CLASSICAL LOGIC (CL)

SYNTAX

$\mathcal{A}$ ... countable set of propositional variables
$\mathcal{L} = \{\to, \mathbf{0}\}$ ... the set of logical connectives:
$\to$ ... (binary) implication
$\mathbf{0}$ ... (nulary) false

Formulas

- all elements of $\mathcal{A}$ are formulas
- $\mathbf{0}$ is a formula
- if A, B are formulas, then $A \to B$ is a formula

more exactly, we use brackets like $(A) \to (B)$

Derived connectives:
$\neg A = A \to \mathbf{0}$ ... (unary) negation
$\mathbf{1} = \neg\mathbf{0} = \mathbf{0} \to \mathbf{0}$ ... (nulary) true
$A \wedge B = \neg(A \to \neg B)$ ... (binary) conjunction
$A \vee B = \neg A \to B$ ... (binary) disjunction
$A \leftrightarrow B = (A \to B) \wedge (B \to A)$
... (binary) equivalence

Logical axioms

(C1)	$A \to (B \to A)$
(C2)	$[A \to (B \to C)] \to [(A \to B) \to (A \to C)]$
(C3)	$(\neg A \to \neg B) \to (B \to A)$

Deduction rule: Modus Ponens

r_{MP} : $\quad \frac{A,\ A \to B}{B}$

Theory $\mathcal{T}$ ... set of formulas (**special axioms**)

Provable formula (=**theorem**) in theory $\mathcal{T}$ is a formula which admits a **proof**, i.e., a finite sequence of formulas such that each of them is

- a special axiom (=element of $\mathcal{T}$), or
- an instance of a logical axiom (obtained by a substitution), or
- a result of application of a deduction rule to preceding formulas in the proof.

Notation: $\mathcal{T} \vdash A$, $B \vdash A$ (for $\mathcal{T} = \{B\}$), $\vdash A$ (for $\mathcal{T} = \emptyset$)

Example Cl1

$B \vdash A \to B$

$$\begin{aligned} \text{(C1), } A :=: B : \quad & B \to (A \to B) \\ \text{Special axiom} : \quad & B \\ \text{Modus Ponens} : \quad & A \to B \end{aligned}$$

Example Cl2

$\vdash A \to A$

For brevity, let B denote a provable formula, e.g., axiom (C1).

$$\begin{aligned} \text{(C2)}, C := A : \quad & \\ & [A \to (B \to A)] \to [(A \to B) \to (A \to A)] \\ \text{(C1)} : \quad & A \to (B \to A) \\ \text{MP} : \quad & (A \to B) \to (A \to A) \\ \text{(C1)} : \quad & B \\ \text{Ex. Cl1} : \quad & A \to B \\ \text{MP} : \quad & A \to A \end{aligned}$$

Example Cl3

$\{B, \neg B\} \vdash A$ for all A

$$
\begin{aligned}
\text{(C3)}: &\quad (\neg A \to \neg B) \to (B \to A)\\
\text{SA}: &\quad \neg B\\
\text{Ex. Cl1}: &\quad \neg A \to \neg B\\
\text{MP}: &\quad B \to A\\
\text{SA}: &\quad B\\
\text{MP}: &\quad A
\end{aligned}
$$

Deduction theorem in classical logic

$\mathcal{T}$ ... a theory

A, B ... formulas

$\mathcal{T} \cup \{A\} \vdash B$ iff $\mathcal{T} \vdash A \to B$

<u>SEMANTICS</u>

In general: a Boolean algebra, it is enough to consider

<u>Standard semantics</u>

the set of truth values ... $\{0,1\}$
$\rightarrow$ ... Boolean implication $\Rightarrow$
$\mathbf{0}$ ... 0

Interpretation of derived connectives:
$\neg$ ... Boolean negation
$\mathbf{1}$ ... 1
$\wedge$ ... conjunction
$\vee$ ... disjunction
$\leftrightarrow$ ... Boolean equality $\Leftrightarrow$

An **evaluation** (**truth assignment**) can be arbitrarily chosen on propositional variables, then it extends uniquely to all formulas.

Tautology is a formula A which is always evaluated to 1

Notation: $\models A$

Moreover, for any theory $\mathcal{T}$, $\mathcal{T} \models A$ means that $e(A) = 1$ for each evaluation such that $\forall B \in \mathcal{T}$: $e(B) = 1$.

Soundness

Each provable formula is a tautology, i.e., if $\vdash A$, then $\models A$.

Moreover, for any theory $\mathcal{T}$, if $\mathcal{T} \vdash A$, then $\mathcal{T} \models A$.

Weak completeness

Each tautology is provable, i.e., if $\models A$, then $\vdash A$.

Completeness

For any finite theory $\mathcal{T}$, if $\mathcal{T} \models A$, then $\mathcal{T} \vdash A$.

How can we simplify our proofs?

$B \leftrightarrow C \vdash (A \rightarrow B) \leftrightarrow (A \rightarrow C)$
$B \leftrightarrow C \vdash (B \rightarrow A) \leftrightarrow (C \rightarrow A)$

BASIC LOGIC (BL) AS AN EXAMPLE OF MANY-VALUED PROPOSITIONAL LOGIC

SYNTAX

$\mathcal{A}$ ... countable set of propositional variables
$\mathcal{L} = \{\rightarrow, \mathbf{0}, \wedge\}$ ... the set of logical connectives:
$\rightarrow$ ... (binary) implication
$\mathbf{0}$ ... (nulary) false
$\wedge$ ... (binary) conjunction (NEW)

Formulas constructed as usual

Derived connectives:
$\neg A = A \rightarrow \mathbf{0}$ ... (unary) negation
$\mathbf{1} = \neg\mathbf{0} = \mathbf{0} \rightarrow \mathbf{0}$ ... (nulary) true
$A \leftrightarrow B = (A \rightarrow B) \wedge (B \rightarrow A)$
... (binary) equivalence
$A \underset{\mathsf{S}}{\wedge} B = A \wedge (A \rightarrow B)$
$A \overset{\mathsf{S}}{\vee} B = ((A \rightarrow B) \rightarrow B) \underset{\mathsf{S}}{\wedge} ((B \rightarrow A) \rightarrow A)$
no $A \vee B$ in general

Logical axioms

$$
\begin{array}{rl}
\text{(A1)} & (A \to B) \to [(B \to C) \to (A \to C)] \\
\text{(A2)} & A \wedge B \to A \\
\text{(A3)} & A \wedge B \to B \wedge A \\
\text{(A4)} & A \wedge (A \to B) \to B \wedge (B \to A) \\
\text{(A5a)} & [A \to (B \to C)] \to (A \wedge B \to C) \\
\text{(A5b)} & (A \wedge B \to C) \to [A \to (B \to C)] \\
\text{(A6)} & [(A \to B) \to C] \to [((B \to A) \to C) \to C] \\
\text{(A7)} & \mathbf{0} \to A
\end{array}
$$

Deduction rule: Modus Ponens

r_{MP} : $\frac{A,\ A \to B}{B}$

Theory ... set of formulas (**special axioms**)

Proofs and **provable formulas** (=**theorems**) are defined as usual

Notation: $\vdash A$, $\mathcal{T} \vdash A$

<u>Example 1</u>

(C1) $\quad A \to (B \to A) \quad$ is provable in BL:

$$\begin{array}{rl}
\text{(A2):} & A \wedge B \to A \\
\text{(A5b), } C := A\text{:} & (A \wedge B \to A) \to (A \to (B \to A)) \\
\text{Modus ponens:} & A \to (B \to A)
\end{array}$$

<u>Proposition 1</u>

Consequence of (A1):

$$\{A \to B,\ B \to C\} \vdash A \to C$$

<u>Example 2</u>

$\vdash (A \to (B \to C)) \to (B \to (A \to C))$

(**Exchange rule**, also called "exchange axiom")

(A1) with substitutions $A := B \wedge A$, $B := A \wedge B$:

$$(B \wedge A \to A \wedge B) \to [(A \wedge B \to C) \to (B \wedge A \to C)]$$

(A3) with $A :=: B$ becomes: $\quad B \wedge A \to A \wedge B$

Modus ponens: $\quad (A \wedge B \to C) \to (B \wedge A \to C)$

Use (A5a), (A5b), and Proposition 1.

Example 3

$\vdash A \to A$

In Example 2, substitute $C := A$:

$$(A \to (B \to A)) \to (B \to (A \to A))$$

From Example 1:

$$A \to (B \to A)$$

Modus ponens:

$$B \to (A \to A)$$

It remains to substitute B (which is arbitrary) with provable formula, e.g., with a logical axiom.

Deduction theorem in basic logic

$\mathcal{T}$ ... a theory
A, B ... formulas

$\mathcal{T} \cup \{A\} \vdash B$ iff $\exists n \in N : (\mathcal{T} \vdash A^n \to B)$,
where $A^n = \underbrace{(A \wedge (A \wedge \cdots (A \wedge A) \cdots))}_{n \text{ times}}$

In view of (A5a), (A5b),
$(A^n \to B) \leftrightarrow \underbrace{(A \to (A \to \cdots (A \to}_{n \text{ times}} B) \cdots))$

SEMANTICS

In general: a BL-algebra, here only

Standard semantics

the set of truth values ... $[0,1]$

$\wedge$ ... continuous t-norm $\underset{\cdot}{\wedge}$

$\to$ ... residuum $\underset{\cdot}{\to}$ of $\underset{\cdot}{\wedge}$

$\mathbf{0}$ ... 0

Even the standard semantics is not unique, it depends on the choice of the continuous t-norm.

Interpretation of derived connectives:

$\neg$ ... $\underset{\cdot}{\neg}$, where $\underset{\cdot}{\neg}\, a = a \underset{\cdot}{\to} 0$

$\mathbf{1}$ ... 1

$\leftrightarrow$ ... $\underset{\cdot}{\leftrightarrow}$, where $a \underset{\cdot}{\leftrightarrow} b = (a \underset{\cdot}{\to} b) \underset{\cdot}{\wedge} (b \underset{\cdot}{\to} a)$

$\underset{\mathsf{S}}{\wedge}$ ... $\underset{\mathsf{S}}{\wedge} = \min$

$\overset{\mathsf{S}}{\vee}$ ... $\overset{\mathsf{S}}{\vee} = \max$

Exercise

Verify that the interpretation of $\underset{\mathsf{S}}{\wedge}, \overset{\mathsf{S}}{\vee}$ is independent of the choice of the t-norm.

An **evaluation** (**truth assignment**) can be arbitrarily chosen on propositional variables, then it extends uniquely to all formulas.

Conjunction $\wedge$ is introduced separately, as its semantics cannot be derived from the implication (as an expression using the other operations).

There are different generalizations of the notion of tautology:
1-tautology is a formula A which is always evaluated to 1 (by all possible evaluations with values with any BL-algebra, in particular, for any continuous t-norm as an interpretation of $\wedge$ and its residuum as an interpretation of $\to$)

Notation: $\models A$

Moreover, for any theory $\mathcal{T}$, $\mathcal{T} \models A$ means that $e(A) = 1$ for each evaluation such that $\forall B \in \mathcal{T}$: $e(B) = 1$.

Soundness
Each provable formula is a 1-tautology, i.e., if $\vdash A$, then $\models A$.

Moreover, for any theory $\mathcal{T}$, if $\mathcal{T} \vdash A$, then $\mathcal{T} \models A$.

Weak completeness

Each 1-tautology is provable, i.e., if $\models A$, then $\vdash A$.

Completeness

[Hájek 1998]
For any finite theory $\mathcal{T}$, if $\mathcal{T} \models A$, then $\mathcal{T} \vdash A$.
(We consider all evaluations with values in BL-algebras.)

Standard completeness

[Cignoli, R., Esteva, F., Godo, L., Torrens, A.: Basic logic is the logic of continuous t-norms. *Soft Computing* **4** (2000), 106–112]
Each formula which is evaluated to 1 by all **standard** evaluations (with values in $[0, 1]$ and an arbitrary continuous t-norm) is provable.

Exercise

Which axioms of the classical logic are 1-tautologies of BL (and hence provable in BL)?

Exchange rule allows us to rewrite (A1):
$(B \to C) \to [(A \to B) \to (A \to C)]$

Example 4

$\vdash A \to (B \to A \wedge B)$

Proof:

$$\begin{aligned} \text{Ex. 3}: &\quad A \wedge B \to A \wedge B \\ \text{(A5b)}: &\quad (A \wedge B \to A \wedge B) \to (A \to (B \to A \wedge B)) \end{aligned}$$

and use modus ponens

Corollary: $\{A, B\} \vdash A \wedge B$

$A \vdash B \to A \wedge B$

Example of deduction

$A \vdash B \to A \wedge (A \wedge B)$

Proof:

$$\begin{aligned} \text{SA}: &\quad A \\ \text{Ex. 4, } B := A \wedge B: &\quad A \to (A \wedge B \to A \wedge (A \wedge B)) \\ \text{MP}: &\quad A \wedge B \to A \wedge (A \wedge B) \\ \text{(A5b)}: &\quad (A \wedge B \to A \wedge (A \wedge B)) \\ \to &\quad (A \to (B \to A \wedge (A \wedge B))) \\ \text{MP}: &\quad A \to (B \to A \wedge (A \wedge B)) \\ \text{MP}: &\quad B \to A \wedge (A \wedge B) \end{aligned}$$

Here $\nvdash A \to (B \to A \wedge (A \wedge B))$
but $\vdash A \wedge A \to (B \to A \wedge (A \wedge B))$
$\quad\vdash A \to (A \to (B \to A \wedge (A \wedge B)))$
$\quad\vdash A \to (A \wedge B \to A \wedge (A \wedge B))$
(by substitution $B := A \wedge B$ in Example 4)

Exercise
Prove in BL: $(A \to B) \wedge (A \to C) \to (A \to B \wedge C)$

1. How do the properties of interpretation of conjunction (t-norm) follow from logical axioms?
2. How can we simplify our proofs?

Corollary of (A1):
$B \leftrightarrow C \vdash (A \to B) \leftrightarrow (A \to C)$
$B \leftrightarrow C \vdash (B \to A) \leftrightarrow (C \to A)$

(We are not yet allowed to work similarly with the conjunction.)

Commutativity of $\underset{\cdot}{\wedge}$ follows directly from (A3).

Boundary condition:

Example 4, $A := \mathbf{1}$: $\quad \mathbf{1} \to (B \to \mathbf{1} \wedge B)$

MP: $\quad B \to \mathbf{1} \wedge B$

the reverse implication follows from (A2)

Associativity of $\underset{\cdot}{\wedge}$:

TFAE (A5):

$(A \wedge B) \wedge C \to D$

$A \wedge B \to (C \to D)$

$A \to (B \to (C \to D))$

Now we use the equivalence of subformulas:

$B \to (C \to D)$

$B \wedge C \to D$

$A \to (B \wedge C \to D)$

$A \wedge (B \wedge C) \to D$

We proved:

$((A \wedge B) \wedge C \to D) \leftrightarrow (A \wedge (B \wedge C) \to D)$

for all D, in particular, for $D := (A \wedge B) \wedge C$:

$((A \wedge B) \wedge C \to (A \wedge B) \wedge C) \leftrightarrow$

$(A \wedge (B \wedge C) \to (A \wedge B) \wedge C)$

MP: $\quad A \wedge (B \wedge C) \to (A \wedge B) \wedge C$

and for $D := A \wedge (B \wedge C)$ the other implication, hence

$A \wedge (B \wedge C) \leftrightarrow (A \wedge B) \wedge C$

Monotonicity of $\underset{\cdot}{\wedge}$ as a crisp property means
$B \to C \vdash B \wedge A \to C \wedge A$

We shall prove monotonicity of $\underset{\cdot}{\wedge}$ as a fuzzy property (this is stronger):
$(B \to C) \to (B \wedge A \to C \wedge A)$

Proof:
TFAE and all are theorems, as the first one is an instance of (A2) (unless other argument is given, (A5) is used):
$(C \to B) \wedge (C \wedge A) \to C \wedge A$
replacing the left-hand side by a formula already known to be equivalent:
$(C \wedge (C \to B)) \wedge A \to C \wedge A$
$C \wedge (C \to B) \to (A \to C \wedge A)$
only now we may replace the left-hand side by an equivalent formula from (A4):
$B \wedge (B \to C) \to (A \to C \wedge A)$
$(B \wedge (B \to C)) \wedge A \to C \wedge A$
replacing the left-hand side by an equivalent formula:
$(B \to C) \wedge (B \wedge A) \to C \wedge A$
$(B \to C) \to (B \wedge A \to C \wedge A)$

Corollary:
$B \leftrightarrow C \vdash B \wedge A \leftrightarrow C \wedge A$
$B \leftrightarrow C \vdash A \wedge B \leftrightarrow A \wedge C$

GÖDEL LOGIC

Syntax:

Axioms (A1)–(A7) and

(G) $A \to A \wedge A$

Corollary: $A \leftrightarrow A \wedge A$

Standard semantics:

$\wedge$ ... idempotent=standard t-norm, $\underset{\mathsf{S}}{\wedge} = \min$

$\to$ ... residuum of $\underset{\mathsf{S}}{\wedge}$, Gödel implication $\underset{\mathsf{S}}{\to}$

$\neg$ ... Gödel generalized negation $\underset{\mathsf{G}}{\neg}$:

$$\underset{\mathsf{G}}{\neg}\, x = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{otherwise} \end{cases}$$

There is no connective in Gödel logic which is interpreted by the standard fuzzy negation.

Only in Gödel logic the classical deduction theorem holds (because $A^n \leftrightarrow A$):

Deduction theorem in Gödel logic

$\mathcal{T}$ ... a finite theory

A, B ... formulas

$\mathcal{T} \cup \{A\} \vdash B$ iff $\mathcal{T} \vdash A \to B$

Standard completeness of Gödel logic

If $\vdash A$, then $\models A$ (i.e., theorems are exactly 1-tautologies w.r.t. $[0,1]$ with Gödel operations).

Moreover, for any finite theory $\mathcal{T}$, if $\mathcal{T} \vdash A$, then $\mathcal{T} \models A$.

$$
\begin{array}{ccc}
\text{theorem of CL} & \Longleftrightarrow & \text{tautology of CL} \\
\Uparrow & & \Uparrow \\
\text{theorem of GL} & \Longleftrightarrow & \text{1-tautology of GL} \\
\Uparrow & & \Uparrow \\
\text{theorem of BL} & \Longleftrightarrow & \text{1-tautology of BL}
\end{array}
$$

PRODUCT LOGIC

Syntax:

Axioms (A1)–(A7) and

(P1) $\neg\neg C \to ((A \wedge C \to B \wedge C) \to (A \to B))$

(P2) $A \wedge (A \to \neg A) \to \mathbf{0}$

Remark: $e(\neg\neg C) = 1$ iff $e(C) \neq 0$ and

$\neg\neg C \vdash (A \wedge C \to B \wedge C) \leftrightarrow (A \to B)$

Alternatively, we may use a single axiom [Cintula] instead of (P1),(P2):

(P) $\neg\neg A \to ((A \to A \wedge B) \to B \wedge \neg\neg B)$

Standard semantics:

$\wedge$ ... product (or any strict) t-norm $\underset{\mathsf{P}}{\wedge} = \cdot$

$\to$ ... residuum of $\underset{\mathsf{P}}{\wedge}$, Goguen implication $\underset{\mathsf{P}}{\to}$

$\neg$ ... Gödel negation $\underset{\mathsf{G}}{\neg}$

There is no connective in product logic which is interpreted by the standard fuzzy negation.

Standard completeness of product logic

If $\vdash A$, then $\models A$ (i.e., theorems are exactly 1-tautologies w.r.t. $[0, 1]$ with product operations).

Moreover, for any finite theory $\mathcal{T}$, if $\mathcal{T} \vdash A$, then $\mathcal{T} \models A$.

<u>LUKASIEWICZ LOGIC</u>

Syntax:

Axioms (A1)–(A7) and

(L) $\qquad \neg\neg A \to A$

Corollary: $\qquad \neg\neg A \leftrightarrow A$

Alternative axiomatization [Lukasiewicz & Tarski] (with only $\to, \mathbf{0}$, conjunction is considered a derived connective $A \wedge B = \neg(A \to \neg B)$):

(L1) $\qquad A \to (B \to A)$

(L2) $\qquad (A \to B) \to [(B \to C) \to (A \to C)]$

(L3) $\qquad (\neg A \to \neg B) \to (B \to A)$

(L4) $\qquad [(A \to B) \to B] \to [(B \to A) \to A]$

(L1)=(C1) $\qquad$ (valid in BL, see Example 1)

(L2)=(A1), $\qquad$ (L3)=(C3)

(L4)=**Lukasiewicz axiom**

Standard semantics:

$\wedge$ ... Lukasiewicz (or any nilpotent) t-norm $\underset{\mathsf{L}}{\wedge}$

$\to$ ... residuum of $\underset{\mathsf{L}}{\wedge}$, Lukasiewicz implication $\underset{\mathsf{L}}{\to}$

$\neg$ ... standard fuzzy negation $\underset{\mathsf{S}}{\neg} x = 1 - x$

$e[(A \to B) \to B] = e(A) \overset{\mathsf{S}}{\vee} e(B) \qquad$ (in (L4))

Standard completeness of Lukasiewicz logic

If $\vdash A$, then $\models A$ (i.e., theorems are exactly 1-tautologies w.r.t. $[0, 1]$ with Lukasiewicz operations).

Moreover, for any finite theory $\mathcal{T}$, if $\mathcal{T} \vdash A$, then $\mathcal{T} \models A$.

CLASSICAL PREDICATE LOGIC

A predicate assigns truth values to objects

$\forall x\, P(x)$ may be understood as a conjunction
$P(x_1) \wedge P(x_2) \wedge \ldots$

$\exists x\, P(x)$ may be understood as a disjunction
$P(x_1) \vee P(x_2) \vee \ldots$

SYNTAX

Logical symbols:

$\mathcal{A} = \{x, y, \ldots\}$ ... countable set of propositional variables

$\mathcal{L} = \{\rightarrow, \mathbf{0}\}$ ... the set of logical connectives

$\forall, \exists$ ... quantifiers

Special symbols:

$\mathcal{P}$ ... nonempty set of **predicates** (with assigned arity)

event. *function symbols* (here not considered)

event. *object constants* (here not considered; may be substituted by predicates)

Formulas

- $P(x_1, \ldots, x_n)$, where P is a predicate of arity n and $x_1, \ldots, x_n$ are variables

- $\mathbf{0}$

- $A \rightarrow B$, where A, B are formulas

- $\forall x\ A$, where A is a formula

- $\exists x\ A$, where A is a formula

Occurence of a variable:
- **bound**
- **free**

Example:
$\forall x\, Q(x, y)$ ... x bound, y free
$x \wedge \forall x\ P(x)$ ... the first occurence of x free, the rest bound

Formula
- **closed (=sentence)**: without free variables
- **open**: without bound variables

Formulas differing only by the notation of bound variables are considered equal.

SEMANTICS

Interpretation M:

- D_M ... nonempty set (**domain, universe**; event. may be distinguished for different variables)

- interpretation of a predicate of arity n ... an n-ary relation $D_M^n \to \{0,1\}$

Evaluation (in interpretation M): $e_M\colon \mathcal{A} \to D_M$ extends uniquely to all formulas
new rules:

$(e_M(\forall x\ A) = 1) \iff e'_M(A) = 1$ for **every** evaluation $e'_M\colon \mathcal{A} \to D_M$ which differs from e_M at most at x
equivalently: $e_M(\forall x\ A) = \inf\{e'_M(A) : e'_M\colon \mathcal{A} \to D_M$ is an evaluation which differs from e_M at most at $x\}$

$(e_M(\exists x\ A) = 1) \iff e'_M(A) = 1$ for **some** evaluation $e'_M\colon \mathcal{A} \to D_M$ which differs from e_M at most at x
equivalently: $e_M(\exists x\ A) = \sup\{e'_M(A) : e'_M\colon \mathcal{A} \to D_M$ is an evaluation which differs from e_M at most at $x\}$

$\models_M A$ ... A is true in all evaluations in interpretation M
$\models A$ ... A is true in all evaluations in **all** interpretations (**tautology**)

Classification of formulas: **tautologies, contradictions, satisfiable formulas**
easy for closed formulas
free variables have to be quantified by the *corresponding* quantifier

Analogously: satisfiability of a set of formulas

Examples of new tautologies:
$(\forall x\, P(x)) \rightarrow P(t)$
$P(t) \rightarrow (\exists x\, P(x))$
$\neg(\forall x\, P(x)) \leftrightarrow (\exists x\, \neg P(x))$
$\neg(\exists x\, P(x)) \leftrightarrow (\forall x\, \neg P(x))$
$(\forall x\, \forall y\, Q(x,y)) \leftrightarrow (\forall y\, \forall x\, Q(x,y))$
$(\exists x\, \exists y\, Q(x,y)) \leftrightarrow (\exists y\, \exists x\, Q(x,y))$

Semantical consequent: $\mathcal{T} \models A$ (without index)
only for sets of closed formulas

<u>Theorem</u>
$(\mathcal{T} \models A) \iff (\mathcal{T} \cup \{\neg A\}$ is unsatisfiable$)$

Deduction theorem in classical predicate logic

$(\mathcal{T} \cup \{A\} \vdash B) \iff (\mathcal{T} \vdash A \to B)$

Corollary

$(A \models\!\!\!\dashv B) \iff (\models A \leftrightarrow B)$

Theorem

$\vdash (\neg \exists x\ A) \leftrightarrow (\forall x\ \neg A)$

$\vdash (\neg \forall x\ A) \leftrightarrow (\exists x\ \neg A)$

$\vdash (\exists x\ A) \leftrightarrow \neg(\forall x\ \neg A)$

$\vdash (\forall x\ A) \leftrightarrow \neg(\exists x\ \neg A)$

Completeness

$(\mathcal{T} \models A) \iff (\mathcal{T} \vdash A)$

BASIC (FUZZY) PREDICATE LOGIC, BL$\forall$

Derived from BL as classical predicate logic from classical propositional logic

What differs:

SEMANTICS

interpretation of a predicate of arity n ... an n-ary **fuzzy** relation $D_M^n \to [\mathbf{0}, \mathbf{1}]$

Evaluation of quantifiers in interpretation M:

$e_M(\forall x\ A) =$ inf$\{e'_M(A) : e'_M \colon \mathcal{A} \to D_M$ is an evaluation which differs from e_M at most at $x\}$

$e_M(\exists x\ A) =$ sup$\{e'_M(A) : e'_M \colon \mathcal{A} \to D_M$ is an evaluation which differs from e_M at most at $x\}$

Remark: For evaluations in more general sets of truth values than $[0, 1]$ (BL-algebras), it becomes necessary to assume the existence of inf, sup in question.

Truth value of formula A in interpretation M:
$||A||_M = \inf\{e_M(A) : e_M$ is an evaluation in $M\}$

SYNTAX

Axioms:

- (A1)–(A7)
- for each formula $P(t)$ originated by substituting t for x in a formula $P(x)$:

$$
\begin{array}{ll}
(\forall 1) & (\forall x\, P(x)) \to P(t) \\
(\exists 1) & P(t) \to (\exists x\, P(x))
\end{array}
$$

- for each formula B not containing x (as a free variable):

$$
\begin{array}{ll}
(\forall 2) & (\forall x\, (B \to A)) \to (B \to \forall x\, A) \\
(\exists 2) & (\forall x\, (A \to B)) \to ((\exists x\, A) \to B) \\
(\forall 3) & (\forall x\, (B \overset{\mathsf{S}}{\vee} A)) \to ((\forall x\, A) \overset{\mathsf{S}}{\vee} B)
\end{array}
$$

Deduction rules:

Modus Ponens: $\frac{A,\ A \to B}{B}$

Generalization: $\frac{A}{\forall x\, A}$

Deduction theorem in basic predicate logic

(for closed formulas)

$(\mathcal{T} \cup \{A\} \vdash B) \iff (\exists n \in N : (\mathcal{T} \vdash A^n \to B))$

Theorem

$\vdash (\exists x \ A) \to \neg(\forall x \ \neg A)$

$\vdash (\neg\exists x \ A) \leftrightarrow (\forall x \ \neg A)$

$\models_M A$ ... A is true in all evaluations in interpretation M

$\models A$ ... A is true in all evaluations in **all** interpretations (**tautology**)

Theorem

$e_M(A) \mathbin{\underset{\cdot}{\wedge}} e_M(A \to B) \leq e_M(B)$

in particular:

$(e_M(A) = 1) \wedge (e_M(A \to B) = 1) \Rightarrow (e_M(B) = 1)$

Corollary

$||A||_M \mathbin{\underset{\cdot}{\wedge}} ||A \to B||_M \leq ||B||_M$

in particular:

$(||A||_M = 1) \wedge (||A \to B||_M = 1) \Rightarrow (||B||_M = 1)$

$||A||_M = ||\forall x \ A||_M$

$(\models_M A) \Rightarrow (\models_M \forall x \ A)$

Interpretation M is a **model** of a theory $\mathcal{T}$ if $||A||_M = 1$ for all $A \in \mathcal{T}$

Soundness and completeness

$(\mathcal{T} \vdash A) \iff (||A||_M = 1$ for each model M of $\mathcal{T})$
(More exactly, we have to assume evaluations with values in linearly ordered BL-algebras.)

RATIONAL PAVELKA PREDICATE LOGIC RPL∀

Derived from RPL as basic predicate logic from basic propositional logic

What differs:

Axioms:

- (L1)–(L4)
- (RPL) (bookkeeping axioms)
- $(\forall 1), (\forall 2)$

This is sufficient, because
(even in Lukasiewicz predicate logic)
$\vdash (\exists x\ A) \leftrightarrow \neg(\forall x\ \neg A)$
and also $(\forall 3)$ is provable.

Soundness and completeness

If $\mathcal{T}$ is a theory (consisting from closed formulas), then
$|A|_{\mathcal{T}} = ||A||_{\mathcal{T}}$,
where
$||A||_{\mathcal{T}}$ is the infimum of all evaluations over all models of $\mathcal{T}$,
$|A|_{\mathcal{T}}$ is the supremum of all provability degrees assuming $\mathcal{T}$.