# WHICH TRIANGULAR NORMS ARE CONVENIENT FOR FUZZY CONTROLLERS?

**Bernhard Moser**
Dept. of Algebra, Stochastics, and
Knowledge-Based Math. Systems
Johannes Kepler University
A-4040 Linz, Austria
moser@flll.uni-linz.ac.at

**Mirko Navara**
Center for Machine Perception
Faculty of Electrical Engineering
Czech Technical University
CZ-166 27 Praha, Czech Republic
navara@cmp.felk.cvut.cz

## Summary

We study some natural requirements for Mamdani and residuum-based fuzzy controllers. We show the importance of the choice of the triangular norm in the fuzzy inference rule. We discuss the questions when triangular norms with/without zero divisors are advantageous.

**Keywords**: Triangular norm, fuzzy relational equation, fuzzy control, fuzzy interpolation, zero divisor.

## 1 MOTIVATION

The concept of approximate reasoning as it was conceived by Zadeh [7] provides a framework which allows to model and process vague linguistic information. The idea is to model linguistic terms by fuzzy sets, their (logical) relationship by fuzzy relations and their composition by the so-called *compositional rule of inference* [7]. As an important field of applications we refer to control processes for which linguistic information of a human expert about the required input-output behaviour of the controller is available (for an introduction see, e.g., [2]).

Let $X$ and $Y$ denote the input and the output space, respectively. The expert's knowledge can be expressed by means of a rule base, $\Theta = (X_i, Y_i)_{i=1}^n$, of if-then rules having the form

$$\textbf{if } x \in X_i \textbf{ then } y \in Y_i,$$

where $i \in \{1, \ldots, n\}$ and $X_i, Y_i$ are fuzzy subsets of $X, Y$, respectively. For a universe of discourse $Z$, let $\mathcal{F}(Z)$ denote the set of all fuzzy subsets of $Z$. According to the paradigm of approximate reasoning, the knowledge from the rule base $\Theta$ can be represented by a fuzzy relation $R \in \mathcal{F}(X \times Y)$. Applying the compositional rule of inference to a fuzzy input $X^* \in \mathcal{F}(X)$ and the relation $R$, a fuzzy output $Y^* \in \mathcal{F}(Y)$ is derived via

$$Y^* = X^* \circ_T R, \tag{1}$$

i.e.,

$$\forall y \in Y : Y^*(y) = \sup_{x \in X} T(X^*(x), R(x, y)),$$

where $T$ is a triangular norm modelling a fuzzy conjunction [3].

Let us now formulate some basic requirements on the rule base. We usually assume that the premises are *normal*, i.e., each premise attains the value 1 at some point.

**Assumption 1.1** *Throughout the paper we assume that $X$ and $Y$ are nonempty compact convex subsets of a finite-dimensional real vector space. The rule base is $\Theta = (X_i, Y_i)_{i=1}^n$, where $n \geq 2$, $X_i \in \mathcal{F}(X)$ and $Y_i \in \mathcal{F}(Y)$, $i = 1, \ldots, n$. We assume that there are pairwise different elements $x_1, \ldots, x_n \in X$ such that $X_i(x_i) = 1$. Throughout the paper, $T$ denotes a fixed continuous triangular norm (t-norm).*

In this paper we are interested in Mamdani's formula and other inference methods as *fuzzy interpolation (approximation)* $\mathit{Int}_\Theta : \mathcal{F}(X) \to \mathcal{F}(Y)$ with respect to the rule base $\Theta = (X_i, Y_i)_{i=1}^n$. We shall concentrate on those inference methods and rule bases which satisfy the following condition: *If the input set $X^*$ coincides with one of the premises, then the resulting output set $Y^* = \mathit{Int}(X^*)$ coincides with the corresponding consequent, i.e.,*

$$\forall i \in \{1, \ldots, n\} : \mathit{Int}_\Theta(X_i) = Y_i.$$

This is a natural requirement for the controller to represent the rules in the rule base. For a compositional rule of inference, this is equivalent to the system of fuzzy relational equations

$$Y_i = X_i \circ_T R, \qquad i \in \{1, \ldots, n\}. \tag{2}$$

## 2 MAMDANI CONTROLLER

The first successful practical applications of fuzzy sets were realized by means of the so-called Mamdani inference [4] which results from formula (1) for

$$R(x, y) = \max_{i=1}^{n} T(X_i(x), Y_i(y)). \tag{3}$$

We denote by $Mam_\Theta$ the induced mapping $\mathcal{F}(X) \to \mathcal{F}(Y)$. Note that Mamdani's approach is not fully coherent with the paradigm of approximate reasoning. For example, by employing a fuzzy conjunction instead of a fuzzy implication it does not represent the logical meaning of a linguistic if-then rule. Also the maximum in (3) represents a disjunction, while we usually consider the rules connected by a conjunction (as they are supposed to be valid simultaneously).

The question when the Mamdani controller satisfies the system (2) was answered by de Baets in [1] (see the following Th. 2.1). It provides a necessary and sufficient condition in terms of an inequality between the "degree of overlapping" of the premises and the "degree of indistinguishability" of the consequents. To formulate it, we use the *residuum* induced by $T$, i.e., the operation $I_T\colon [0,1]^2 \to [0,1]$ such that

$$I_T(a, b) = \sup\{c \in [0,1] : T(a, c) \leq b\}.$$

The corresponding *biimplication* is defined as

$$E_T(a, b) = \min\{I_T(a, b), I_T(b, a)\}.$$

Let $A, B \in \mathcal{F}(Z)$. We define the *degree of overlapping* by

$$\mathcal{D}_T(A, B) = \sup_{z \in Z} T(A(z), B(z))$$

and the *degree of indistinguishability* by

$$\mathcal{E}_T(A, B) = \inf_{z \in Z} E_T(A(z), B(z)).$$

Notice that all these notions are dependent on the choice of the t-norm $T$.

Due to the associativity and the continuity of $T$, the Mamdani inference expressed by (1) and (3) can be effectively computed by means of the degree of overlapping of the input $X^*$ and the premises:

$$Mam_\Theta(X^*)(y) = \max_{i=1}^{n} T\big(\mathcal{D}_T(X^*, X_i), Y_i(y)\big).$$

**Theorem 2.1** [1] *The relation* (3) *satisfies the system* (2) *iff all* $i, j \in \{1, \ldots, n\}$ *satisfy*

$$\mathcal{D}_T(X_i, X_j) \leq \mathcal{E}_T(Y_i, Y_j). \tag{4}$$

Inequality (4) yields a rather restrictive condition on the rule base $\Theta$ for t-norms without (non-trivial) zero divisors. Recall that the *support* of a fuzzy set $A \in \mathcal{F}(Z)$ is $\mathrm{Supp}\, A = \{z \in Z : A(z) > 0\}$.

**Proposition 2.1** *Suppose that $T$ is a t-norm without zero divisors and that the relation* (3) *satisfies the system* (2). *Then $\mathrm{Supp}\, X_i \cap \mathrm{Supp}\, X_j = \emptyset$ whenever $\mathrm{Supp}\, Y_i \neq \mathrm{Supp}\, Y_j$. If, moreover, the consequents have pairwise different supports and all premises are continuous, then there is a normal input $X^*$ giving the empty set as an output, $Mam_\Theta(X^*) = 0$.*

As an empty output does not give any information about the output value of the controller, the Mamdani inference for t-norms without zero divisors cannot be considered as a good fuzzy interpolation in the sense of the fuzzy relational equations (2). This case covers many of the standardly used shapes of fuzzy premises and consequents and t-norms without zero divisors, including the t-norms most commonly used in the field of fuzzy control applications, the minimum $T_{\mathbf{M}}$ and the algebraic product $T_{\mathbf{P}}$.

Concluding this section, we may state that under reasonable conditions the Mamdani inference meets the requirements of a fuzzy interpolation only if the t-norm $T$ has zero divisors (e.g., the Łukasiewicz t-norm $T_{\mathbf{L}}(a, b) = \max\{a + b - 1, 0\}$). However, for t-norms having zero divisors the degrees of indistinguishability of the consequents $Y_1, \ldots, Y_n$ often vanish. This leads to the restriction that $T(X_i(x), X_j(x)) = 0$, and, in particular, $X_i(x) = 1$ implies $X_j(x) = 0$, whenever $i \neq j$. (This restriction is weaker than that obtained for t-norms without zero divisors.)

## 3 RESIDUUM-BASED FUZZY CONTROLLER

In this section we consider formula (1) with the fuzzy relation $R \in \mathcal{F}(X \times Y)$ of the form

$$R(x, y) = \min_{i=1}^{n} I_T(X_i(x), Y_i(y)). \tag{5}$$

This formula expresses better the original meaning of the rule base as a conjunction of implications. On the other hand, it does not allow a way of computing as efficient as the formula for the Mamdani controller (given before Th. 2.1). Again the question arises under which conditions on the rule base $\Theta$ the relation (5) satisfies the system of relational equations (2). It is known that if the system (2) has a solution at all, then the relation (5) is a solution (for details see, e.g., [3]). Therefore the restrictions induced by (2) are as weak as possible provided that a compositional rule of inference of the form (1) is used. In particular, inequality (4) is a sufficient criterion for (5) to satisfy (2).

Contrary to the case of $Mam_{\Theta}$, no nice necessary and sufficient condition for $Res_{\Theta}$ to satisfy (2) is known. The reasons why this seems to be impossible are analysed in detail in [5], where one can also find an example of a simple rule base for which (2) cannot be satisfied by means of any compositional rule of inference of the form (1).

Being the maximal possible solution of (2), the residuum-based fuzzy controller gives the best chance to satisfy (2) without getting empty outputs for some normal inputs. Nevertheless, we encounter the following problem arising only for t-norms *with* zero divisors. As the input-output correspondence of a controller can be considered as an interpolation of "fuzzy points" given by the rule base, the following requirement seems to be quite natural: *If the input set $X^*$ belongs to the convex hull of the premises $\{X_i : i \in F\}$ for some $F \subseteq \{1, \ldots, n\}$, then the corresponding output $Y^*$ belongs to the convex hull of the corresponding consequents $\{Y_i : i \in F\}$.*

The latter condition has rather restrictive consequences if $T$ has zero divisors. Typically we may obtain output sets which are nonzero on the whole output space $Y$:

**Proposition 3.1** *Suppose that $\gamma := \inf\{a \in [0,1] : I_T(a, 0) = 0\} > 0$ and*

$$\inf_{x \in X} \max_{i=1}^{n} X_i(x) < \gamma.$$

*Then there is a constant $\delta > 0$ and a normal input $X^* \in \mathcal{F}(X)$ such that $Res_{\Theta}(X^*)(y) \geq \delta$ for all $y \in Y$.*

If the output set is greater than some $\delta > 0$ everywhere, this means that "any output value is at least partially good". This is not a desirable output, depending, e.g., on the bounds of the output space $Y$ (for fixed consequents $Y_1, \ldots, Y_n$). The output set is not localized to the respective consequents. We often encounter situations in which such an output set occurs and it does not belong to the convex hull of the corresponding consequents. This means that, for t-norms with zero divisors, the residuum-based inference rule does not express an interpolation properly.

## 4 CONCLUSION

We present arguments which show the disadvantages of the use of t-norms without zero divisors in Mamdani controllers. On the other hand, we show that t-norms with zero divisors cause problems when used in residuum-based controllers. These problems are inevitable when we restrict attention to (well-motivated) compositional inference rules. Generalizing the inference rule, one can obtain satisfactory behaviour in both cases; one such approach is suggested in [6].

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