Why quantum logic cannot be classical

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Classical event structure

σ-**algebra** of sets, $\mathcal{L} \subseteq 2^U$:

- $U \in \mathcal{L}$
- $\{A_i \mid i \in \mathbb{N}\} \subseteq \mathcal{L} \Longrightarrow \bigcup_{i\in\mathbb{N}} A_i \in \mathcal{L}$

State (=**probability measure**) $s\colon \mathcal{L} \to [0,1]$:

- $s(U) = 1$
- $\{A_i \mid i \in \mathbb{N}\} \subseteq \mathcal{L},\ A_i \cap A_j = \emptyset$ for $i \neq j \Longrightarrow s\left(\bigcup_{i\in\mathbb{N}} A_i\right) = \sum_{i\in\mathbb{N}} s(A_i)$

$\mathcal{S}(\mathcal{L})$:= **state space** of $\mathcal{L}$; it is a Choquet simplex
Pure states: extreme points of $\mathcal{S}(\mathcal{L})$
Two-valued states: $\mathcal{S}(\mathcal{L}) \cap \{0,1\}^{\mathcal{L}}$

For σ-algebras:

- pure states = two-valued states = points in the Stone space
- state space (even the space of two-valued states) determines the whole structure

We need **disjoint**, not all unions!

σ-**class** of sets, $\mathcal{L} \subseteq 2^U$:

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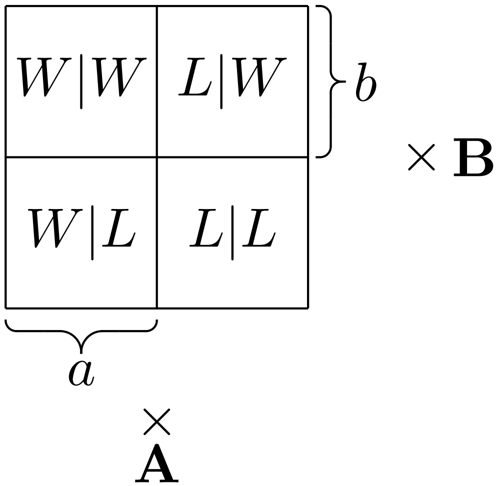
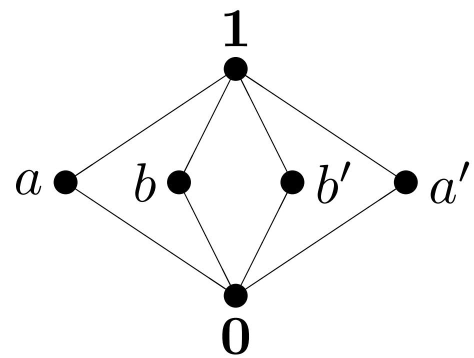
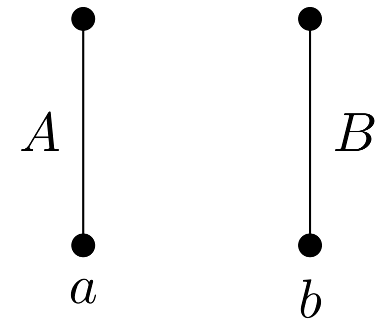
- $\{A_i \mid i \in \mathbb{N}\} \subseteq \mathcal{L},\ A_i \cap A_j = \emptyset$ for $i \neq j \Longrightarrow \bigcup_{i\in\mathbb{N}} A_i \in \mathcal{L}$

Example 1

W/L = Wins/Loses with|without player JJ







$U = \{W|W, W|L, L|W, L|L\}$

$A = \{\emptyset, \underbrace{\{W|W, W|L\}}_{a}, \underbrace{\{L|W, L|L\}}_{a'}, U\}$

$B = \{\emptyset, \underbrace{\{W|W, L|W\}}_{b}, \underbrace{\{W|L, L|L\}}_{b'}, U\}$

$\mathcal{L} = A \cup B = \{\emptyset, a, a', b, b', U\}$

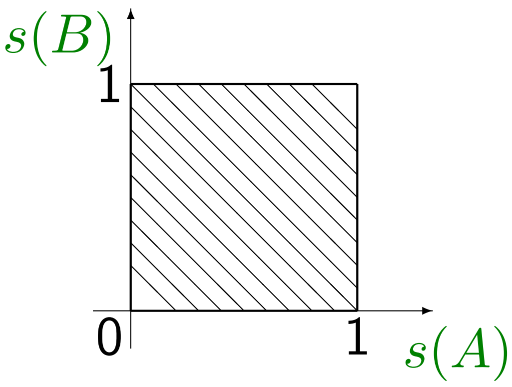
$\mathcal{L}$ is a (nondistributive modular) lattice called $MO2$

Pure states:
$s : \mathcal{L} \longrightarrow \{0, 1\}$

$s(A)$	$s(B)$
0	0
0	1
1	0
1	1

(S0) $s(U) = 1$
(S1) $s(x') = 1 - s(x)$

All states:
$s : \mathcal{L} \longrightarrow [0, 1]$, satisfy (S0), (S1)
$s(A) = p,\ s(B) = q,\ p, q \in [0, 1]$ arbitrary



Example 2

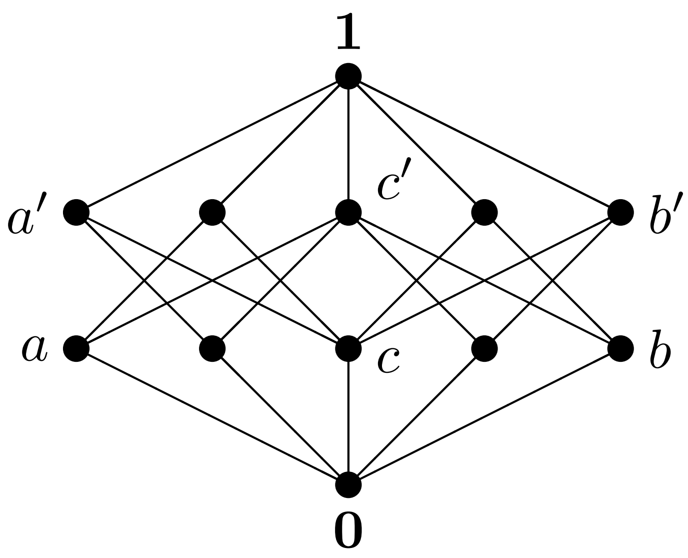
Example 1 with one more result, c = match cancelled

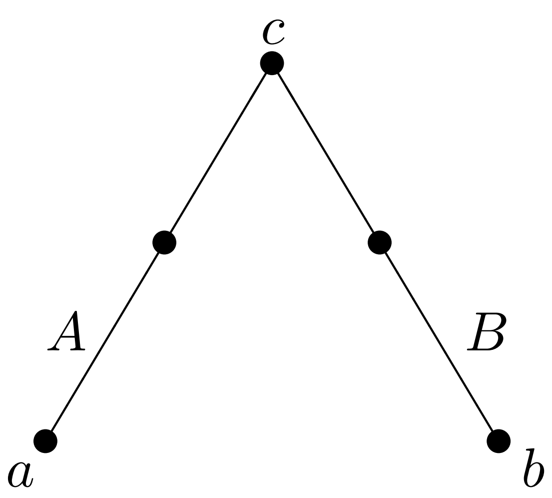
$A = \{\mathbf{0}, a, c, (a \vee c)', a \vee c, a', c', \mathbf{1}\}$

$B = \{\mathbf{0}, b, c, (b \vee c)', b \vee c, b', c', \mathbf{1}\}$

$A \cap B = \{\mathbf{0}, c, c', \mathbf{1}\}$

$\mathcal{L} = A \cup B = \{\mathbf{0}, a, b, c, a \vee c, b \vee c, (a \vee c)', (b \vee c)', a', b', c', \mathbf{1}\}$



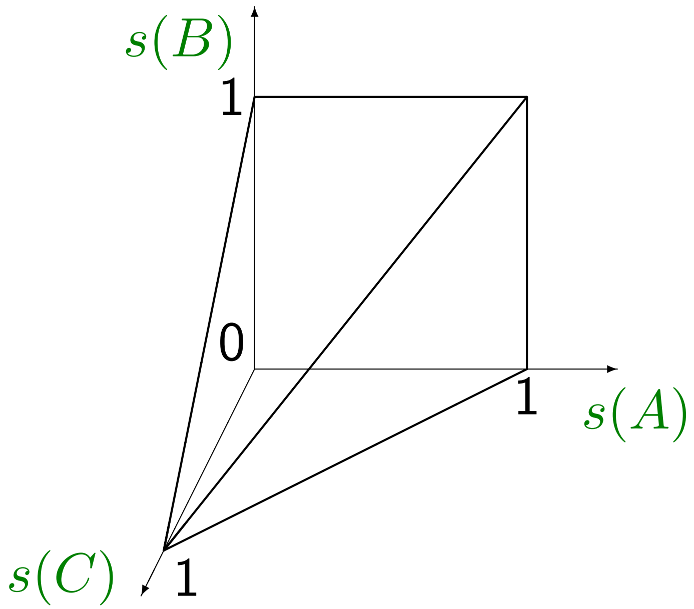


Pure states:

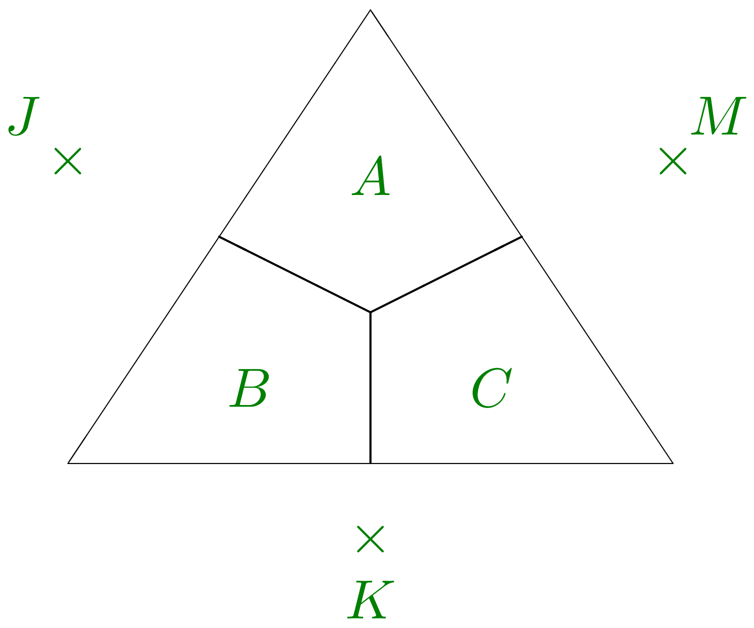
$s(A)$	$s(B)$	$s(C)$
0	0	0
0	1	0
1	0	0
1	1	0
0	0	1

All states:

$s(A) = p,\ s(B) = q,\ s(C) = r,\ \ r \in [0, 1]$ arbitrary, $p, q \in [0, 1 - r]$

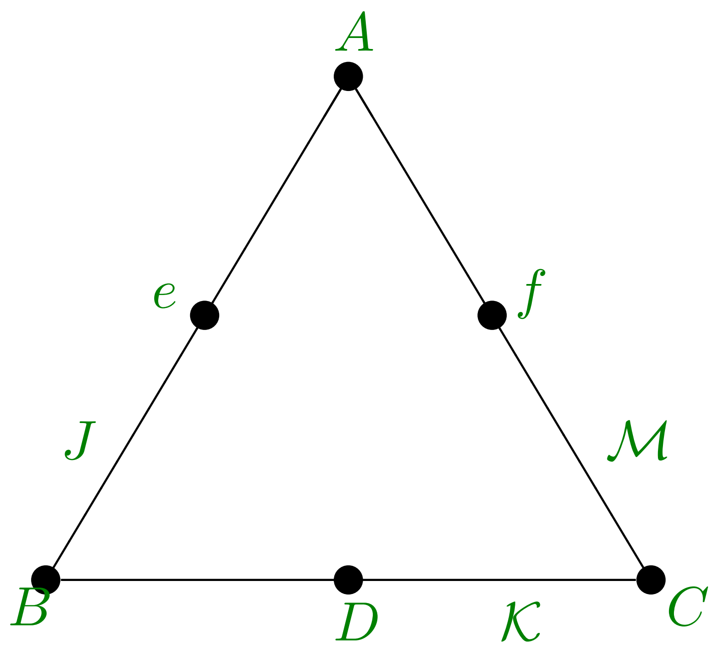


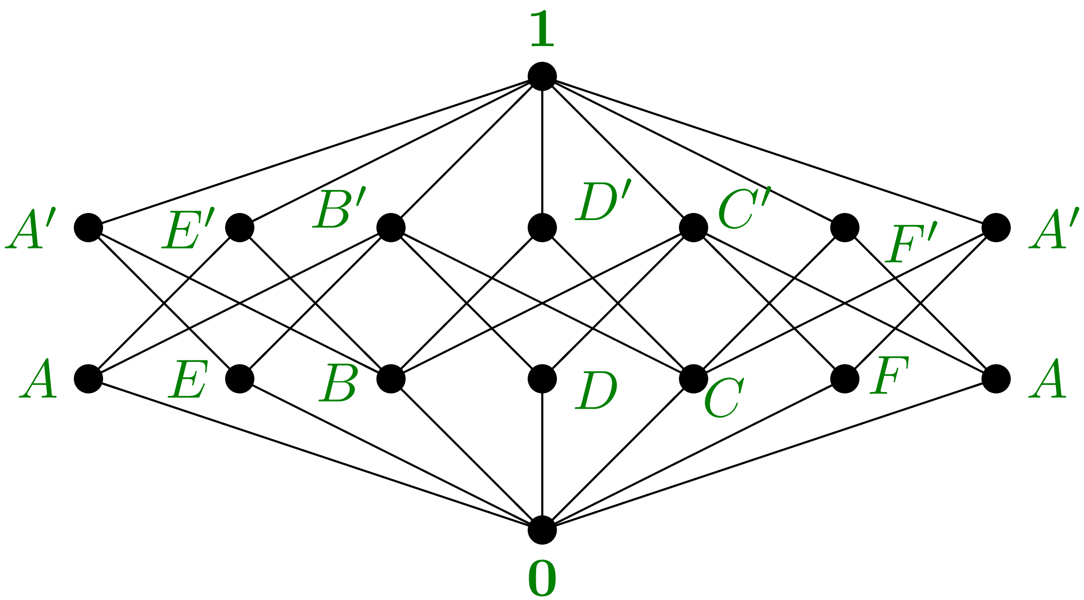
Example 3:



$\mathcal{K} = \{\mathbf{0}, B, C, D, B', C', D', \mathbf{1}\}$, where D means "the fire-fly is not observed from K"
$\mathcal{M} = \{\mathbf{0}, A, C, E, A', C', E', \mathbf{1}\}$, where A means "the fire-fly is observed in the upper part"
$\mathcal{J} = \{\mathbf{0}, A, B, F, A', B', F', \mathbf{1}\}$
$\mathcal{K} \cup \mathcal{M} \cup \mathcal{J} = \{\mathbf{0}, A, B, C, D, E, F, A', B', C', D', E', F', \mathbf{1}\}$

This is **not** a lattice.





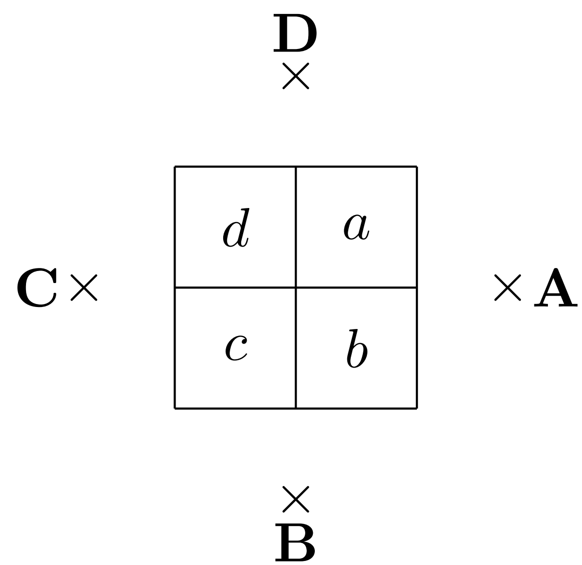
Pure states:

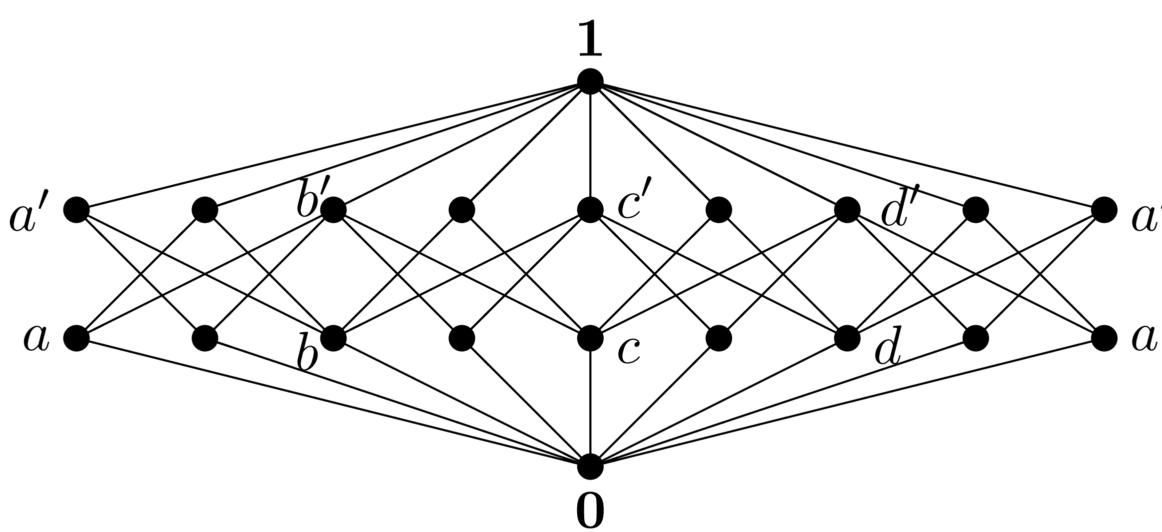
$s(A)$	$s(B)$	$s(C)$
1	0	0
0	1	0
0	0	1
0	0	0
1/2	1/2	1/2

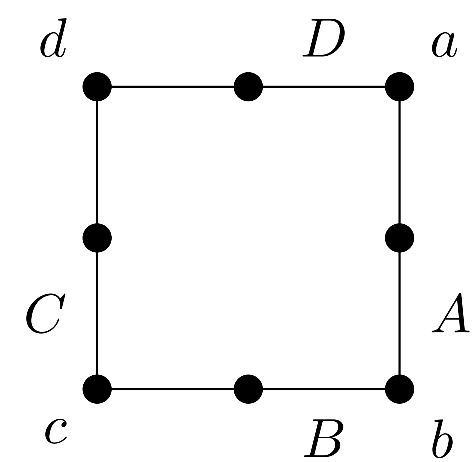
All states:

$s(A) = p,\ s(B) = q,\ s(C) = r, \quad p, q, r \in [0, 1],\ p + q \leq 1,\ p + r \leq 1,\ q + r \leq 1$

Example 3 (non-transparent barriers)







Pure states:

$s(a)$	$s(b)$	$s(c)$	$s(d)$
1	0	1	0
0	1	0	1
1	0	0	0
0	1	0	0
0	0	1	0
0	0	0	1
0	0	0	0

Orthomodular lattices

Definition: An **orthomodular lattice** is a lattice with bounds $\mathbf{0}$, $\mathbf{1}$ equipped with a unary operation $' : \mathcal{L} \to \mathcal{L}$ (**orthocomplementation**) such that, for all $a, b \in \mathcal{L}$,

- $a'' = a$
- $a \leq b \Longrightarrow b' \leq a'$
- $a \wedge a' = \mathbf{0}$
- $a \leq b \Longrightarrow b = a \vee (a' \wedge b)$ (**orthomodular law**)

Orthogonality: $a \perp b \iff a \leq b'$
(This condition is strictly stronger than the usual $a \wedge b = \mathbf{0}$.)

Example: A σ-class of subsets needs not be a lattice; if it is, it is an OML.

Structure of orthomodular lattices

Boolean subalgebra: $\mathcal{M} \subseteq \mathcal{L}$ such that

- $\mathbf{0}, \mathbf{1} \in \mathcal{M}$,
- $a \in \mathcal{M} \implies a' \in \mathcal{M}$,
- $(\mathcal{M},\ \leq\restriction_{\mathcal{M}},\ {}'\restriction_{\mathcal{M}})$ is a Boolean algebra.

Compatibility: $a \leftrightarrow b \iff \exists$ Boolean subalgebra $\mathcal{M}$: $a, b \in \mathcal{M}$

Block: a maximal Boolean subalgebra

Center: The set of all $a \in \mathcal{L}$ such that $\forall b \in \mathcal{L} : a \leftrightarrow b$
= the set of all "absolutely compatible" elements
= the classical part of the system
= the intersection of all blocks

Atom: $a \in \mathcal{L} \setminus \{\mathbf{0}\}$ such that there is no b satisfying $\mathbf{0} < b < a$
$\mathcal{A}(\mathcal{L}) :=$ the set of all atoms of $\mathcal{L}$

(σ-additivite) **state:** $s\colon \mathcal{L} \to [0,1]$ such that

- $s(\mathbf{1}) = 1$
- $\{a_i \mid i \in \mathbb{N}\} \subseteq \mathcal{L},\ a_i \perp a_j$ for $i \neq j \implies s\Big(\bigvee_{i \in \mathbb{N}} a_i\Big) = \sum_{i \in \mathbb{N}} s(a_i)$

Orthomodular lattices as families of Boolean algebras

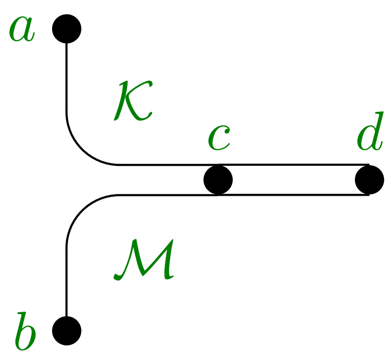
Every OML is the union of its maximal Boolean subalgebras (=blocks)

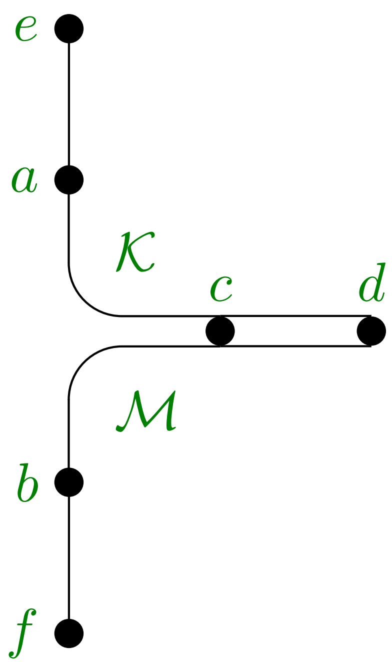
Hypergraph: a nonempty set (of **vertices**) and its covering by nonempty subsets (**edges**)

Greechie diagram: hypergraph whose vertices are atoms and edges are blocks

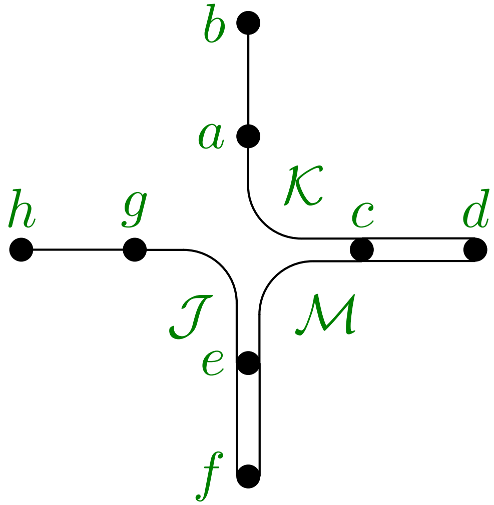
State on a hypergraph: evaluation of vertices such that the sum over each edge is 1

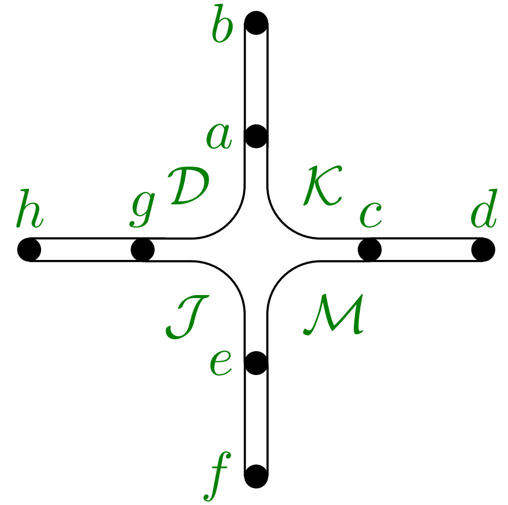
Problem: Which hypergraphs are Greechie diagrams of OMLs?



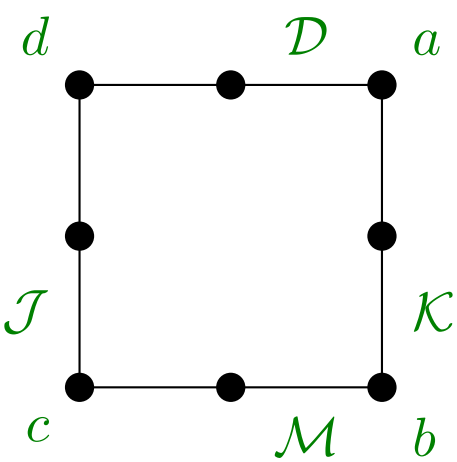


$a = (c \vee d)' = b$ $\qquad$ $a \vee e = b \vee f$



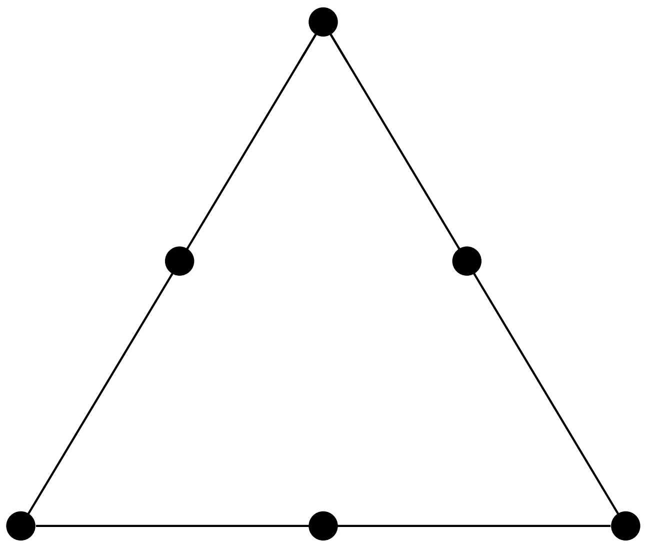


$$a \vee b = e \vee f \perp g \vee h$$



Orthoalgebras

Allowed



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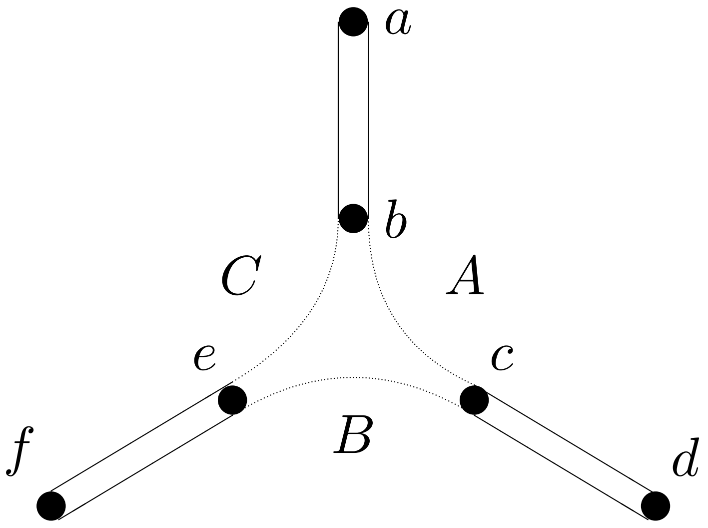
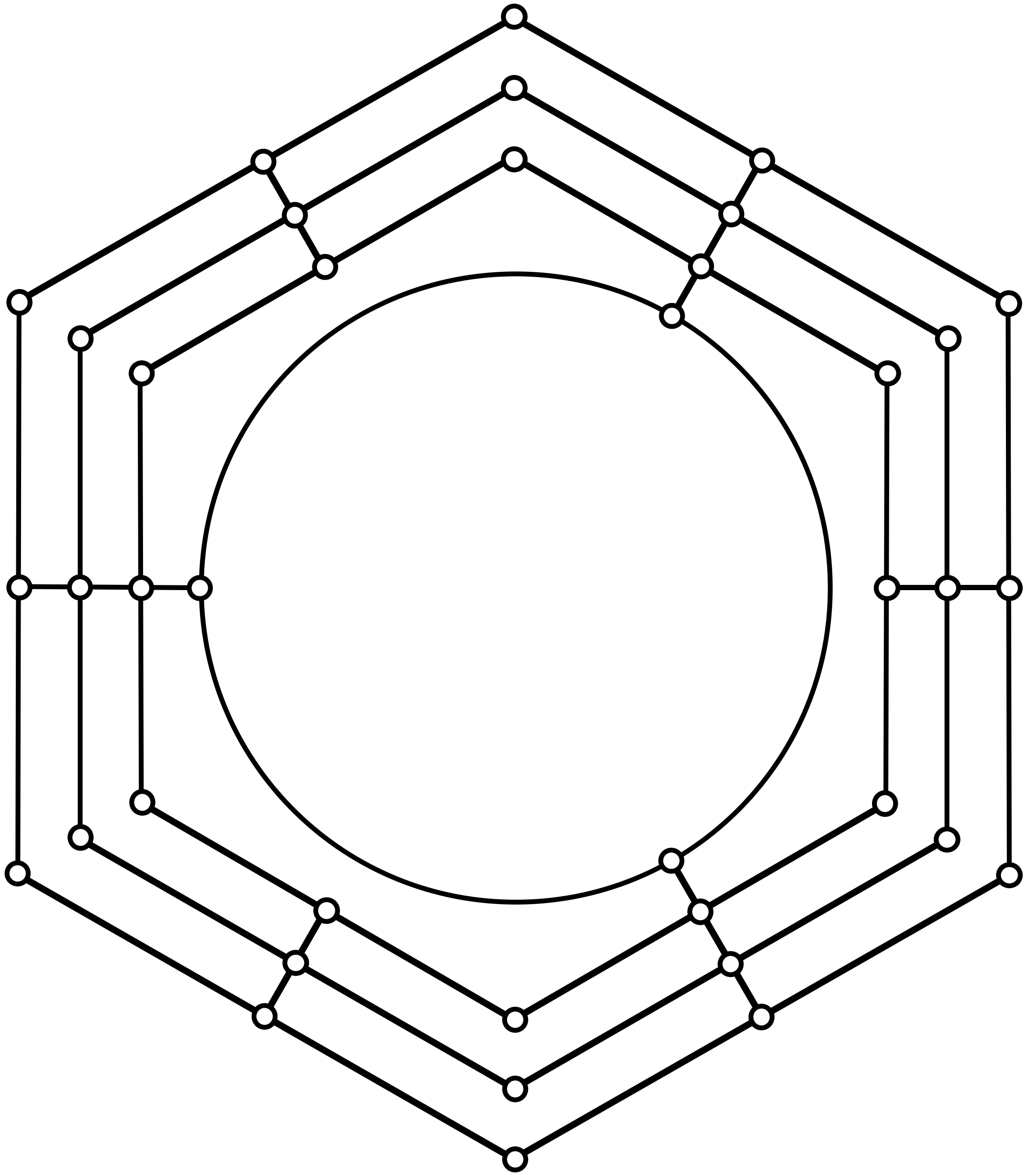
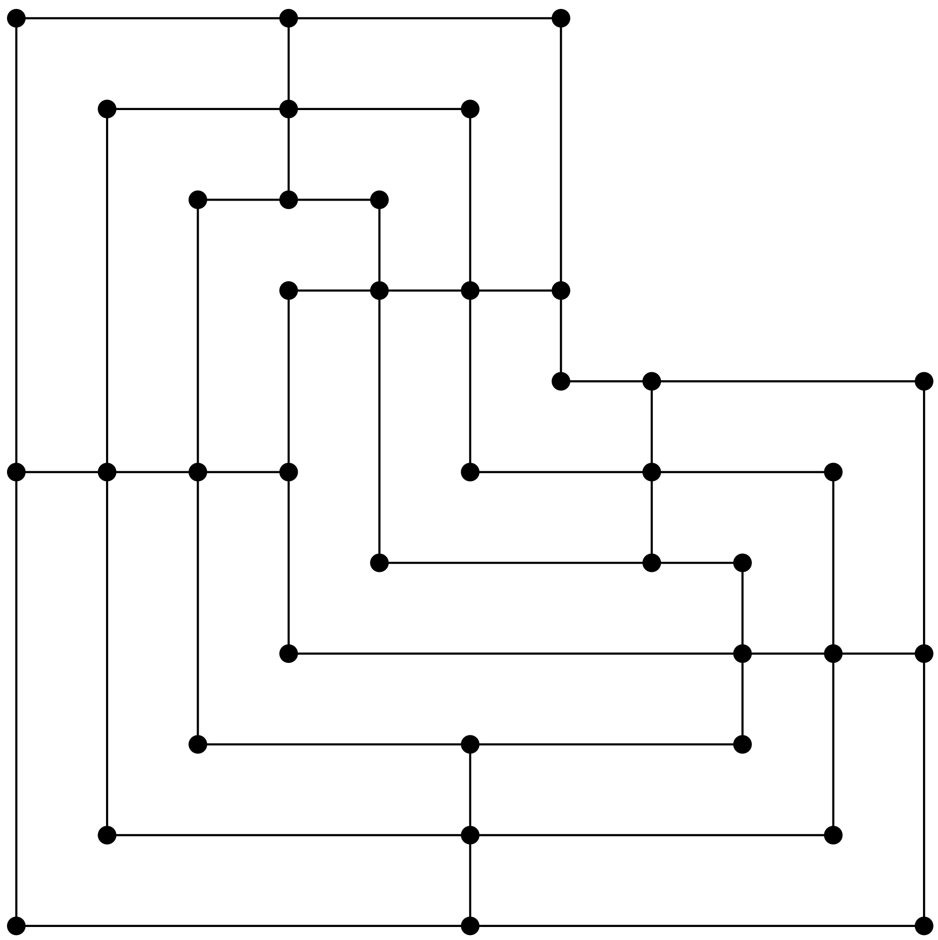
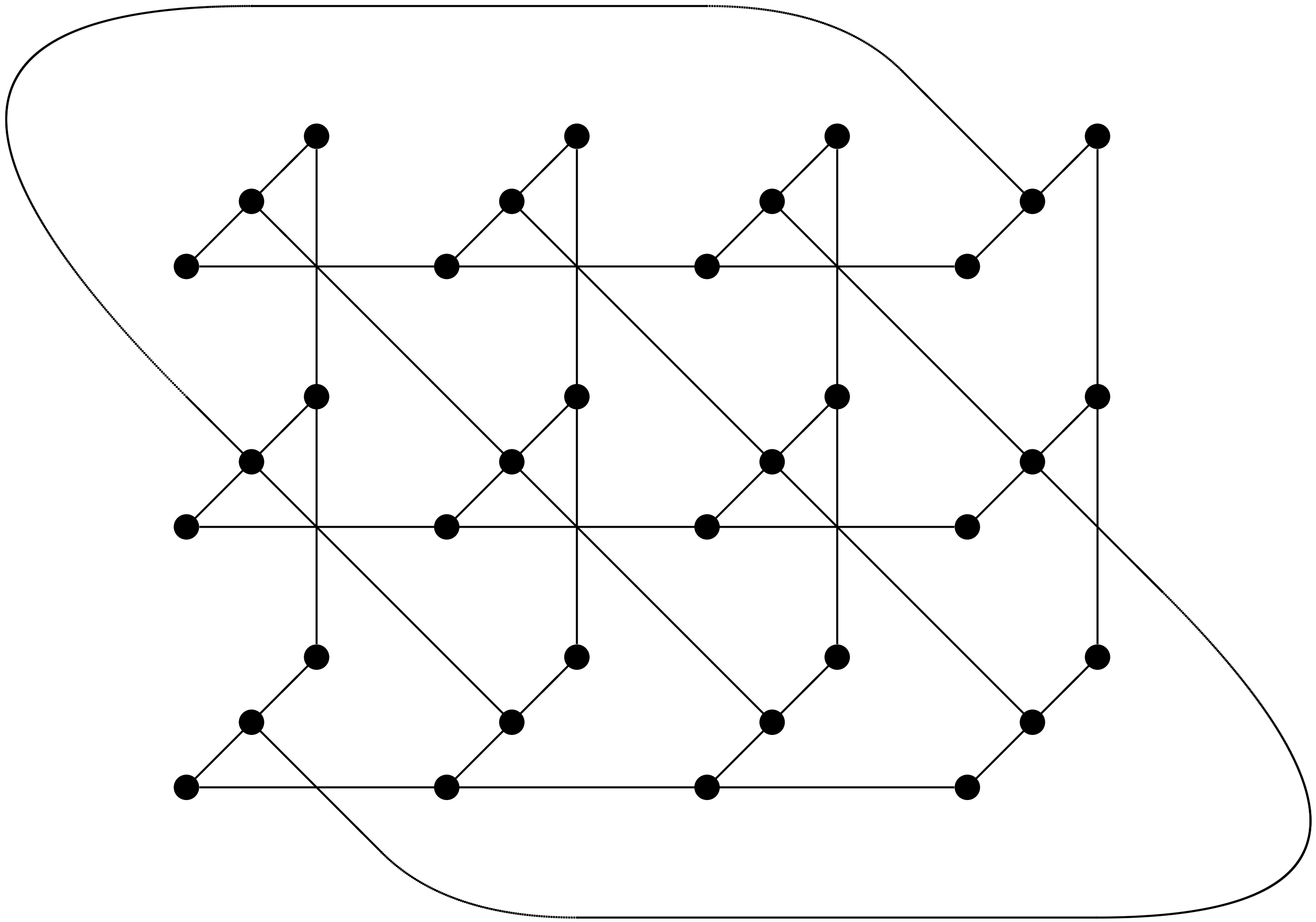


In particular, the state space may be empty [Rogalewicz]:

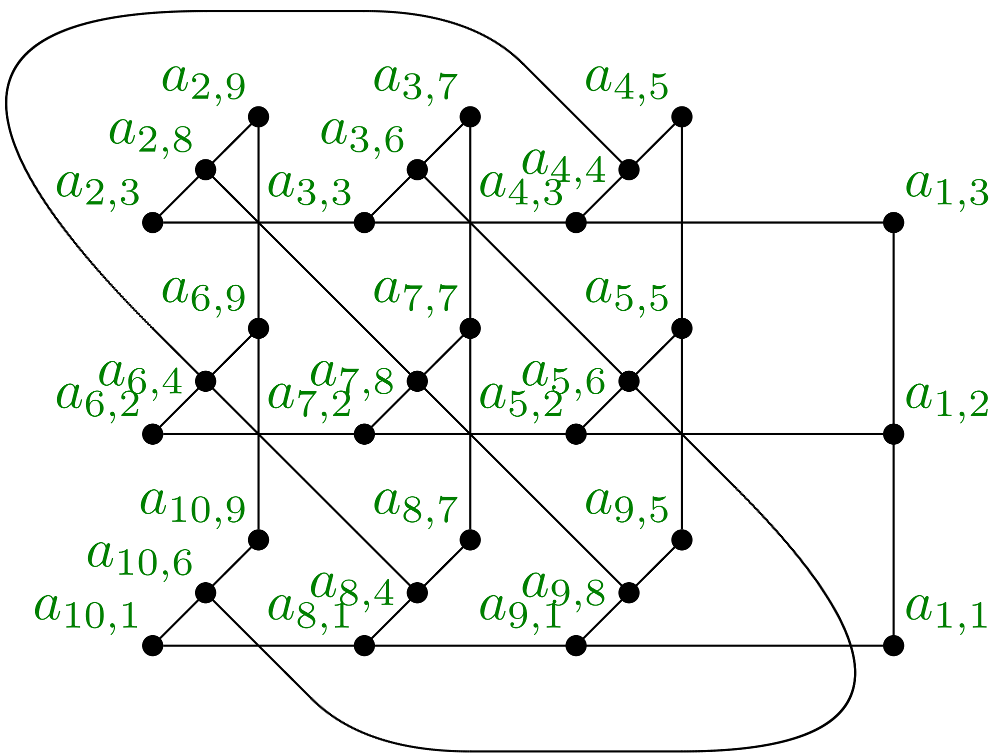


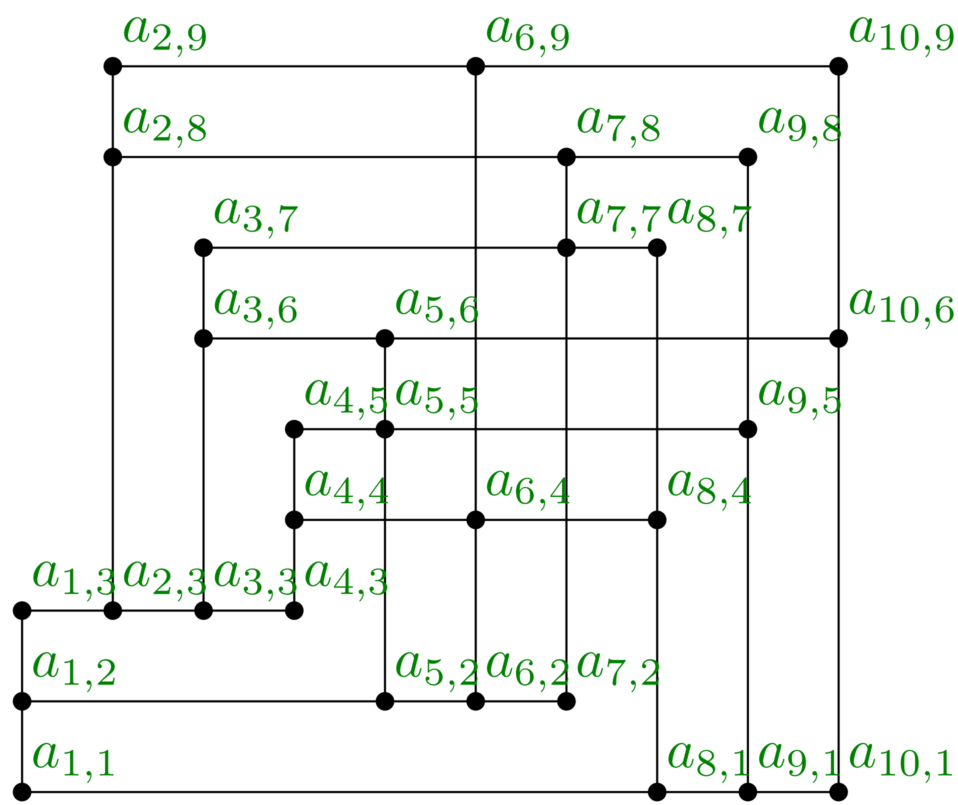


Smaller example with empty state space [Greechie]:



Even smaller example with empty state space [R. Mayet]:





This is the smallest example with empty state space obtained by this technique and it is not unique [MN 08]; it has 19 blocks.

OMLs with ≤ 5 blocks admit states [Riečanová 07].

Bell inequalities

$s(a) + s(b) - s(a \wedge b) \leq 1$

$0 \geq s(a \wedge b) + s(b \wedge c) + s(c \wedge d) - s(a \wedge d) - s(b) - s(c)$

$s(a) + s(b) + s(c) - s(a \wedge b) - s(a \wedge c) - s(b \wedge c) \leq 1$

$s(a \wedge b) + s(b \wedge c) + s(c \wedge d) - s(a \wedge d) - s(b) - s(c) \geq -1$

The first is equivalent to the **valuation property**:

$s(a \wedge b) + s(a \vee b) = s(a) + s(b)$

If the OML is not a Boolean algebra and admits a rich set of states, all Bell inequalities are violated.

Crucial example of a quantum structure: Hilbert lattice

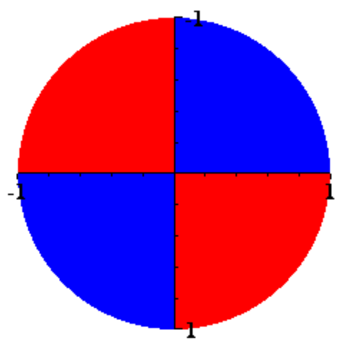
H ... a separable Hilbert space (real or complex)
$L(H)$ ... the set of all closed subspaces of H (equivalently, all projectors of H)

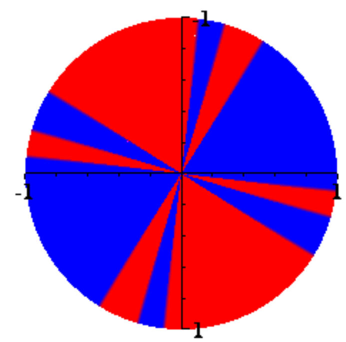
$$\begin{aligned} \mathbf{0} &= \{0\}, \\ \mathbf{1} &= H, \\ A \le B &\iff A \subseteq B, \\ A \wedge B &= A \cap B, \\ A' &= \{x \in H \mid \forall y \in A : y \perp x\}, \\ A \vee B &= Lin(A \cup B), \end{aligned}$$

where Lin denotes the closed linear hull

The only restriction of states for $\dim P = 1$: $\quad s(P') = 1 - s(P)$

Many two-valued states = colourings of non-zero vectors by two colors (blue, red) such that each orthogonal basis contains exactly one red vector





1. For $q \in H$, $||q|| = 1$, define a **vector state**

$$s_q(\mathit{Lin}(\{y_1, \ldots, y_n\})) = \sum_{i=1}^{n} (q \cdot y_i)^2 = \sum_{i=1}^{n} \cos^2 \sphericalangle(q, y_i)$$

for any orthonormal basis $(y_1, \ldots, y_n)$ of H

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Corollary: $s_q(q) = 1$

States on Hilbert lattice $\mathbb{R}^3$

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2. **Mixture** of vector states $s(P) = \sum_i c_i s_{q_i}(P)$, where $c_i > 0$, $\sum_i c_i = 1$.

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2. **Mixture** of vector states $s(P) = \sum_i c_i s_{q_i}(P)$, where $c_i > 0$, $\sum_i c_i = 1$.

3. What else?

Nothing!

Gleason's Theorem [Gleason 57]: For $\dim H \geq 3$, all states are mixtures of vector states.

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30

Crucial case: $H = \mathbb{R}^3$ (simplified proof by [Cooke, Keane, Moran 85]).

Corollary 1: The restriction of a state to 1D subspaces is continuous (proved by [von Neumann 1932] even for $\mathbb{R}^2$, error found by [Hermann 1935],

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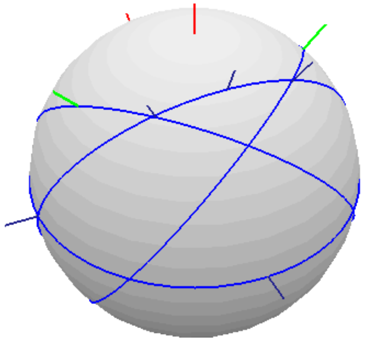
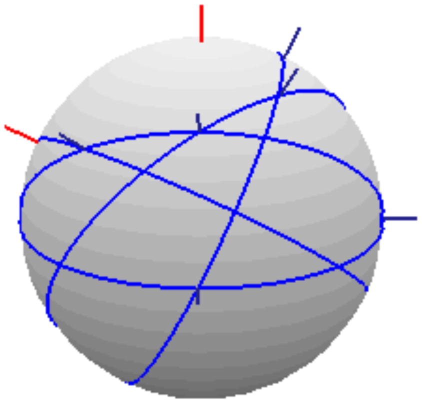
There is no colouring of non-zero vectors by two colors (blue, red) such that each orthogonal basis contains exactly one red vector

Constructions proving Geometrical Lemmas

BGL

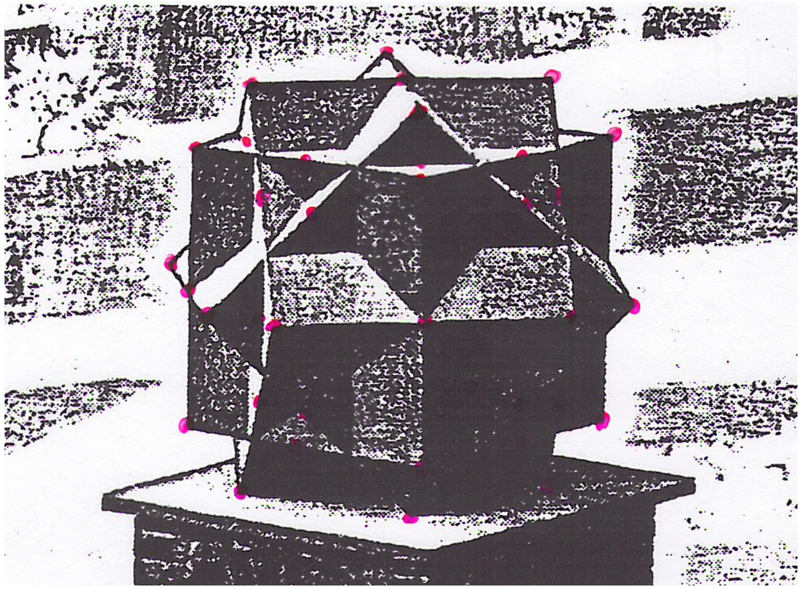


MGL



It is possible to find a finite set of vectors whose orthogonality relations exclude the possibility of a two-valued state.

The smallest example known uses 31 vectors, the following uses 33 vectors:



Constructions proving the non-existence of two-valued states in $\mathbb{R}^4$

Theorem: [Cabello] There is no two-valued state on $\mathcal{L}(\mathbb{R}^4)$.

Take 36 vectors in $\mathbb{R}^4$ ($\bar{1}$ denotes -1):

1000	1000	0100	1111	1111	$111\bar{1}$	$11\bar{1}\bar{1}$	$111\bar{1}$	$11\bar{1}1$
0100	0010	0010	$11\bar{1}\bar{1}$	$1\bar{1}1\bar{1}$	$11\bar{1}1$	$1\bar{1}1\bar{1}$	$1\bar{1}11$	$1\bar{1}11$
0011	0101	1001	$1\bar{1}00$	$10\bar{1}0$	$1\bar{1}00$	1001	$10\bar{1}0$	$100\bar{1}$
$001\bar{1}$	$010\bar{1}$	$100\bar{1}$	$001\bar{1}$	$010\bar{1}$	0011	0110	0101	0110

Each of the 9 column represents an orthogonal basis of $\mathbb{R}^4$ and each vector **occurs twice**. The number of vectors of unit state in this table must be both even and odd (9)—a contradiction.

Corollaries for group-valued states

Theorem: There is no nontrivial $\mathbb{Z}_2$-valued state, s, on $\mathcal{L}(R^4)$ which satisfies $s(\mathbf{1}) = 1$.

Theorem: If $n \geq 4$, then there is no nontrivial $\mathbb{Z}_2$-valued state on $\mathcal{L}(R^n)$.

For $n \geq 5$ it follows from the above construction.
The refinement for $n = 4$ is due to [Harding, Jager, Smith]

Open problem: Are there nontrivial $\mathbb{Z}_2$-valued states on $\mathcal{L}(R^3)$?

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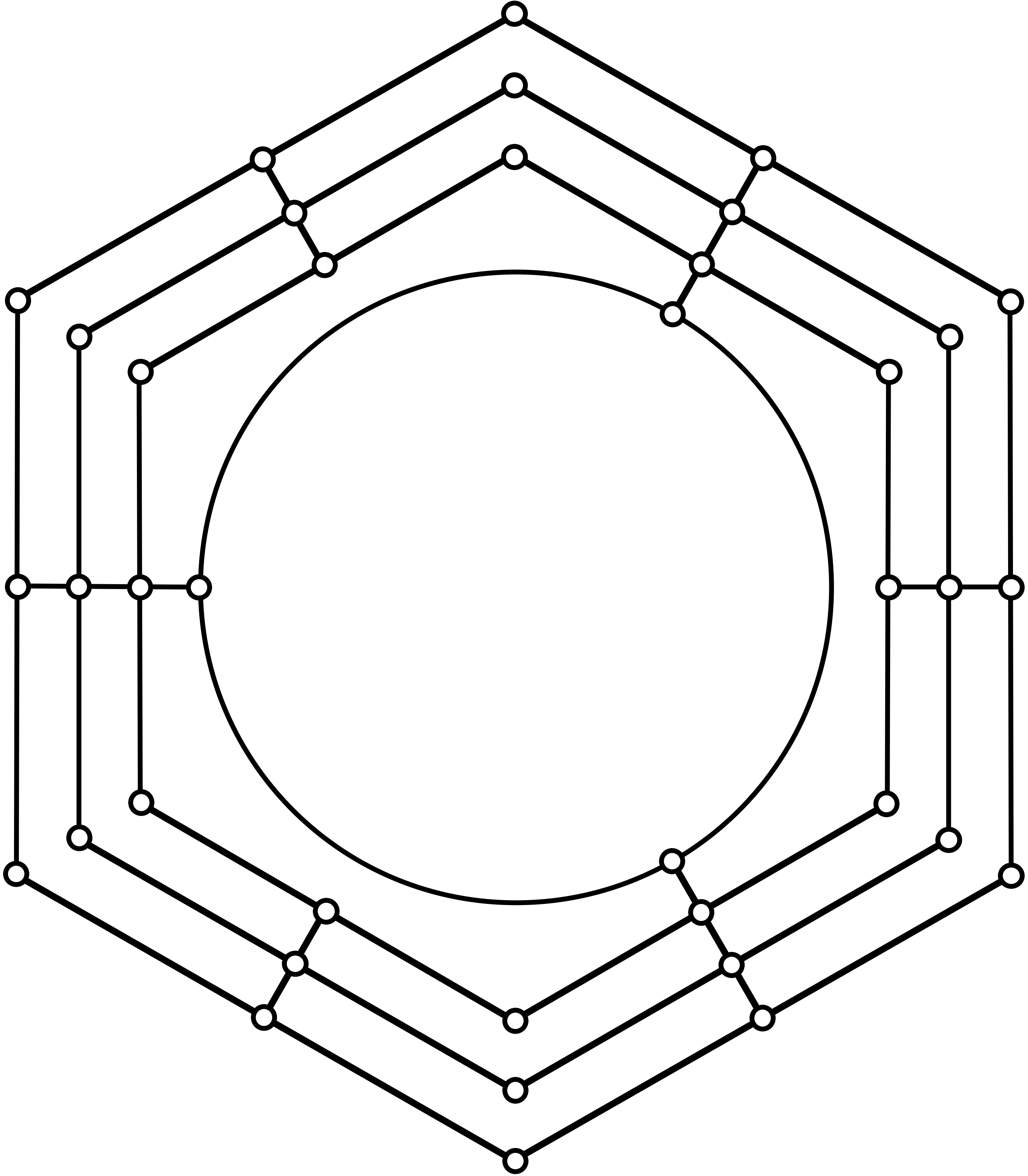
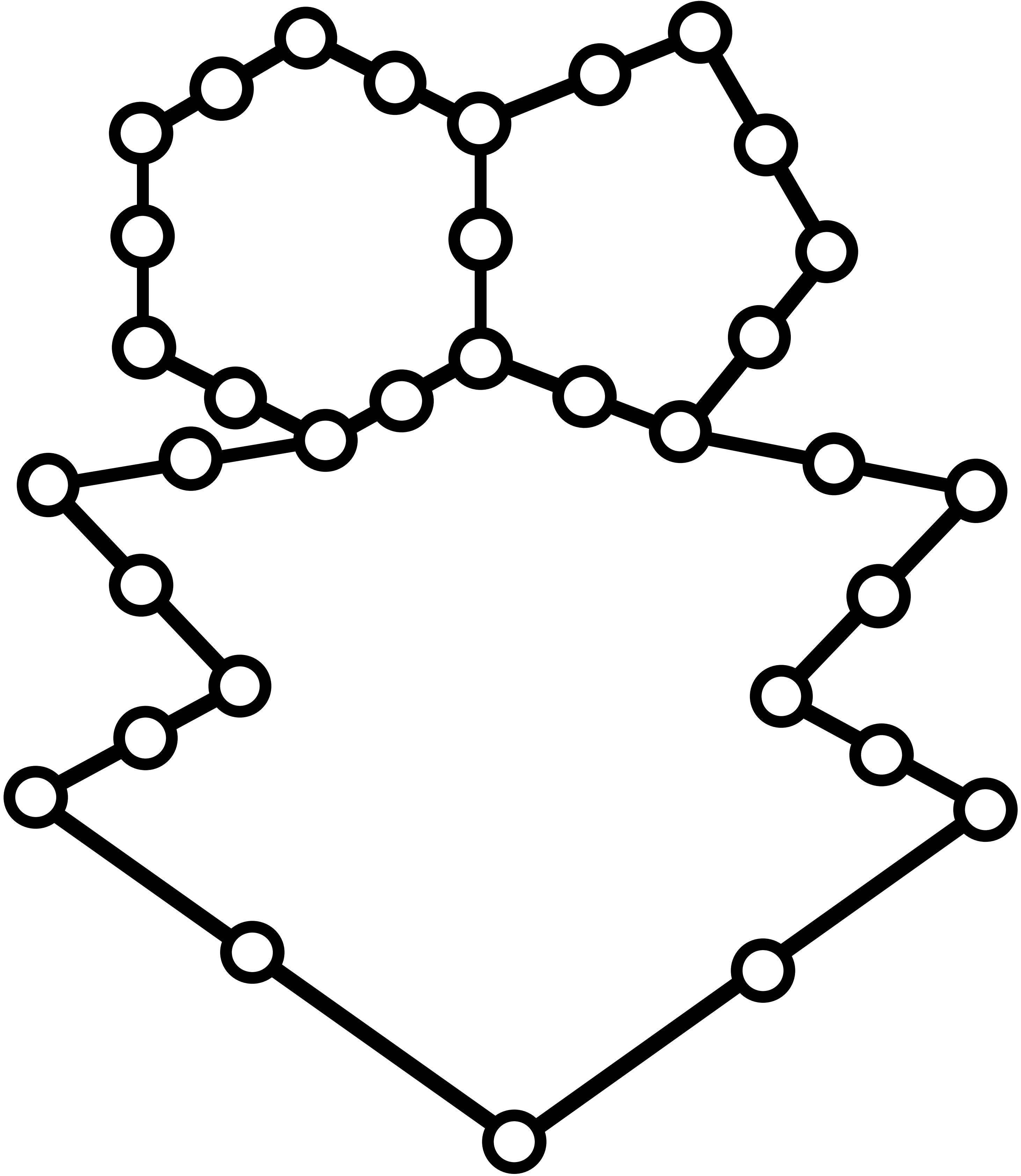
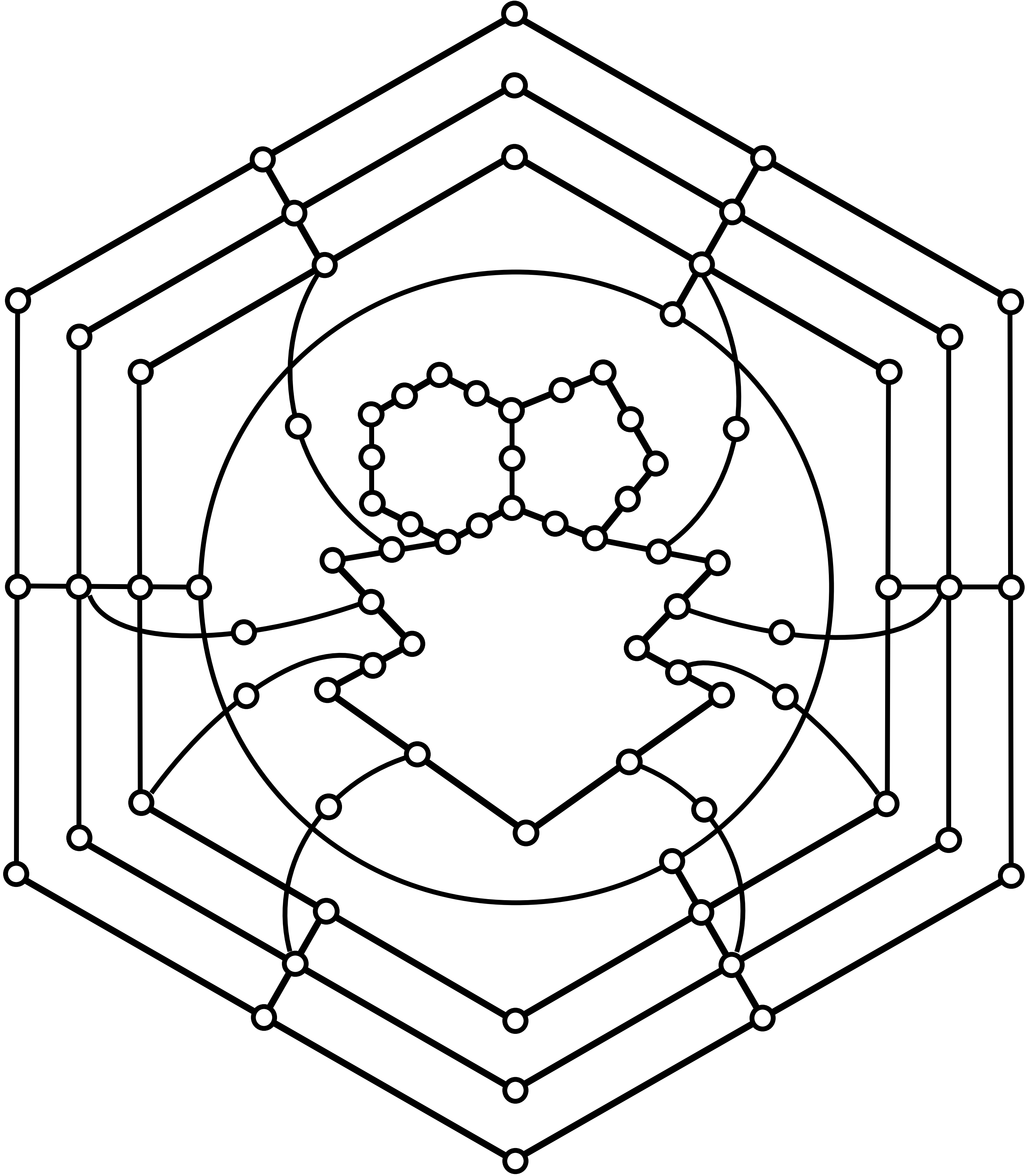
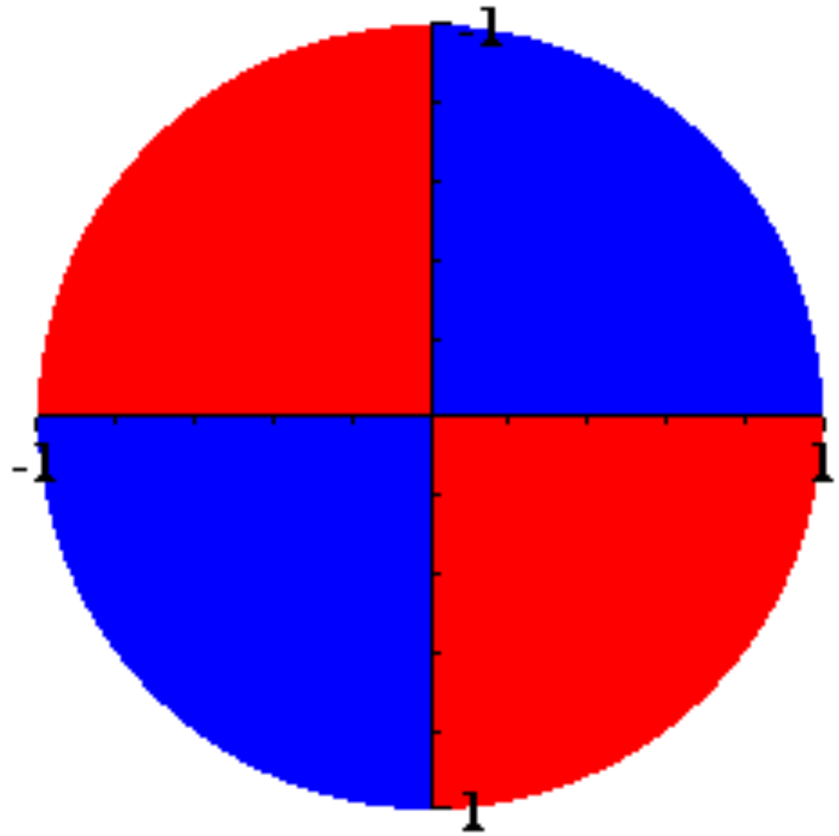
Details of pictures





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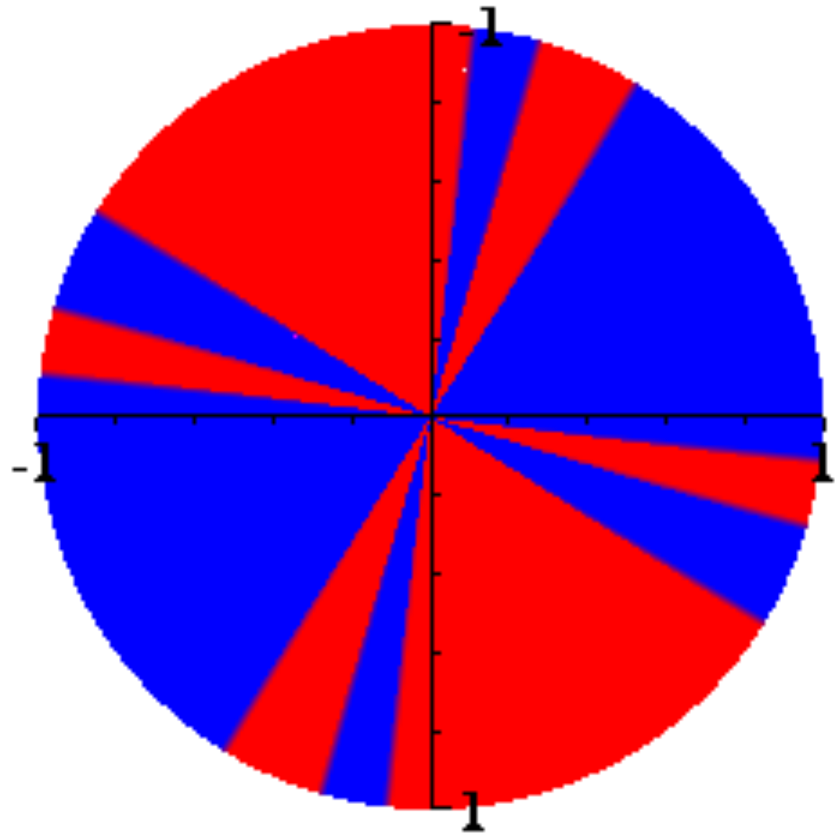
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