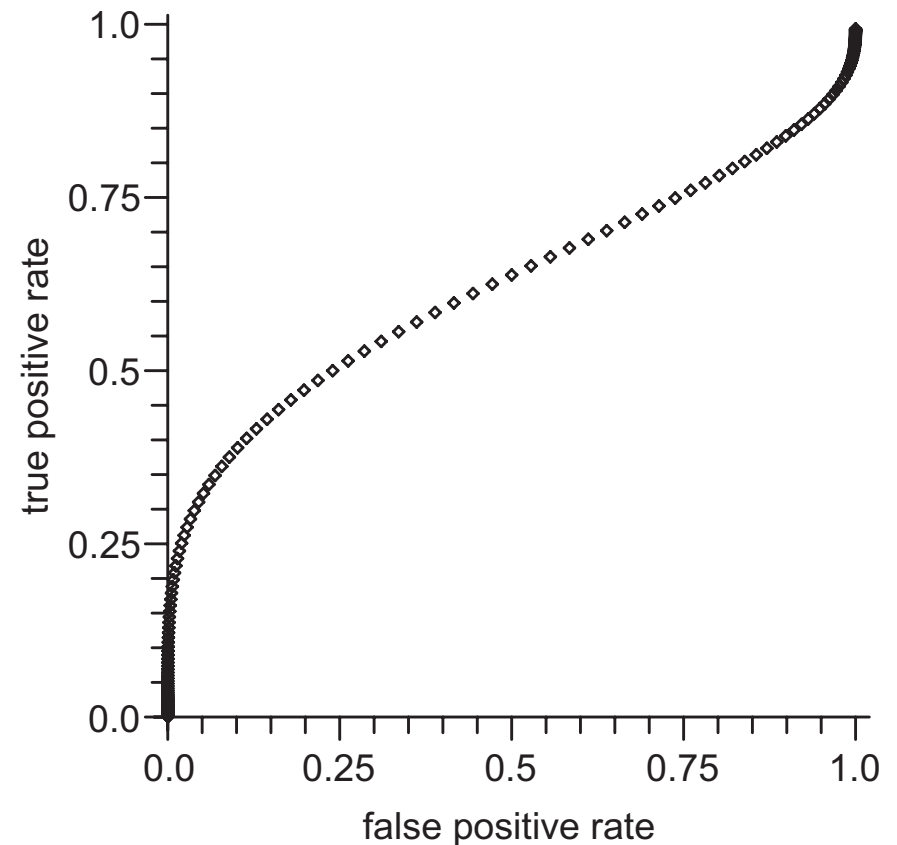
## Two Gaussians with different variances

- ◆ Two Gaussians,  $\mu_1 = 4.0$ ,  $\mu_2 = 4.5$ ,  $\sigma_1 = 1.0$ ,  $\sigma_2 = 2.0$
- ◆ More overlap, worse discriminability.
- ◆ In general, ROC is not convex.

Stochastic model, Gaussians, different variances



ROC curve

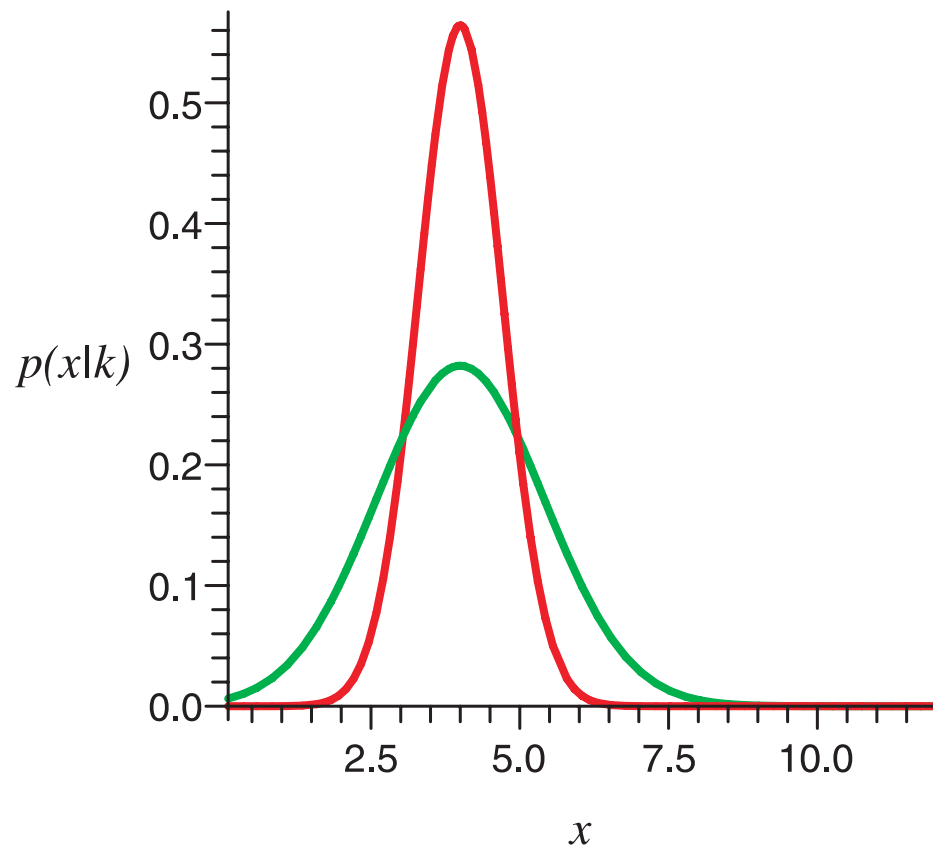


# ROC – Example F

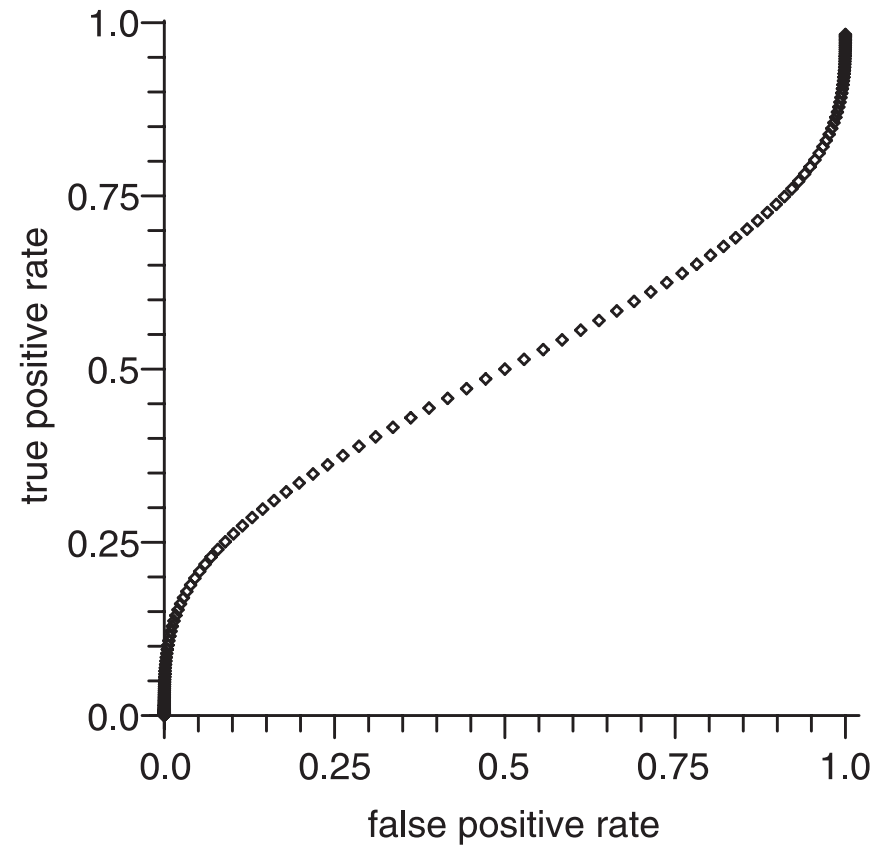
## Two Gaussians with different variances

- ◆ Two Gaussians,  $\mu_1 = 4.0$ ,  $\mu_2 = 4.0$ ,  $\sigma_1 = 1.0$ ,  $\sigma_2 = 2.0$
- ◆ Maximal overlap, the worst discriminability.
- ◆ In general, ROC is not convex.

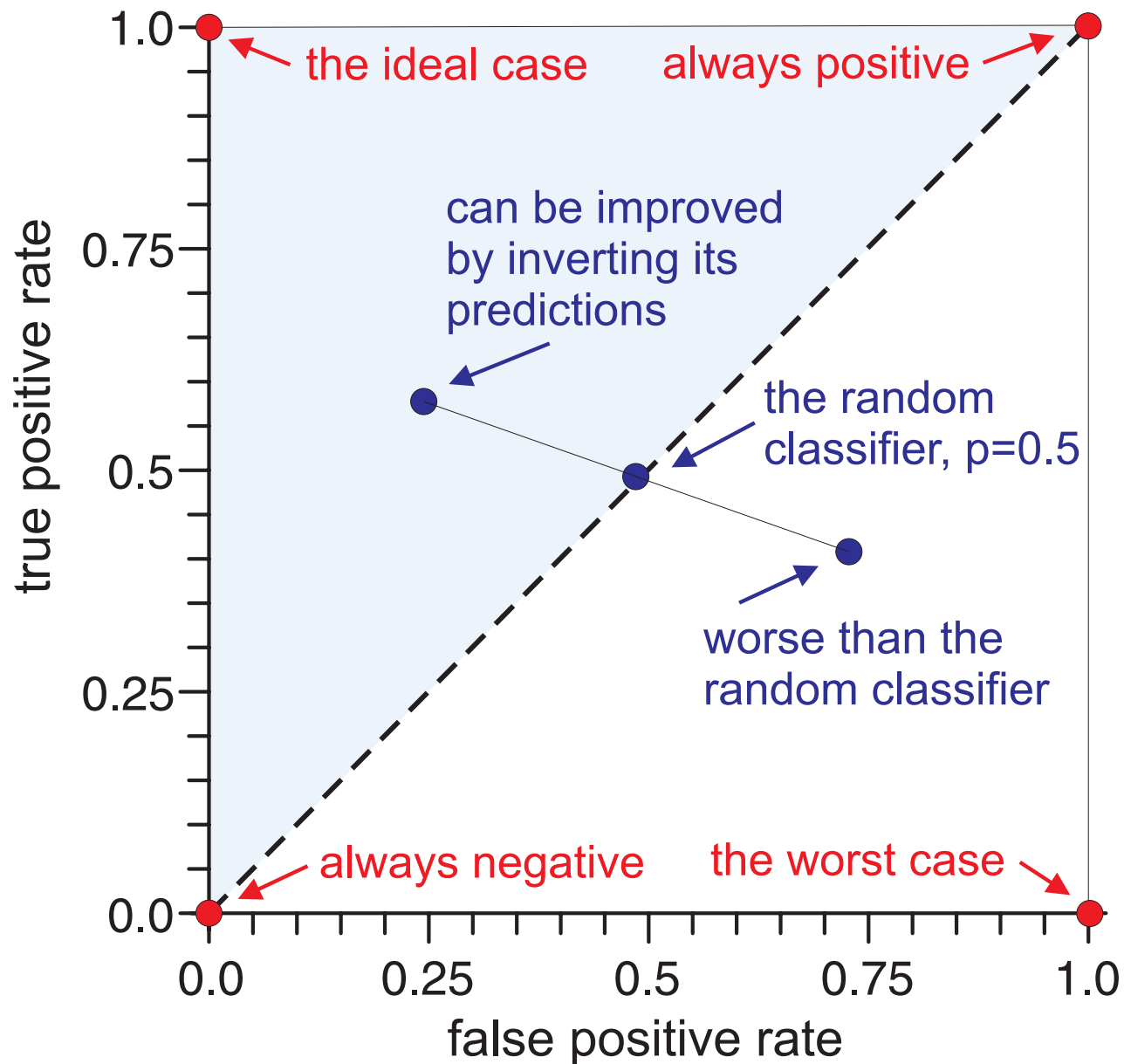
Stochastic model, Gaussians, different variances



ROC curve



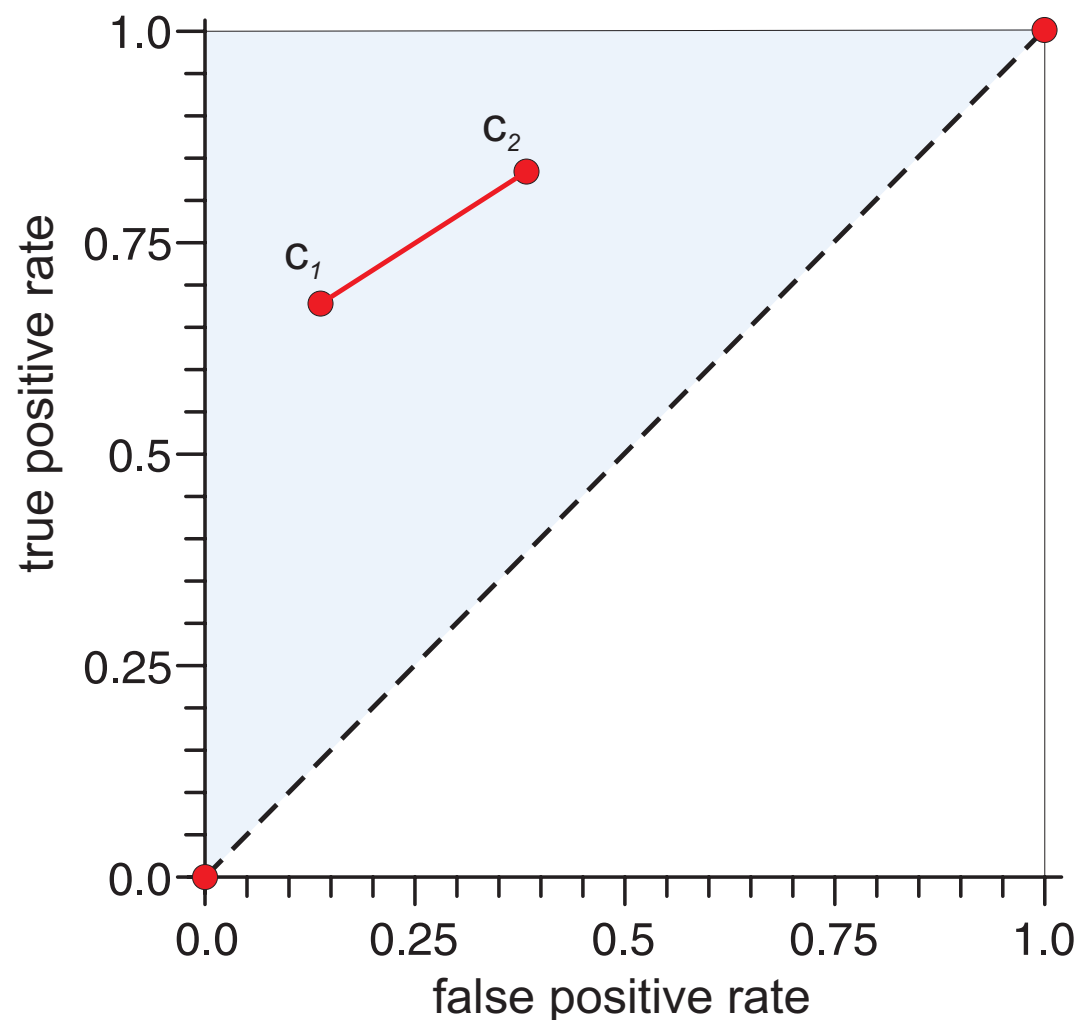
# Properties of the ROC space



# 'Continuity' of the ROC

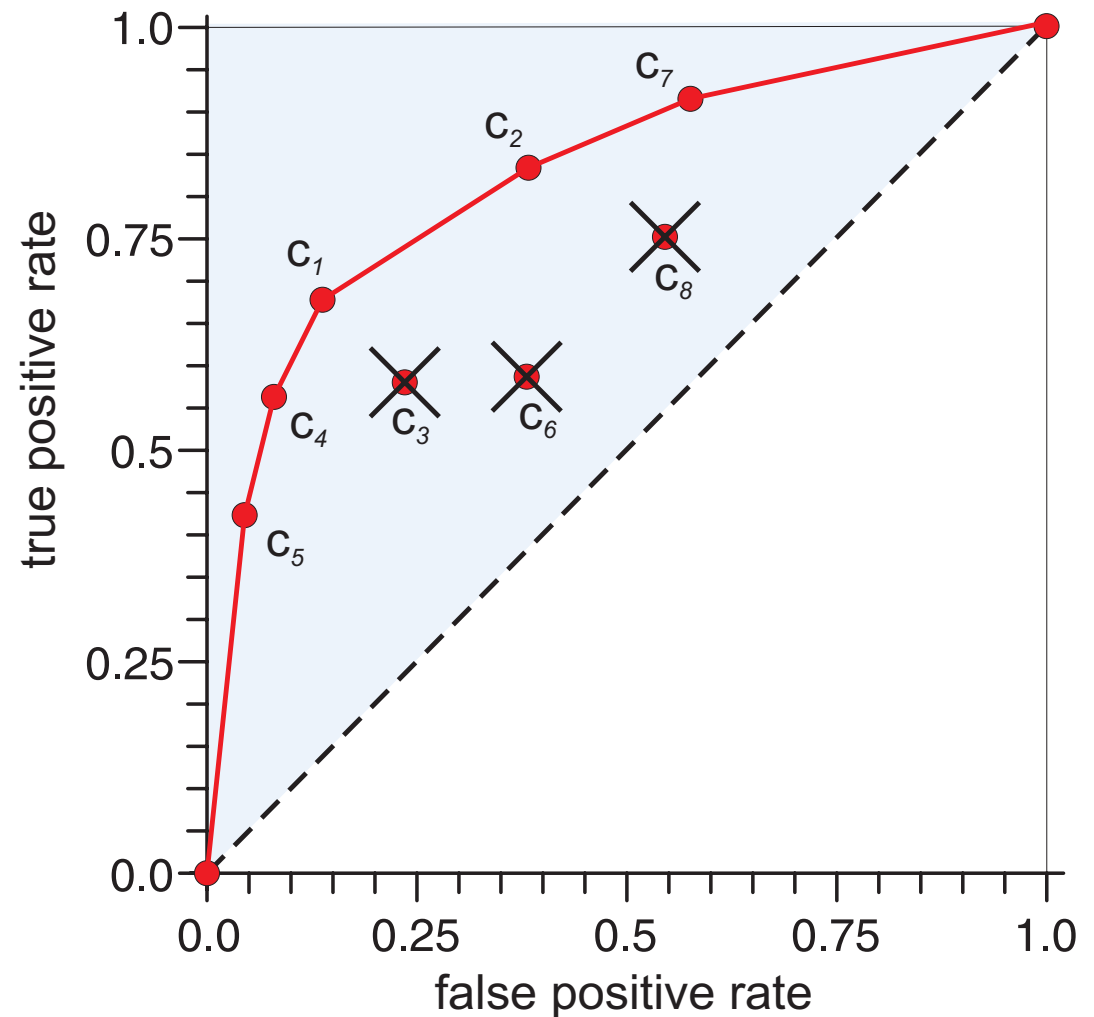
Given two classifiers  $c_1$  and  $c_2$ .

- ◆ Classifiers  $c_1$  and  $c_2$  can be linearly weighted to create 'intermediate' classifiers and
- ◆ In such a way, a continuum of classifiers can be imagined which is denoted by a red line in the figure.



# More classifiers, ROC construction

- ◆ The convex hull covering classifiers in the ROC space is constructed.
- ◆ Classifiers on the convex hull achieve always the best performance for some class probability distributions (i.e., the ratio of positive and negative examples).
- ◆ Classifiers inside the convex hull perform worse and can be discarded.
- ◆ No penalties have been involved so far. To come ...



## ROC convex hull and iso-accuracy

- ◆ Each line segment on the convex hull is an iso-accuracy line for a particular class distribution.
  - Under that distribution, the two classifiers on the end-points achieve the same accuracy.
  - For distributions skewed towards negatives (steeper slope than  $45^\circ$ ), the left classifier is better.
  - For distributions skewed towards positives (flatter slope than  $45^\circ$ ), the right classifier is better.
- ◆ Each classifier on convex hull is optimal for a specific range of class distributions.

# Accuracy expressed in the ROC space

Accuracy  $a$

$$a = pos \cdot TPR + neg \cdot (1 - FPR).$$

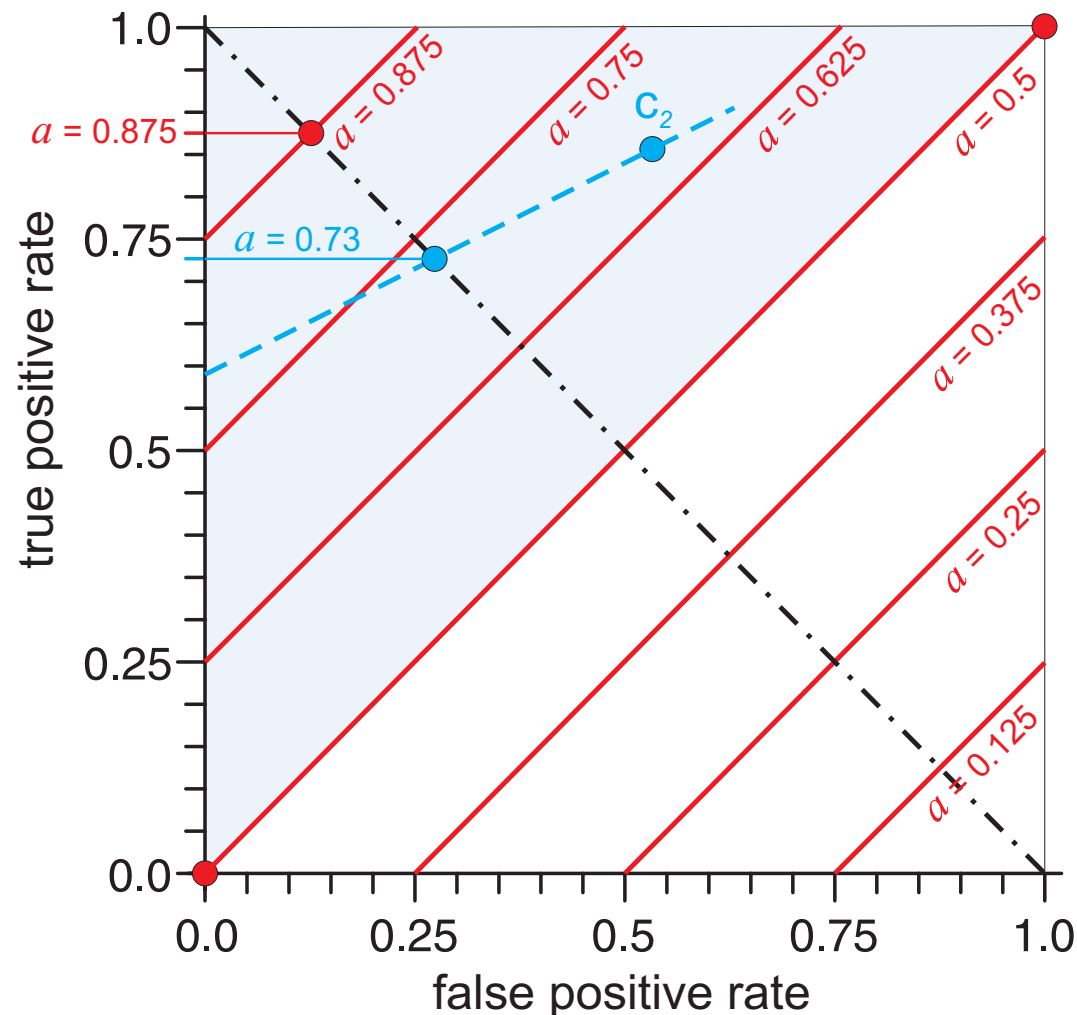
Express the accuracy in the ROC space as  $TPR = f(FPR)$ .

$$TPR = \frac{a - neg}{pos} + \frac{neg}{pos} \cdot FPR.$$

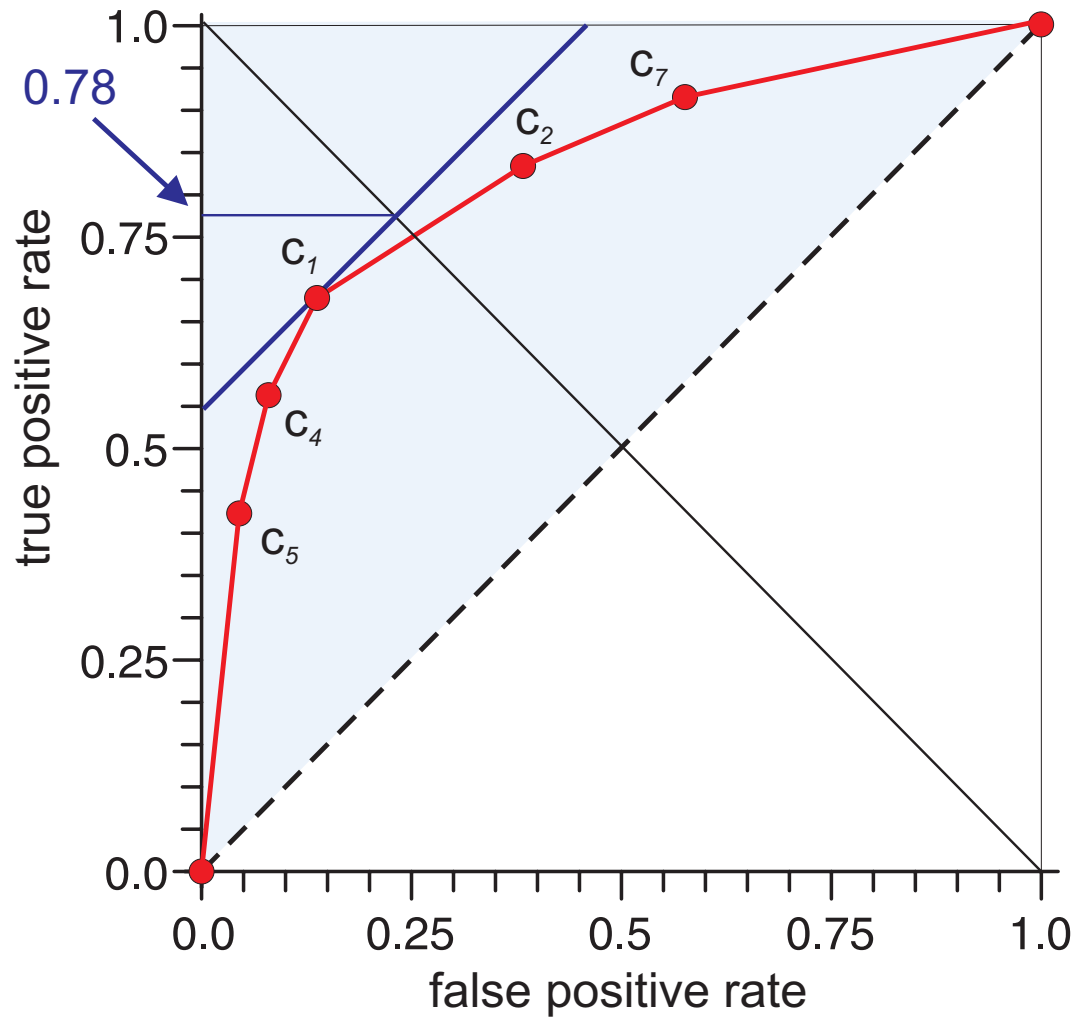
Iso-accuracy are given by straight lines with the slope  $neg/pos$ , i.e., by a degree of balance in the test set. The balanced set  $\Leftrightarrow 45^\circ$ .

The diagonal  $\sim TPR = FPR = a$ .

A blue line is a iso-accuracy line for a general classifier  $c_2$ .



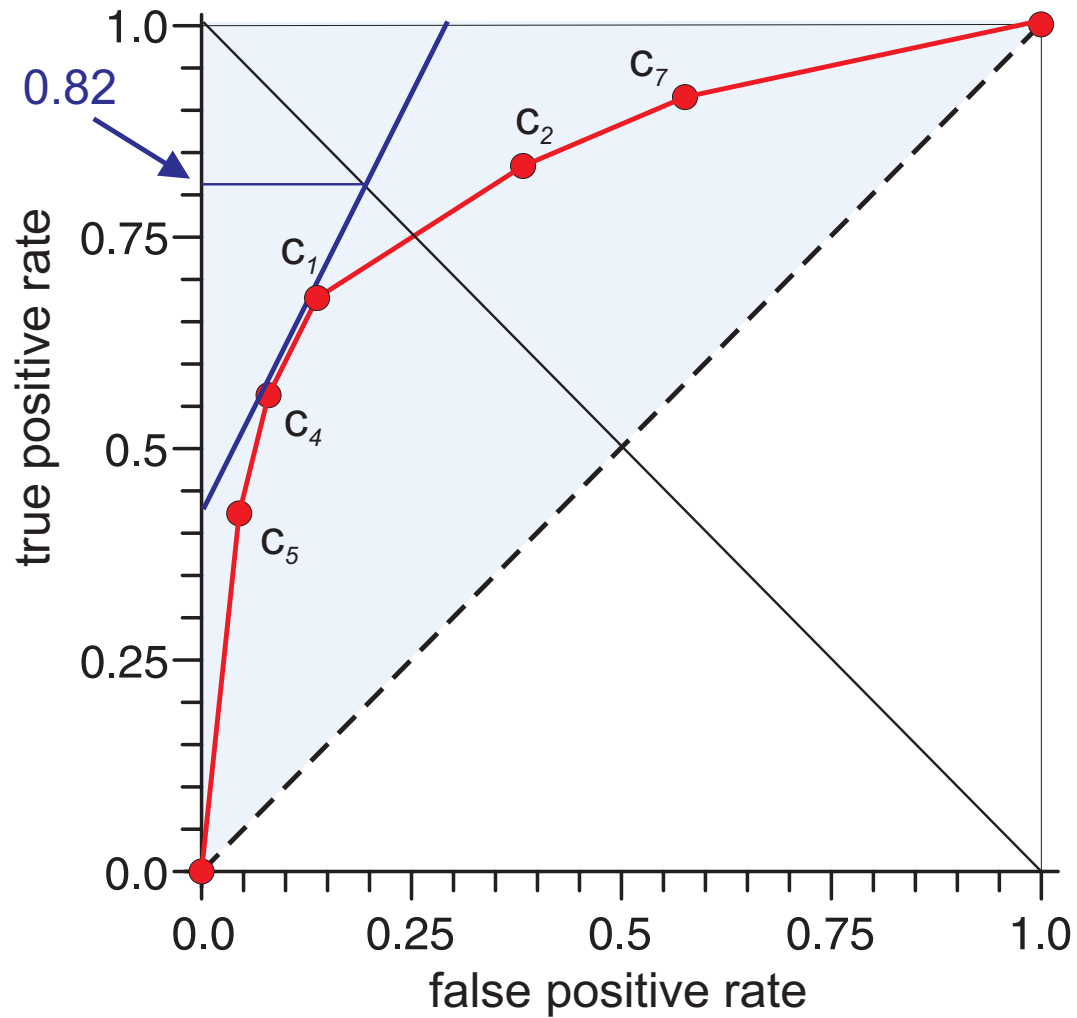
# Selecting the optimal classifier (1)



For a balanced training set (i.e., as many +ves as -ves), see solid blue line.

Classifier  $c_1$  achieves  $\approx 78\%$  true positive rate.

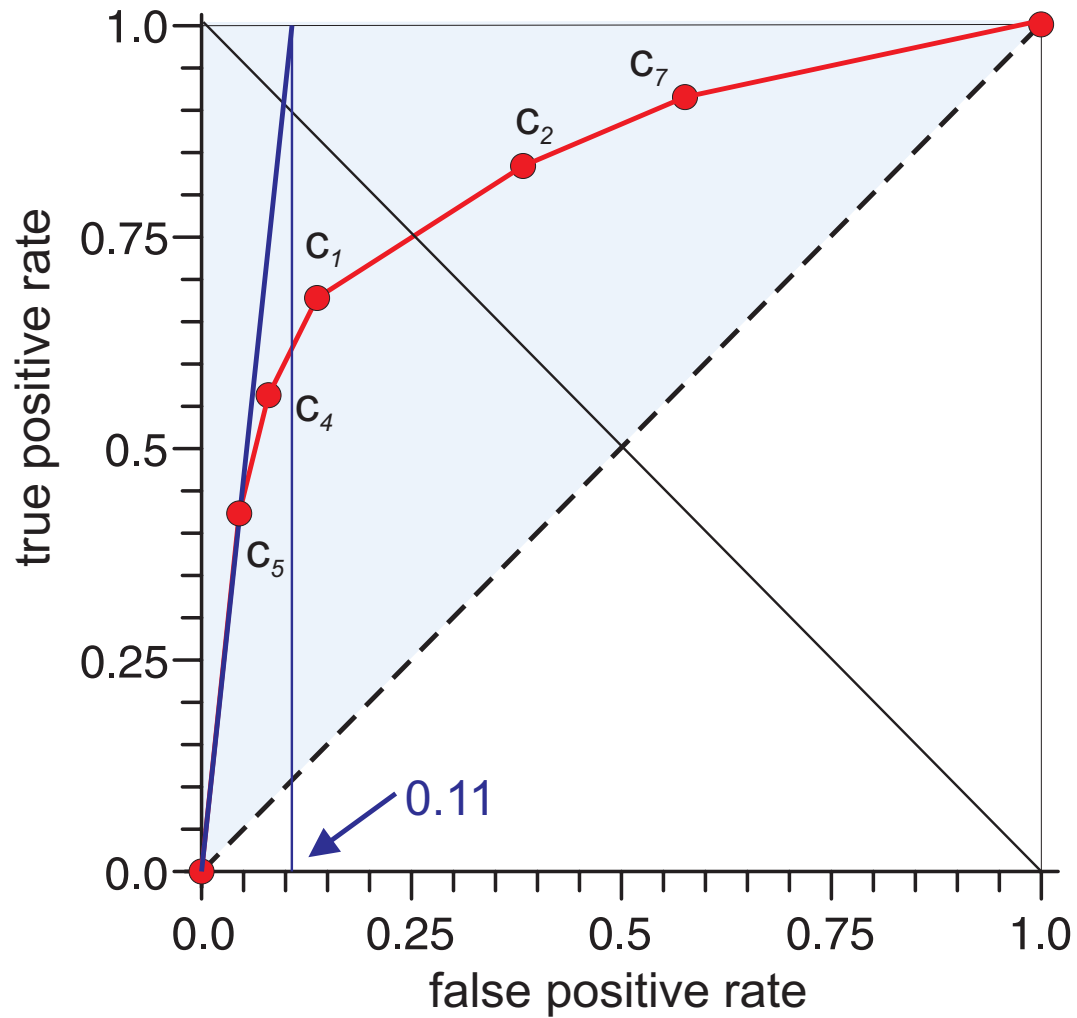
# Selecting the optimal classifier (2)



For a training set with half +ves than -ves), see solid blue line.

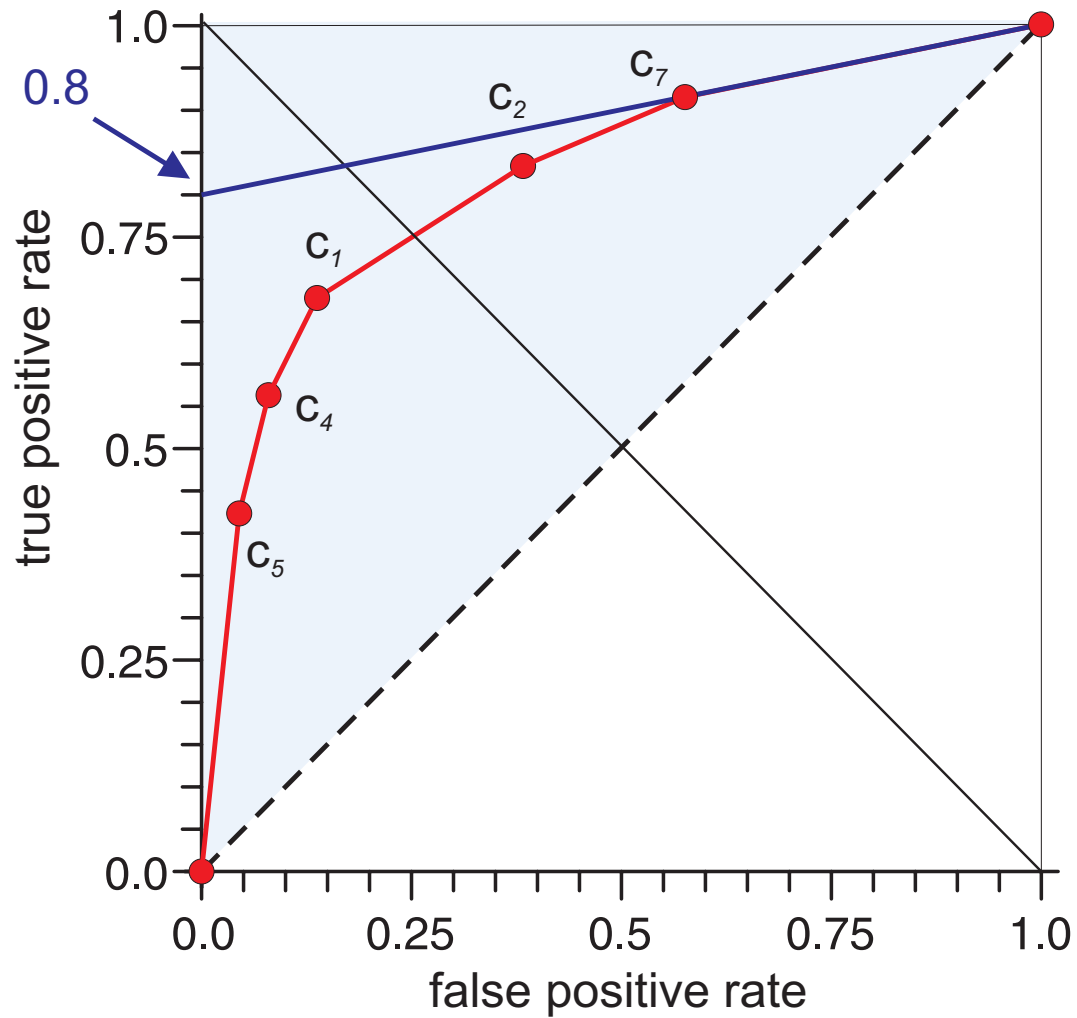
Classifier  $c_1$  achieves  $\approx 82\%$  true positive rate.

# Selecting the optimal classifier (3)



For less than 11 % -ves, the always negative is the best.

# Selecting the optimal classifier (4)



For less than 80 % +ves, the always positive is the best.

## Benefit of the ROC approach

- ◆ It is possible to distinguish operationally between
  - Discriminability** – inherent property of the detection system.
  - Decision bias** – implied by the loss function changable by the user.
- ◆ The discriminability can be determined from ROC curve.
- ◆ If the Gaussian assumption holds then the Bayesian error rate can be calculated.

# ROC – generalization to arbitrary multidimensional distribution

- ◆ The approach can be used for any two multidimensional distributions  $p(\mathbf{x}|k_1)$ ,  $p(\mathbf{x}|k_2)$  provided they overlap and thus have nonzero Bayes classification error.
- ◆ Unlike in one-dimensional case, there may be many decision boundaries corresponding to particular true positive rate, each with different false positive rate.
- ◆ Consequently, we cannot determine discriminability from true positive and false positive rate without knowing more about underlying decision rule.

# Optimal true/false positive rates? Unfortunately not

- ◆ This is a rarely attainable ideal for a multidimensional case.
- ◆ We should have found the decision rule of all the decision rules giving the measured true positive rate, the rule which has the minimum false negative rate.
- ◆ This would need huge computational resources (like Monte Carlo).
- ◆ In practice, we forgo optimality.