Geometry for robotics

Václav Hlaváč

Czech Technical University in Prague

Czech Institute of Informatics, Robotics and Cybernetics

166 36 Prague 6, Jugoslávských partyzánů 3, Czech Republic

http://people.ciirc.cvut.cz/hlavac, vaclav.hlavac@cvut.cz

Courtesy: J. Xiao, R. Möller, T. Pajdla

Outline of the talk:

- Formalisms, notation, rehersal.
- Point in the 3D vector space.
- Rotation in the 3D vector space.
- Rotation matrix, its inversion.
- Rotation and translation jointly.
- Euler, Cardan angles.

Where and why is geometry needed in robotics ?

- Motion in robotics is often approximated by a movement of a rigid body in a 3D space.
- We briefly review a needed mathematical formalism(s), i.e. geometry of motion.
- Three main application areas in robotics from a geometric point of view are:
  1. Open kinematic chain manipulators.
  2. Closed kinematic chain mechanisms.
  3. Mobile robots.

The item 2 will not be tackled because it is too complicated for this overview course.

Formalisms

- Vector space.
- Projective space ($\Rightarrow$ homogeneous coordinates).
- *Quaternions. (not explained here)*

We start with a quick math review.

Notation

The notation of the subject B3M33PRO (Advanced robotics, lectured by Dr Tomas Pajdla for the Robotics study branch in the coming semester) is used to maintain consistency.

$\vec{x}$	...	vector
A	...	matrix
A_{ij}	...	element ij of A_{ij}
$\mathsf{A}^\top$	...	A transposed
$\lvert\mathsf{A}\rvert$	...	determinant of A
I	...	identity matrix
R	...	rotation matrix
$\vec{x} \times \vec{y}$	...	vector (cross) product of $\vec{x}, \vec{y}$

β	...	basis, the ordered triple $\beta = [\vec{b}_1, \vec{b}_2, \vec{b}_3]$ of independent generator vectors
$\vec{x}_\beta$	...	column matrix of coordinates w.r.t. the basis β
$\vec{x} \cdot \vec{y}$	...	scalar product of vectors $\vec{x}, \vec{y}$
$\lVert\vec{x}\rVert$	...	Euclidean norm of $\vec{x}$, $\lVert\vec{x}\rVert = \sqrt{\vec{x} \cdot \vec{x}}$

Dot product

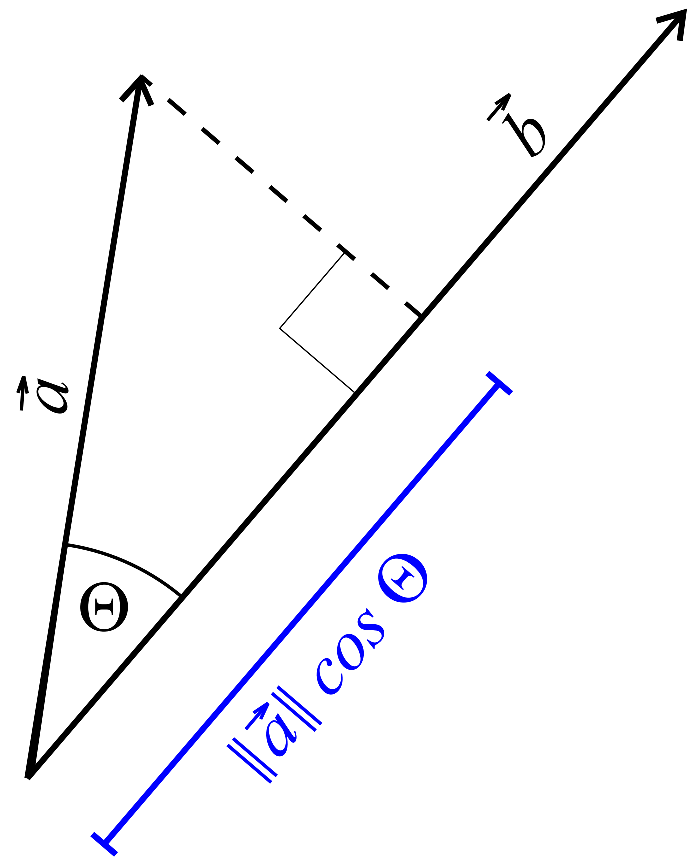
Dot product of vectors $\vec{a}, \vec{b}$
(also scalar or inner product)

- Geometric definition:
  $\vec{a} \cdot \vec{b} = \|\vec{a}\| \; \|\vec{b}\| \; \cos\Theta$
- Algebraic definition:
  $\vec{a} \cdot \vec{b} = \sum_{i=1}^{n} a_i \, b_i$

A two-dimensional example

$\vec{a} = [a_x, a_y]^\top, \quad \vec{b} = [b_x, b_y]^\top$

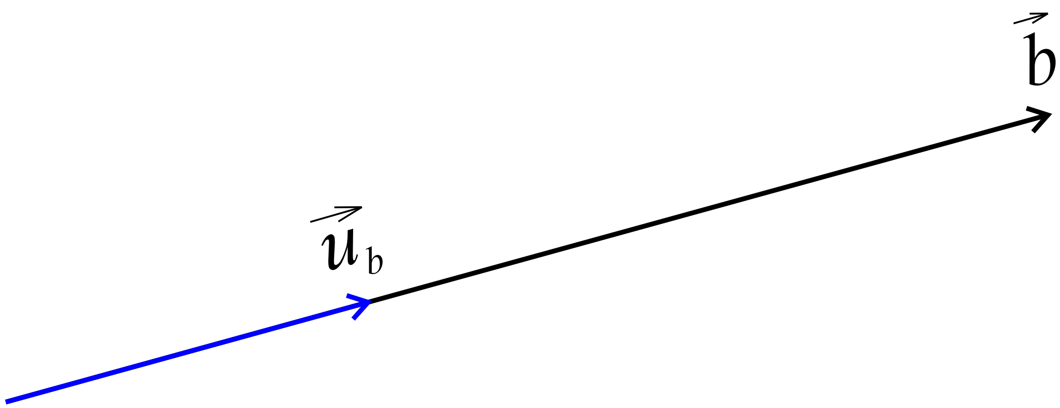
$$\vec{a} \cdot \vec{b} = \begin{bmatrix} a_x \\ a_y \end{bmatrix} \cdot \begin{bmatrix} b_x \\ b_y \end{bmatrix} = a_x \, b_x + a_y \, b_y$$



Unit vector

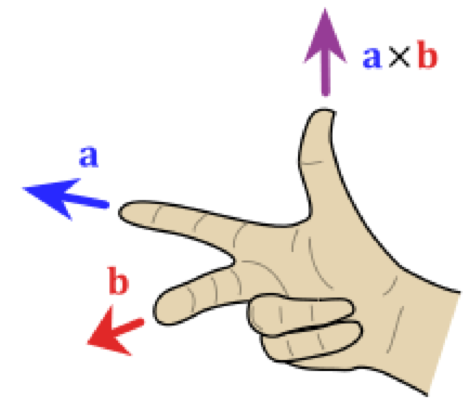
Unit vector $\vec{u}_b$ is a vector in the direction of a chosen vector (in our particular case of the vector $\vec{b}$), the magnitude of which equals to one.

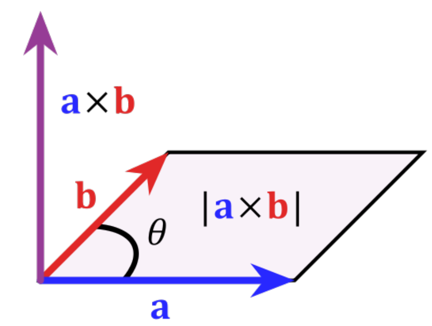
$$\vec{u}_b = \frac{\vec{b}}{\|\vec{b}\|}$$



Cross (vector) product

- The cross product $\vec{a} \times \vec{b}$ is defined as a vector $\vec{c}$ that is perpendicular to both $\vec{a}$ and $\vec{b}$, with a direction given by the right-hand rule and a magnitude equal to the area of the parallelogram that the vectors span, i.e. $||\vec{a}||\,||\vec{b}||\sin\Theta$.
- Alternatively: $\vec{a} \times \vec{b} = ||\vec{a}||\,||\vec{b}||\,\sin\Theta\,\vec{n}$, where $\vec{n}$ is a unit vector perpendicular to the plane containing $\vec{a}$, $\vec{b}$ and the direction given by the right-hand rule.
- The cross-product is anti-commutative, i.e., $\vec{b} \times \vec{a} = -(\vec{a} \times \vec{b})$.





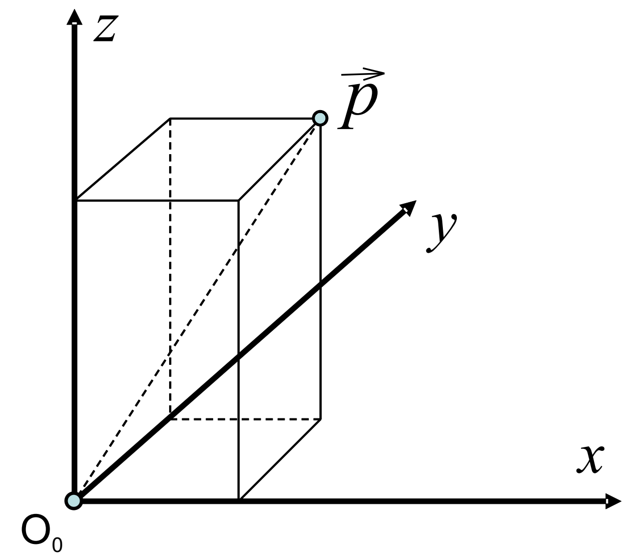
Animation

Cartesian coordinate system

- Specifies the point in an n-dimensional Euclidean space. Coordinates are equal, up to the sign, to distances from the point to n mutually perpendicular hyperplanes.
- In 3D, reference coordinate system O_0xyz.
- Point $\vec{p} = [p_x, p_y, p_z]^\top$ represented in O_0xyz:

  $\vec{p}_{xyz} = p_x\vec{\imath}_x + p_y\vec{\jmath}_y + p_z\vec{k}_z$
- $\vec{\imath} \cdot \vec{\jmath} = 0$, $\vec{\imath} \cdot \vec{k} = 0$, $\vec{k} \cdot \vec{\jmath} = 0$

  $|\vec{\imath}| = 1$, $|\vec{\jmath}| = 1$, $|\vec{k}| = 1$
- Name after René Descartes (latinized: Cartesius), who provided the first systematic link between Euclidean geometry and algebra. Bílá hora battle participant.



René Descartes, 1596-1650

Reference coordinate system
Representation of a point in it

- Reference coordinate system O_0xyz,
  unit coordinate vectors $\vec{x}$, $\vec{y}$, $\vec{z}$.

  Body attached frame O_1uvw,
  unit coordinate vectors $\vec{u}$, $\vec{v}$, $\vec{w}$.

- Point represented in O_0xyz:

  $\vec{p} = [p_x, p_y, p_z]^\top$

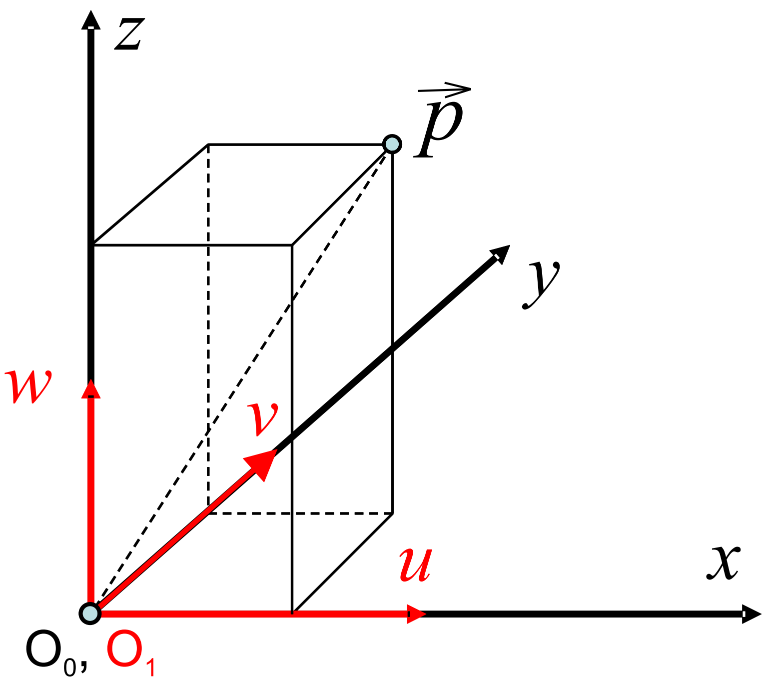
  $\vec{p}_{xyz} = p_x\vec{\imath}_x + p_y\vec{\jmath}_y + p_z\vec{k}_z$

- Point represented in O_1uvw:

  $\vec{p}_{uvw} = p_u\vec{\imath}_u + p_v\vec{\jmath}_v + p_w\vec{k}_w$

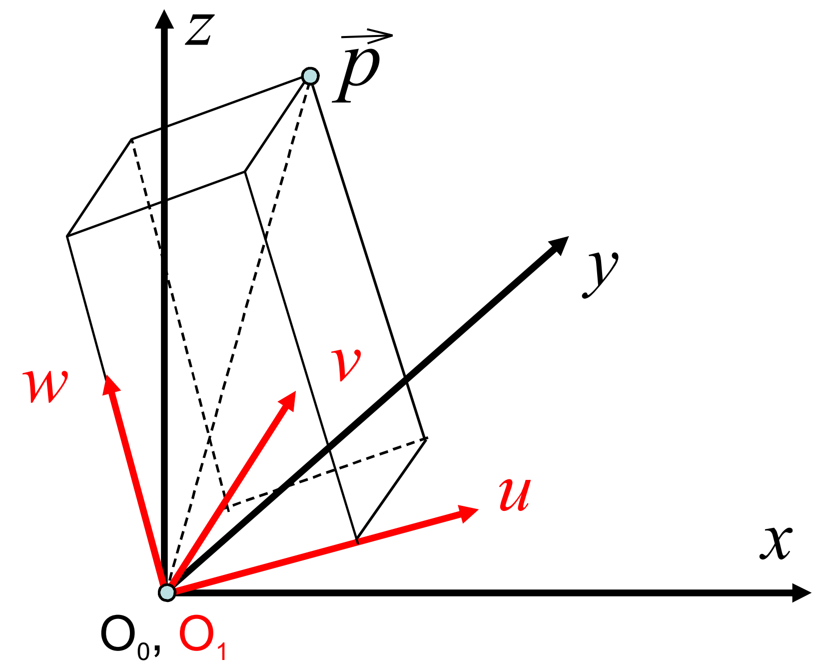
- If these two frames coincide then

  $p_u = p_x$, $p_v = p_y$, $p_w = p_z$



Coordinate transformation, rotation only

- $\vec{p}_{xyz} = p_x \vec{\imath}_x + p_y \vec{\jmath}_y + p_z \vec{k}_z$
- $\vec{p}_{uvw} = p_u \vec{\imath}_u + p_v \vec{\jmath}_v + p_w \vec{k}_w$
- $\vec{p}_{xyz} = \mathsf{R}\, \vec{p}_{uvw}$, where R is a rotation matrix.
- p_x, p_y and p_z represent projections of a point $\vec{p}$ onto O_0x, O_0y, O_0z axes, respectively.
- $$p_x = \vec{\imath}_x \cdot \vec{p} = \vec{\imath}_x \cdot \vec{\imath}_u\, p_u + \vec{\imath}_x \cdot \vec{\jmath}_v\, p_v + \vec{\imath}_x \cdot \vec{k}_w\, p_w$$
  $$p_y = \vec{\imath}_y \cdot \vec{p} = \vec{\imath}_y \cdot \vec{\imath}_u\, p_u + \vec{\imath}_y \cdot \vec{\jmath}_v\, p_v + \vec{\imath}_y \cdot \vec{k}_w\, p_w$$
  $$p_z = \vec{\imath}_z \cdot \vec{p} = \vec{\imath}_z \cdot \vec{\imath}_u\, p_u + \vec{\imath}_z \cdot \vec{\jmath}_v\, p_v + \vec{\imath}_z \cdot \vec{k}_w\, p_w$$

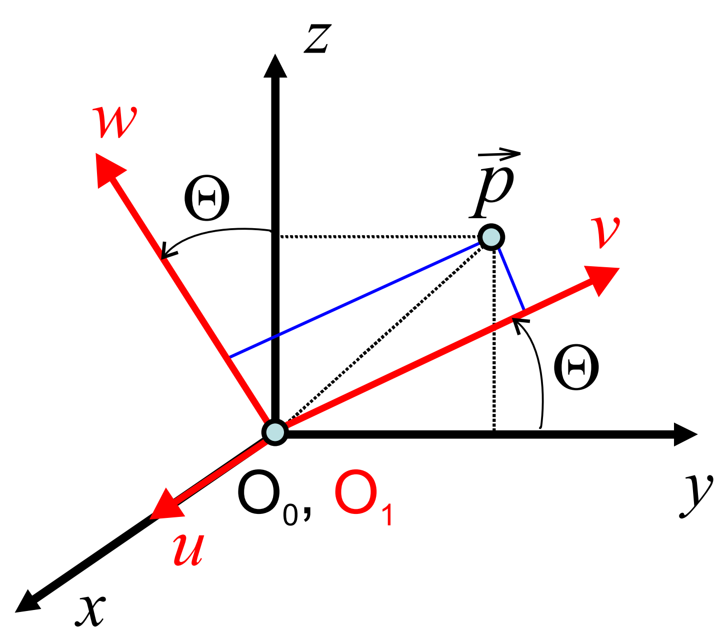


Rotation matrix

- Repeated from the previous slide:

$$p_x = \vec{\imath}_x \cdot \vec{p} = \vec{\imath}_x \cdot \vec{\imath}_u \, p_u + \vec{\imath}_x \cdot \vec{\jmath}_v \, p_v + \vec{\imath}_x \cdot \vec{k}_w \, p_w$$
$$p_y = \vec{\imath}_y \cdot \vec{p} = \vec{\imath}_y \cdot \vec{\imath}_u \, p_u + \vec{\imath}_y \cdot \vec{\jmath}_v \, p_v + \vec{\imath}_y \cdot \vec{k}_w \, p_w$$
$$p_z = \vec{\imath}_z \cdot \vec{p} = \vec{\imath}_z \cdot \vec{\imath}_u \, p_u + \vec{\imath}_z \cdot \vec{\jmath}_v \, p_v + \vec{\imath}_z \cdot \vec{k}_w \, p_w$$

Example, rotation about axis x by Θ:



- Expressed as a matrix multiplication:

$$\begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = \begin{bmatrix} \vec{\imath}_x \cdot \vec{\imath}_u & \vec{\imath}_x \cdot \vec{\jmath}_v & \vec{\imath}_x \cdot \vec{k}_w \\ \vec{\jmath}_y \cdot \vec{\imath}_u & \vec{\jmath}_y \cdot \vec{\jmath}_v & \vec{\jmath}_y \cdot \vec{k}_w \\ \vec{k}_z \cdot \vec{\imath}_u & \vec{k}_z \cdot \vec{\jmath}_v & \vec{k}_z \cdot \vec{k}_w \end{bmatrix} \begin{bmatrix} p_u \\ p_v \\ p_w \end{bmatrix}$$

- Example, rotation about axis x by Θ:

$$\mathsf{R} = \mathsf{R}(x, \Theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\Theta & -\sin\Theta \\ 0 & \sin\Theta & \cos\Theta \end{bmatrix}$$

$$p_x = p_u$$
$$p_y = p_v \cos\Theta - p_w \sin\Theta$$
$$p_z = p_v \sin\theta + p_w \cos\Theta$$

Rotation about coordinate axes

- Rotation about axis x by Θ:

$$\vec{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} ; \ \vec{v} = \begin{bmatrix} 0 \\ \cos\Theta \\ \sin\Theta \end{bmatrix} ; \ \vec{w} = \begin{bmatrix} 0 \\ -\sin\Theta \\ \cos\Theta \end{bmatrix} ; \quad \mathsf{R} = \mathsf{R}(x,\Theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\Theta & -\sin\Theta \\ 0 & \sin\Theta & \cos\Theta \end{bmatrix}$$

- Rotation about axis y by Θ:

$$\mathsf{R} = \mathsf{R}(y,\Theta) = \begin{bmatrix} \cos\Theta & 0 & \sin\Theta \\ 0 & 1 & 0 \\ -\sin\Theta & 0 & \cos\Theta \end{bmatrix}$$

- Rotation about axis z by Θ:

$$\mathsf{R} = \mathsf{R}(z,\Theta) = \begin{bmatrix} \cos\Theta & -\sin\Theta & 0 \\ \sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Inverting rotation matrix

- $\vec{p}_{xyz} = \mathsf{R}\,\vec{p}_{uvw}$

$$\begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = \begin{bmatrix} \vec{\imath}_x \cdot \vec{\imath}_u & \vec{\imath}_x \cdot \vec{\jmath}_v & \vec{\imath}_x \cdot \vec{k}_w \\ \vec{\jmath}_y \cdot \vec{\imath}_u & \vec{\jmath}_y \cdot \vec{\jmath}_v & \vec{\jmath}_y \cdot \vec{k}_w \\ \vec{k}_z \cdot \vec{\imath}_u & \vec{k}_z \cdot \vec{\jmath}_v & \vec{k}_z \cdot \vec{k}_w \end{bmatrix} \begin{bmatrix} p_u \\ p_v \\ p_w \end{bmatrix}$$

- $\vec{p}_{uvw} = \mathsf{Q}\,\vec{p}_{xyz}$. Notice: dot product is commutative.

$$\begin{bmatrix} p_u \\ p_v \\ p_w \end{bmatrix} = \begin{bmatrix} \vec{\imath}_u \cdot \vec{\imath}_x & \vec{\imath}_u \cdot \vec{\jmath}_y & \vec{\imath}_u \cdot \vec{k}_z \\ \vec{\jmath}_v \cdot \vec{\imath}_x & \vec{\jmath}_v \cdot \vec{\jmath}_y & \vec{\jmath}_v \cdot \vec{k}_z \\ \vec{k}_w \cdot \vec{\imath}_x & \vec{k}_w \cdot \vec{\jmath}_y & \vec{k}_w \cdot \vec{k}_z \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$$

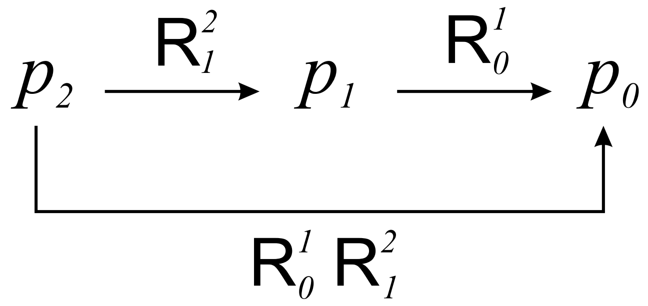
- Rotation matrices are orthogonal, i.e.
  $\mathsf{Q} = \mathsf{R}^{-1} = \mathsf{R}^\top \Rightarrow \mathsf{QR} = \mathsf{R}^\top\mathsf{R} = \mathsf{R}\mathsf{R}^\top = \mathsf{R}^{-1}\mathsf{R} = \mathsf{I}$.
  (a) Column vector are mutually perpendicular unit vectors; (b) $\det \mathsf{R} = \pm 1$ ($+1$ for right-hand coordinates); (c) $\mathsf{R} \in \mathrm{SO}(3)$, i.e. special orthogonal group of rotational matrices of the third order.

Composite rotation matrix

- A sequence of finite rotations.
- Matrix multiplications do not commute $\Rightarrow$ the correct order is important.
- Point $\vec{p}$ is represented as $\vec{p}_0$ w.r.t. to its coordinates $Oi_0j_0k_0$.

  Point $\vec{p}_1$ similarly as $\vec{p}_1$ w.r.t. $Oi_1j_1k_1$.

  Point $\vec{p}_2$ similarly as $\vec{p}_2$ w.r.t. $Oi_2j_2k_2$.
- $\vec{p}_0 = \mathsf{R}_0^1\,\vec{p}_1$ and $\vec{p}_1 = \mathsf{R}_1^2\,\vec{p}_2$
- $\mathsf{R}_0^2 = \mathsf{R}_0^1\,\mathsf{R}_1^2$, consequently $\vec{p}_0 = \mathsf{R}_0^2\,\vec{p}_2$



Example, a composite rotation, around z-axis first, around y axis next

1. Rotation around the current z-axis by the angle Θ.
2. Rotation around the current y-axis by the angle Φ.

$$\mathsf{R} = \mathsf{R}(y,\Phi)\,\mathsf{R}(z,\Theta) = \begin{bmatrix} \cos\Phi & 0 & \sin\Phi \\ 0 & 1 & 0 \\ -\sin\Phi & 0 & \cos\Phi \end{bmatrix} \begin{bmatrix} \cos\Theta & -\sin\Theta & 0 \\ \sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\Phi\ \cos\Theta & -\cos\Phi\ \sin\Theta & \sin\Phi \\ \sin\Theta & \cos\Theta & 0 \\ -\sin\Phi\ \cos\Theta & \sin\Phi\ \sin\Theta & \cos\Phi \end{bmatrix}$$

Example, a composite rotation, around y-axis first, around z-axis next

1. Rotation around the current y-axis by the angle Φ.
2. Rotation around the current z-axis by the angle Θ.

$$\mathsf{R} = \mathsf{R}(z, \Theta)\, \mathsf{R}(y, \Phi) = \begin{bmatrix} \cos\Theta & -\sin\Theta & 0 \\ \sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\Phi & 0 & \sin\Phi \\ 0 & 1 & 0 \\ -\sin\Phi & 0 & \cos\Phi \end{bmatrix}$$

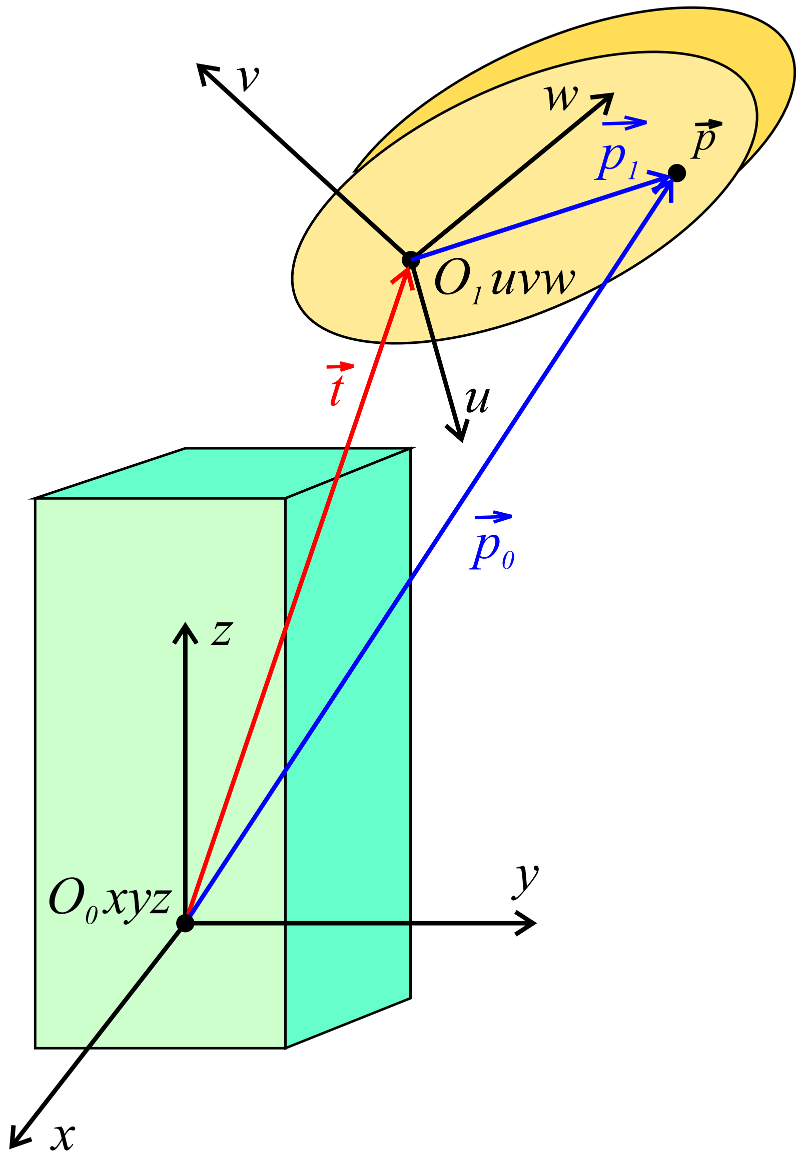
$$= \begin{bmatrix} \cos\Theta\ \cos\Phi & -\sin\Theta & \cos\Theta\ \sin\Phi \\ \sin\Theta\ \cos\Phi & \cos\Theta & 0 \\ -\sin\Phi & 0 & \cos\Theta \end{bmatrix}$$

Rotation and translation jointly

- A point (vector) $\vec{p}$ originally expressed with respect to the coordinate system $O_1 uvw$ as $\vec{p}_1$ is newly represented with respect to the coordinate system $O_0 xyz$ as $\vec{p}_0$.
- The transformation writes as

$$\vec{p}_0 = \mathsf{R}\,\vec{p}_1 + \vec{t}\,,$$

where R is the rotation matrix aligning the coordinate system $O_0 xyz$ to $O_1 uvw$ and $\vec{t}$ is a translation vector bringing the origin O_0 to the origin O_1.



Euler angles, the minimal representation

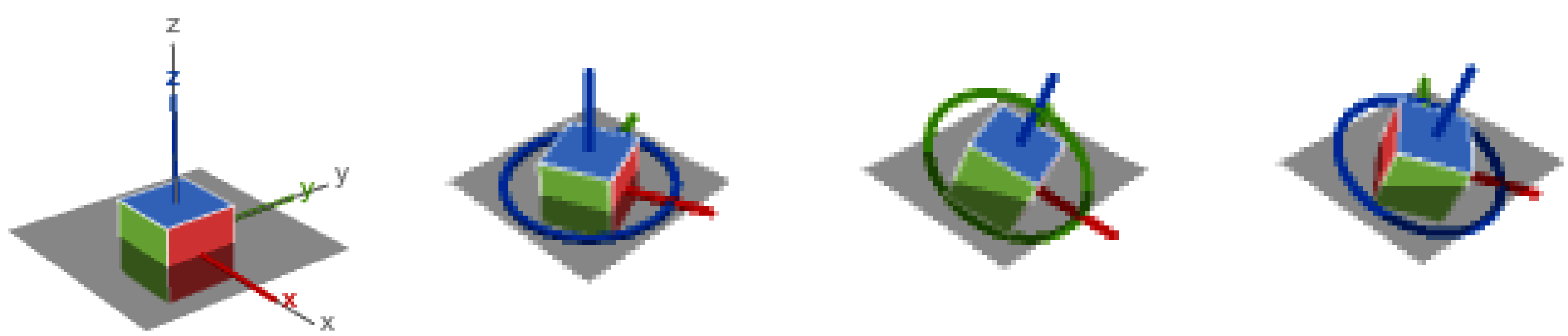
- Rotation matrices provide a redundant representation of the frame orientation. They are given by nine elements.
- These elements are not independent because they are related by the orthogonality condition $\mathsf{R}^\top \mathsf{R} = \mathsf{I}$.
- This implies that three parameters suffice to express orientation of a rigid body in 3D space.
- Orientation expressed by three parameters constitutes a minimal representation.
- There are 12 possible sequences of rotation axes, divided into two groups:
  - Euler angles: $z\,x\,z$, $x\,y\,x$, $y\,z\,y$, $z\,y\,z$, $x\,z\,x$, $y\,x\,y$
  - Cardan angles (after Jerome Cardan or Gerolamo Cardano, also called Tait-Brian, nautical, yaw-pitch-roll):

    $x\,y\,z$, $y\,z\,x$, $z\,x\,y$, $x\,z\,y$, $z\,y\,x$, $y\,x\,z$

$z\ y\ z$ Euler angles

Composition of three elementary rotations

- Rotate the reference frame by the angle ϕ about z-axis.
- Rotate the current frame by the θ about (tranformed) axis y'.
- Rotate the current frame by the angle ψ about axis z''.



Input ϕ θ ψ

$z\,y\,z$ Euler angles, rotation matrices

The rotation described by $z\,y\,z$ composes three rotations of the current frame $R = R_z(\phi)\,R_{y'}(\theta)\,R_{z''}(\psi)$.

- Rotation by the angle ϕ around axis z: $R_z = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$
- Rotation by the angle θ around axis y': $R_{y'} = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$
- Rotation by the angle ψ around axis z'': $R_{z''} = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$z\,y\,z$ Euler angles, the direct solution

Given: Three $z\ y\ z$ Euler angles.

Task: Rotate (1) by the angle ϕ along the axis z giving the new axes x', y' and $z' \equiv z$; (2) by the angle θ along the axis y' giving new axes x'', y'', z'' and (3) by the angle ψ around the axis z''.

Outcome: The rotation matrix R.

$$\mathsf{R} = \mathsf{R}_z(\phi)\,\mathsf{R}_{y'}(\theta)\,\mathsf{R}_{z''}(\psi) =$$

$$= \begin{bmatrix} c_\phi\, c_\theta\, c_\psi - s_\phi\, s_\psi\,, & -c_\phi\, c_\theta\, s_\psi - s_\phi\, c_\psi\,, & c_\phi\, s_\theta \\ s_\phi\, c_\theta\, c_\psi + c_\phi\, s_\psi\,, & -s_\phi\, c_\theta\, s_\psi + c_\phi\, c_\psi\,, & s_\phi\, s_\theta \\ -s_\theta\, c_\psi\,, & s_\theta\, s_\psi\,, & c_\theta \end{bmatrix}$$

$z\, y\, z$ Euler angles, the inverse solution

$$\mathsf{R} = \begin{bmatrix} r_{11}, & r_{12}, & r_{13} \\ r_{21}, & r_{22}, & r_{23} \\ r_{31}, & r_{32}, & r_{33} \end{bmatrix} = \begin{bmatrix} c_\phi\, c_\theta\, c_\psi - s_\phi\, s_\psi\,, & -c_\phi\, c_\theta\, s_\psi - s_\phi\, c_\psi\,, & c_\phi\, s_\theta \\ s_\phi\, c_\theta\, c_\psi + c_\phi\, s_\psi\,, & -s_\phi\, c_\theta\, s_\psi + c_\phi\, c_\psi\,, & s_\phi\, s_\theta \\ -s_\theta\, c_\psi\,, & s_\theta\, s_\psi\,, & c_\theta \end{bmatrix}$$

The solution to the inverse problem, i.e. calculating Euler angles from the rotation matrix R, is given by explicit formulas as

- $\theta = \cos^{-1}(r_{33})$ because $r_{33} = \cos\theta$
- $\phi = \tan^{-1}\left(\frac{r_{23}}{r_{13}}\right)$ because $r_{13} = \cos\phi\, \sin\theta$; $r_{23} = \sin\phi\, \sin\theta$
  $\tan\phi = \frac{\sin\phi}{\cos\phi} = \frac{r_{23}}{\sin\theta} / \frac{r_{13}}{\sin\theta} = \frac{r_{23}}{r_{13}}$
- $\psi = \tan^{-1}\left(\frac{r_{32}}{-r_{31}}\right)$ because $r_{31} = -\sin\theta\, \sin\psi$; $r_{32} = \sin\theta\, \sin\psi$
  analogically to ϕ

Note: A little more care is needed due to multiple solutions and singularities in practice.

$x\ y\ z$ Cardan angles, yaw-pitch-roll

Composition of three elementary rotations

- Rotate the reference frame by the angle ψ about x-axis (yaw).
- Rotate the reference frame by the angle θ about axis y' (pitch).
- Rotate the reference frame by the angle ϕ about axis z'' (roll).